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ON POLYNOMIAL-STRUCTURE OF RINGS OF MODULAR FORMS FOR $\Gamma_0(N)$

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Abstract. In this note, we show that $\mathcal{M}(\Gamma_0(N))$ is a weighted polynomial ring if and only if $N = 1, 2, 4$, where $\mathcal{M}(\Gamma_0(N))$ is the graded ring of integral-weighted modular forms for the congruence subgroup $\Gamma_0(N)$.

1. Introduction

Let $\mathcal{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > 0\}$ be the complex upper half plane and Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. Denote $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ the set of natural numbers, the ring of rational integers and the field of rational numbers. For $k \in \mathbb{Z}$, denote $M_k(\Gamma)$ the finite dimensional vector space of modular forms of weight k for Γ. Since $M_k(\Gamma) \cdot M_l(\Gamma) \subset M_{k+l}(\Gamma)$, the direct sum

$$\mathcal{M}(\Gamma) = \bigoplus_{k\in\mathbb{Z}} M_k(\Gamma)$$

forms a graded ring. For $\Gamma = SL_2(\mathbb{Z})$, it is well-known that $\mathcal{M}(\Gamma) = \mathbb{C}[E_4, E_6]$ is a weighted polynomial ring generated by E_4 and E_6, that is,

$$M_k(SL_2(\mathbb{Z})) = \bigoplus_{4a+6b=k} \mathbb{C} \cdot E_4^a E_6^b,$$

where, for $k \geq 4$ and even, $E_k \in M_k(SL_2(\mathbb{Z}))$ is the normalized Eisenstein series of weight k defined by

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad \tau \in \mathcal{H},$$

with

$$q = e^{2\pi i\tau}, \quad \sigma_{k-1}(n) = \sum_{d|n, d>0} d^{k-1},$$

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and B_k the k-th Bernoulli number, i.e., B_k-s satisfy

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}B_k\frac{t^k}{k!}.$$

The structure of graded ring $\mathcal{M}(SL_2(\mathbb{Z}))$ is simple and elegant. So it is natural to ask the following interesting question.

Question. For which congruence subgroup Γ, $\mathcal{M}(\Gamma)$ is a weighted polynomial ring?

In this note, we will give a complete answer to the above question for the case: $\Gamma=\Gamma_0(N)$.

Theorem 1.1. $\mathcal{M}(\Gamma_0(N))$ *is a weighted polynomial ring if and only if* $N=1,2,4$.

The congruence subgroup $\Gamma_0(4)$ is very important. It plays the same role in the theory of half-integral weighted modular forms as $SL_2(\mathbb{Z})$ does in the theory of integral weighted modular forms (For details, see Chapter 4 of [3] and Section 1.3 of [4]). As a consequence of Proposition 4 (p. 184) of [3], or equivalently, Theorem 1.49 of [4], we know that $\mathcal{M}(\Gamma_0(4))=\mathbb{C}[\theta^4,F]$ is also a weighted polynomial ring generated by $\theta^4,\ F\in M_2(\Gamma_0(4))$, where, for $\tau\in\mathcal{H}$,

$$\theta(\tau)=\sum_{n=-\infty}^{+\infty}q^{n^2},$$
$$F(\tau)=\sum_{n>0,\mathrm{odd}}\sigma_1(n)q^n,$$

with

$$\sigma_1(n)=\sum_{d|n,d>0}d.$$

It is less well-known that $\mathcal{M}(\Gamma_0(2))=\mathbb{C}[E_{2,2},E_4]$ is a weighted polynomial ring, where, for $\tau\in\mathcal{H}$,

$$E_{2,2}(\tau)=1+24\sum_{n=1}^{\infty}\sigma_{1,1}(n;2)q^n,$$

(see p. 19 of [1]) with

$$\sigma_{1,1}(n;2)=\sum_{d|n,2\nmid d}d$$

(see Exercise (2) on p. 55 of [2] and note that our notation $E_{2,2}(\tau)$ equals to their notation $-E_2^*(q)$).

To prove Theorem 1.1, it remains to show that these are the only cases. This will be done in the next section.

2. Proof of the main result

We need several lemmas to prove Theorem 1.1, especially, the dimension formula of $M_k(\Gamma_0(N))$. To state them, we introduce some notations first.

Let $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$. Then $GL_2^+(\mathbb{Q})$ acts on $\mathcal{H}^*$ by linear fractional transformation, i.e.,

$$\alpha(\tau) = \frac{a\tau + b}{c\tau + d}, \text{ for } \tau \in \mathcal{H} \text{ and } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q}).$$

For a congruence subgroup, the quotient space $\Gamma \backslash \mathcal{H}^*$ is a Riemmanian surface, which is usually called a modular curve, and denoted by $X(\Gamma)$. Let $g(\Gamma)$ be the genus of $X(\Gamma)$ as a Riemannian surface.

To compute the dimension of $M_k(\Gamma)$, we need several other quantities $d(\Gamma)$, $\epsilon_2(\Gamma)$, $\epsilon_3(\Gamma)$ and $\epsilon_\infty(\Gamma)$. Now we explain their meanings. We denote $d(\Gamma)$ the degree of the morphism of Riemmanian surfaces $X(\Gamma) \to X(SL_2(\mathbb{Z}))$, which is explicitly given by

$$d(\Gamma) = [SL_2(\mathbb{Z}) : \{\pm I\}\Gamma],$$

where I is the identity matrix of $SL_2(\mathbb{Z})$. For each point $\tau \in \mathcal{H}$, let

$$\Gamma_\tau = \{\gamma \in \Gamma | \gamma(\tau) = \tau\}$$

be the τ-fixing subgroup of Γ and $h_\tau = [\{\pm I\}\Gamma_\tau : \{\pm I\}]$ be the period of τ. A point $\tau \in \mathcal{H}$ is elliptic if and only if $h_\tau > 1$. Denote $\epsilon_2(\Gamma)$ (respectively, $\epsilon_3(\Gamma)$) the number of equivalent classes of elliptic points τ of period $h_\tau = 2$ (respectively, period $h_\tau = 3$) under the action of Γ. Finally, let $\epsilon_\infty(\Gamma)$ be the number of cusps of $X(\Gamma)$, that is, the number of equivalent classes of $\mathbb{Q} \cup \{\infty\}$ under the action of Γ. We may write g, d, ϵ_2, ϵ_3 and ϵ_∞ instead of $g(\Gamma)$, $d(\Gamma)$, $\epsilon_2(\Gamma)$, $\epsilon_3(\Gamma)$ and $\epsilon_\infty(\Gamma)$, for short.

Lemma 2.1 (Theorem 3.1.1 of [1])**.** *Let Γ, g, ϵ_2, ϵ_3, and ϵ_∞ be as above. Then*

$$g(\Gamma) = 1 + \frac{d(\Gamma)}{12} - \frac{\epsilon_2(\Gamma)}{4} - \frac{\epsilon_3(\Gamma)}{3} - \frac{\epsilon_\infty(\Gamma)}{2}.$$

Lemma 2.2. *Let $\Gamma = \Gamma_0(N)$. Then*

$$d(\Gamma_0(N)) = N \prod_{p|N} \left(1 + \frac{1}{p}\right);$$

$$\epsilon_2(\Gamma_0(N)) = \begin{cases} \prod_{p|N} (1 + \left(\frac{-1}{p}\right)) & \textit{if } \ 4 \nmid N, \\ 0 & \textit{if } \ 4|N, \end{cases}$$

where $(-1/p)$ is ± 1 if $p \equiv \pm 1 \pmod 4$ and is 0 if $p = 2$;

$$\epsilon_3(\Gamma_0(N)) = \begin{cases} \prod_{p|N} (1 + \left(\frac{-3}{p}\right)) & \textit{if } \ 9 \nmid N, \\ 0 & \textit{if } \ 9|N, \end{cases}$$

where $(-3/p)$ *is* ± 1 *if* $p \equiv \pm 1 \pmod 3$ *and is* 0 *if* $p = 3$ *and*

$$\epsilon_\infty(\Gamma_0(N)) = \sum_{d|N} \phi\left(\gcd(d, N/d)\right),$$

where ϕ *is the Euler* ϕ*-function.*

Proof. See pp.106–107 and Corollary 3.7.2 of [1]. □

Since $-I \in \Gamma_0(N)$, we have $M_k(\Gamma_0(N)) = 0$ for odd k. For k even, we have the following result.

Lemma 2.3 (Theorem 3.5.1 of [1])**.** *Let* k *be an even integer. Then*

$$\dim(M_k(\Gamma_0(N))) = \begin{cases} (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \epsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \epsilon_3 + \frac{k}{2}\epsilon_\infty & \text{if } k \geq 2, \\ 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0, \end{cases}$$

where $\lfloor \cdot \rfloor$ *is the greatest integer function.*

Proof of Theorem 1.1. Assume $\mathcal{M}(\Gamma_0(N)) = \mathbb{C}[f_1, \ldots, f_n]$ is a weighted polynomial ring with independent variables $f_i \in M_{k_i}(\Gamma_0(N))$ $(1 \leq i \leq n)$. We claim that $n = 2$.

First, look at the case: $n = 3$. We have

$$M_k(\Gamma_0(N)) = \bigoplus_{ak_1+bk_2+ck_3=k} \mathbb{C} \cdot f_1^a f_2^b f_3^c.$$

Therefore,

$$\dim(M_k(\Gamma_0(N))) = \#\{(a,b,c) \mid ak_1 + bk_2 + ck_3 = k,\ a,b,c \in \mathbb{N} \cup \{0\}\}.$$

Let $k = mk_1k_2k_3$ with $m \in \mathbb{N}$. Then $\dim M_k(\Gamma_0(N))$ is greater than or equals to

$$\begin{aligned} &\#\{(m_1,m_2,m_3) \mid (k_2k_3m_1)k_1 + (k_1k_3m_2)k_2 + (k_1k_2m_3)k_3 \\ &\qquad = k,\ m_1,m_2,m_3 \in \mathbb{N} \cup \{0\}\} \\ =&\#\{(m_1,m_2,m_3) \mid m_1 + m_2 + m_3 = m,\ m_1,m_2,m_3 \in \mathbb{N} \cup \{0\}\} \\ =&\sum_{m_1=0}^{m} \sum_{m_2=0}^{m-m_1} 1 \\ =&\frac{1}{2}(m+1)(m+2) \\ =&\frac{1}{2}\left(\frac{k}{k_1k_2k_3} + 1\right)\left(\frac{k}{k_1k_2k_3} + 2\right), \end{aligned}$$

which is a quadratic function of k, and by Lemma 2.3, obviously larger than $\dim M_k(\Gamma_0(N))$. A contradiction!

The same arguments show that the case: $n \geq 4$ is impossible, too. The case: $n = 1$ is also excluded since, otherwise, $\dim M_k(\Gamma_0(N))$ will be either 1 or 0. Thus, the only possibility is that $n = 2$.

As a consequence of the claim: $n = 2$, we have

$$\dim M_2(\Gamma_0(N)) = g - 1 + \epsilon_\infty \leq 2. \tag{1}$$

Since $g \geq 0$, we have $\epsilon_\infty \leq 3$. From Lemma 2.2, we know $\epsilon_\infty \geq 4$ if n has four distinct divisors and, otherwise,

$$\epsilon_\infty(\Gamma_0(N)) = \begin{cases} 1 & \text{if } N = 1, \\ 2 & \text{if } N = p, \\ p+1 & \text{if } N = p^2, \end{cases}$$

where p is a prime number. Now assume $N \neq 1, 2, 4$. Then the only possible case that $\epsilon_\infty(\Gamma_0(N)) \leq 3$ is that $N = p$ is a prime number. Since $\epsilon_\infty(\Gamma_0(p)) = 2$, by Eq. (1), we have $g = 0$ or $g = 1$, in this case.

Now assume $N = p$ is a prime number. From Lemma 2.2, we know

$$d = p + 1,\ \epsilon_2 = 1 + \left(\frac{-1}{p}\right), \epsilon_3 = 1 + \left(\frac{-3}{p}\right) \text{ and } \epsilon_\infty = 2.$$

Then from Lemma 2.1, we have

$$\begin{aligned} g(\Gamma_0(p)) &= \frac{p}{12} - \frac{1}{2} - \frac{1}{4}\left(\frac{-1}{p}\right) - \frac{1}{3}\left(\frac{-3}{p}\right) \\ &= \begin{cases} 0 & \text{if } p = 2 \text{ or } 3, \\ \frac{p-13}{12} & \text{if } p \equiv 1 \mod 12, \\ \frac{p-5}{12} & \text{if } p \equiv 5 \mod 12, \\ \frac{p-7}{12} & \text{if } p \equiv 7 \mod 12, \\ \frac{p+1}{12} & \text{if } p \equiv 11 \mod 12. \end{cases} \end{aligned} \tag{2}$$

From Eq. (2), we have

$$\begin{aligned} &g(\Gamma_0(p)) = 0 \Leftrightarrow p = 2, 3, 5, 7 \text{ and } 13; \\ &g(\Gamma_0(p)) = 1 \Leftrightarrow p = 11, 17 \text{ and } 19. \end{aligned} \tag{3}$$

For $2 \leq p \leq 19$, a detailed calculation for $\dim(M_k(\Gamma_0(p)))$ is made in the following table:

TABLE. $\dim(M_k(\Gamma_0(p)))$ for $2 \leq p \leq 19$, $k = 2, 4, 6$

p	$\dim(M_k(\Gamma_0(p)))$	$k=2$	$k=4$	$k=6$
2	$1 + \lfloor k/4 \rfloor$	1	2	2
3	$1 + \lfloor k/3 \rfloor$	1	2	3
5	$1 + 2\lfloor k/4 \rfloor$	1	3	3
7	$1 + 2\lfloor k/3 \rfloor$	1	3	5
11	k	2	4	6
13	$1 + 2\lfloor k/3 \rfloor + 2\lfloor k/4 \rfloor$	1	5	7
17	$k + 2\lfloor k/4 \rfloor$	2	6	8
19	$k + 2\lfloor k/3 \rfloor$	2	6	10

From the above table, we can see that the only case that $\mathcal{M}(\Gamma_0(p))$ is a weighted polynomial ring happens when $p = 2$. This concludes the proof. $\square$

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