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ON $\mathcal{M}$-SUBHARMONICITY IN THE BALL

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Abstract. We establish an easy proof of an integral identity using the unitary invariance, which is applied to compare harmonicity and M-harmonicity.

1. Introduction

1.1. Let $B = B_n$ denote the open unit ball of the complex n-dimensional space $\mathbb{C}^n$ and S denote the boundary of B: $S = \{z \in \mathbb{C}^n : |z| = 1\}$. Let $\mathcal{M}$ denote the group of all automorphisms, that is, one to one biholomorphic onto maps, of B. Let ν and σ denote respectively the Lebesgue volume measure on B and the surface measure on S normalized to be $\nu(B) = \sigma(S) = 1$, and τ denote the $\mathcal{M}$-invariant volume measure of B: $d\tau(z) = (1 - |z|^2)^{-(n+1)} d\nu(z)$.

1.2. $\mathcal{M}$ consists of all maps of the form $U\varphi_a$, where U is a unitary transformation of $\mathbb{C}^n$ and φ_a is defined by

$$\varphi_a(z) = \begin{cases} \frac{a - P_a z - \sqrt{1-|a|^2} Q_a z}{1 - \langle z, a\rangle}, & \text{if } a \neq 0 \\ 0, & \text{if } a = 0. \end{cases} \tag{1.1}$$

Here $\langle , \rangle$ is the Hermitian inner product of $\mathbb{C}^n$: $\langle z, w\rangle = \sum_{j=1}^n z_j \bar{w}_j,\ z,\ w \in \mathbb{C}^n$, $P_a z$ is the projection of $\mathbb{C}^n$ onto the subspace generated by B:

$$P_a z = \frac{\langle z, a\rangle}{\langle a, a\rangle} a, \quad \text{if } a \neq 0 \quad \text{and} \quad P_0 z = 0,$$

and $Q_a(z) = z - P_a z$.

1.3. An upper semicontinuous function $f : B \to [-\infty,\ \infty)$, $f \not\equiv -\infty$, satisfying the inequality

$$f(a) \le \int_S f(a + r\zeta)\ d\sigma(\zeta)$$

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for all $a \in B$ and for all r such that $a + r\bar{B} \subset B$ is called subharmonic (in B). An upper semicontinuous function $f : B \to [-\infty,\ \infty)$, $f \not\equiv -\infty$, satisfying

$$f(a) \le \int_S f \circ \varphi_a(r\zeta)\ d\sigma(\zeta)$$

for all $a \in B$ and for all r sufficiently small is called $\mathcal{M}$-subharmonic. Also, an upper semicontinuous function $f : B \to [-\infty,\ \infty)$, is called plurisubharmonic if the functions

$$\lambda \to f(a + \lambda b)$$

are subharmonic in neighborhoods of the origin in $\mathbb{C}$ for all $a \in B$, $b \in \mathbb{C}^n$.

1.4. Let Δ denote the complex Laplacian of $\mathbb{C}^n$ and $\widetilde{\Delta}$ denote the Laplace-Beltrami operator associated with the Bergman kernel of B, that is, for $f \in C^2(B)$

$$\Delta f = 4 \sum_{j=1}^{n} D_j \bar{D}_j f$$

and

$$\widetilde{\Delta} f(a) = 4(1 - |a|^2) \sum_{i,j=1}^{n} (\delta_{i,j} - \bar{a}_i a_j)(\bar{D}_i D_j f)(a), \quad a = (a_1, a_2, \ldots, a_n) \in B,$$

where $D_j = \frac{\partial}{\partial z_j}$ and $\bar{D}_j = \frac{\partial}{\partial \bar{z}_j}$, $j = 1, 2, \ldots, n$. Let f_a, for $a \in B$, be defined by $f_a(\lambda) = f(\lambda a), \lambda \in B_1$. Then $\widetilde{\Delta} f(a)$ is equivalently expressible as

$$\widetilde{\Delta} f(a) = (1 - |a|^2) \left\{ \Delta f(a) - \Delta f_a(1) \right\}. \tag{1.2}$$

See [1, Theorem 4.1.3-(iii)].

1.5. A $C^2(B)$ function f is said to be harmonic if $\Delta f \equiv 0$, $\mathcal{M}$-harmonic if $\widetilde{\Delta} f \equiv 0$, pluriharmonic if $\Delta f \equiv 0 \equiv \widetilde{\Delta} f$ in B. For real valued $f \in C^2(B)$, $\Delta f \ge 0$ if and only if f is subharmonic, and $\widetilde{\Delta} f \ge 0$ if and only if f is $\mathcal{M}$-subharmonic (see [1, Proposition 4.1]).

It is generally known on B that the harmonicity and the $\mathcal{M}$-harmonicity are neither inclusive nor exclusive when $n \ge 2$. Also $\Delta f \ge 0$ and $\widetilde{\Delta} f \ge 0$ does not imply that f is plurisubharmonic (see [1, 7.2.1]).

2. Harmonicity vis $\mathcal{M}$-harmonicity

2.1. In B, that $\Delta f \equiv 0$ does not imply $\widetilde{\Delta} f \equiv 0$ when $n \ge 2$. Neither is the converse. But harmonicity and $\mathcal{M}$-harmonicity are equal in the following sense of mean:

Theorem 2.1. *Let $f \in C^2(B)$. Then, for a fixed $r : 0 < r < 1$,*

$$\int_{rB} \Delta f\ d\nu\ =\ 0 \iff \int_{rB} \widetilde{\Delta} f\ d\tau\ =\ 0.$$

2.2. As in 1.5, a $C^2(B)$ function harmonic and simultaneously $\mathcal{M}$-harmonic is pluriharmonic. But 1.5 also says that this is no more true when 'harmonic' replaced by 'subharmonic', that is, subharmonicity together with $\mathcal{M}$-subharmonicity can not imply plurisubharmonicity. Between these two different qualities, we have:

Theorem 2.2. *A $C^2(B)$ function harmonic and simultaneously $\mathcal{M}$-subharmonic is pluriharmonic; a $C^2(B)$ function $\mathcal{M}$-harmonic and simultaneously subharmonic is pluriharmonic.*

Theorem 2.1 and Theorem 2.2 will be proven in Section 3.

3. Consequence of the unitary invariance

3.1. We note that the measure σ is unitary invariant in the sense that

$$\int_S f(U\zeta)\, d\sigma(\zeta) = \int_S f(\zeta)\, d\sigma(\zeta)$$

for unitary transformations U of $\mathbb{C}^n$. Δ and $\widetilde{\Delta}$ are also unitary invariant as in the following lemma, whose simple proof we include for the reader's convenience.

Lemma 3.1. *Let $f \in C^2(B)$. Then Δf and $\widetilde{\Delta} f$ are unitary invariant in the sense that*

$$(\Delta f)(Ua) = \Delta(f \circ U)(a) \tag{3.1}$$

and

$$\left(\widetilde{\Delta} f\right)(Ua) = \widetilde{\Delta}(f \circ U)(a) \tag{3.2}$$

for all unitary transformations U of $\mathbb{C}^n$ and $a \in B$.

Proof. Fix $a \in B$. Taylor expansion of $C^2(B)$ function $f(z)$ about $z = a$ gives

$$f(a+\rho\eta)-f(a) = \sum_{j=1}^{n} \left(\rho\eta_j D_j f + \rho\bar{\eta}_j \bar{D}_j f\right)(a) + \frac{1}{2}\sum_{j,k=1}^{n} \rho^2 \eta_j \bar{\eta}_k \left(D_j \bar{D}_k f\right)(a+\delta\rho\eta)$$

for some δ provided $a + \rho\eta \in B,\ \rho > 0$. Integrating with respect to $d\sigma(\eta)$ over S and dividing by ρ^2, we have

$$\frac{1}{\rho^2}\int_S \{f(a+\rho\eta) - f(a)\}\, d\sigma(\eta) = \frac{1}{2}\sum_{j,k=1}^{n}\int_S \eta_j\bar{\eta}_k \left(D_j\bar{D}_k f\right)(a+\delta\rho\eta)\, d\sigma(\eta).$$

By letting $\rho \to 0$, we obtain

$$\lim_{\rho\to 0}\frac{1}{\rho^2}\int_S \{f(a+\rho\eta) - f(a)\}\, d\sigma(\eta) = \frac{1}{2}\sum_{j,k=1}^{n}\left(D_j\bar{D}_k f\right)(a)\int_S \eta_j\bar{\eta}_k\, d\sigma(\eta).$$

Straightforward calculation of the last integral (see [1, 1.4.8 and 1.4.9]) gives

$$\Delta f(a) = \lim_{\rho\to 0}\frac{4n}{\rho^2}\int_S \{f(a+\rho\eta) - f(a)\}\, d\sigma(\eta). \tag{3.3}$$

Now, applying the unitary invariance of the measure σ to (3.3), we have (3.1).

Also, by a process similar to the process we used to obtain (3.3), we have the representation

$$\widetilde{\Delta} f(a) = \lim_{\rho\to 0} \frac{4n}{\rho^2} \int_S \{f\circ\varphi_a(\rho\eta) - f(a)\}\, d\sigma(\eta) \tag{3.4}$$

(see [1, 4.1.3-(a)]). Applying the identity $U\varphi_a U^{-1} = \varphi_{Ua}$, which is obvious from (1.1), to the equation (3.4), we have (3.2). □

3.2. By help of Lemma 3.1, we can have an identity linking Δ and $\widetilde{\Delta}$ as the following.

Theorem 3.2. *If $f \in C^2(B)$ and $0 < r < 1$, then*

$$\int_{rB} \Delta f(z)\, d\nu(z) \;=\; (1-r^2)^{n-1} \int_{rB} \widetilde{\Delta} f(z)\, d\tau(z). \tag{3.5}$$

Proof. Let $\mathcal{U}$ denote the group of unitary transformations of $\mathbb{C}^n$. Since $f\circ U \in C^2(B)$ for $U \in \mathcal{U}$, it follows from Lebesgue's dominated convergence theorem that

$$D_i \bar{D}_j \left(\int_{\mathcal{U}} f\circ U(z)\, dU \right) \;=\; \left(\int_{\mathcal{U}} D_i\bar{D}_j\,(f\circ U)\,(z)\, dU \right)$$

for $z \in B$ and $i, j = 1, 2, \ldots, n$. Thus, by [1, Proposition 1.4.7] and Lemma 3.1

$$\begin{aligned}
\int_S (\Delta f)\,(|z|\zeta)\; d\sigma(\zeta) &= \int_{\mathcal{U}} (\Delta f)\,(Uz)\; dU \\
&= \int_{\mathcal{U}} \Delta\,(f\circ U)\,(z)\; dU \\
&= \Delta \left(\int_{\mathcal{U}} (f\circ U)\,(z)\; dU \right) \\
&= \Delta \left(\int_S f(|z|\zeta)\; d\sigma(\zeta) \right).
\end{aligned}$$

Similarly,

$$\int_S \left(\widetilde{\Delta} f\right)(|z|\zeta)\; d\sigma(\zeta) \;=\; \widetilde{\Delta} \left(\int_S f(|z|\zeta)\; d\sigma(\zeta) \right).$$

Therefore to obtain (3.5), we are sufficient to prove

$$\int_0^r \rho^{2n-1} \Delta\phi(\rho)\; dr \;=\; (1-r^2)^{n-1} \int_0^r \frac{\rho^{2n-1}}{(1-\rho^2)^{n+1}} \widetilde{\Delta}\phi(\rho) d\rho, \tag{3.6}$$

where

$$\phi(\rho) = \int_S f(\rho\zeta)\; d\sigma(\zeta), \quad 0 < \rho < r.$$

It follows from a straightforward differentiation that

$$\Delta\phi(\rho) \;=\; \phi''(\rho) + \frac{2n-1}{\rho}\phi'(\rho)$$

and

$$\widetilde{\Delta}\phi(\rho) = (1-\rho^2)^2\Delta\phi(\rho) + 2(n-1)\rho(1-\rho^2)\phi'(\rho).$$

Now elementary calculation gives

$$\begin{aligned}
&\int_0^r \rho^{2n-1}\Delta\phi(\rho)dr \\
&= \int_0^r \rho^{2n-1}\left\{\phi''(\rho) + \frac{2n-1}{\rho}\phi'(\rho)\right\}d\rho \\
&= \int_0^r \frac{d}{d\rho}\left\{\rho^{2n-1}\phi'(\rho)\right\}d\rho \\
&= r^{2n-1}\phi'(r) \\
&= (1-r^2)^{n-1}\int_0^r \frac{d}{d\rho}\left\{\frac{1}{(1-\rho^2)^{n-1}}\rho^{2n-1}\phi'(\rho)\right\}d\rho \\
&= (1-r^2)^{n-1}\int_0^r \frac{1}{(1-\rho^2)^n}\left\{(1-\rho^2)\frac{d}{d\rho}\left(\rho^{2n-1}\phi'(\rho)\right) + 2(n-1)\rho^{2n}\phi'(\rho)\right\}d\rho \\
&= (1-r^2)^{n-1}\int_0^r \frac{\rho^{2n-1}}{(1-\rho^2)^{n+1}}\widetilde{\Delta}\phi(\rho)d\rho,
\end{aligned}$$

which verifies (3.6). □

3.3. Theorem 2.1 now follows from Theorem 3.2.

3.4. Theorem 2.2 follows from Theorem 2.1 by noting that a non-negative continuous function whose values integrated to be zero should be zero function.

4. Further remarks

4.1. More extensively, Theorem 2.1 and the formula (1.2) gives the followings. Here 'g has same sign in B' means that $g(a) \geq 0$ for all $a \in B$ or $g(a) \leq 0$ for all $a \in B$ or $g(a) = 0$ for all $a \in B$.

Theorem 4.1. *Let f be real-valued and $f \in C^2(B)$. If $\Delta f = 0$ in B, then each one of the following property are equivalent.*

(a) *f is pluriharmonic in B.*
(b) *$\widetilde{\Delta} f$ has same sign in B.*
(c) *$\Delta f_a(1)$ has same sign in B.*

4.2. Exactly same way, we have a similar result with Δf and $\widetilde{\Delta} f$ interchanged in Theorem 4.1.

References

[1] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York,1980.

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