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ON NIL GENERALIZED POWER SERIESWISE ARMENDARIZ RINGS

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Abstract. We in this note introduce a concept, so called nil generalized power serieswise Armendariz ring, that is a generalization of both S-Armendariz rings and nil power serieswise Armendariz rings. We first observe the basic properties of nil generalized power serieswise Armendariz rings, constructing typical examples. We next study the relationship between the nilpotent property of R and that of the generalized power series ring $[[R^{S,\leq}]]$ whenever R is nil generalized power serieswise Armendariz.

1. Introduction

Throughout this paper R denotes an associative ring with identity and $nil(R)$ stands for the set of all nilpotent elements of R. A ring R is called an NI ring if $nil(R)$ forms an ideal, and a ring R is said to be semicommutative if for all a, $b \in R$, $ab = 0$ implies $aRb = 0$. Let I be an ideal of R, I is said to be semicommutative if I is considered as a semicommutative ring without identity, and I is said to be nilpotent if $I^n = 0$ for some positive integer n. Let U be a subset of R. We denote by $U[[x]]$ the set $\{f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]] \mid a_i \in U, i = 0, 1, \ldots\}$.

A ring R is called *Armendariz* if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each i, j. The term Armendariz was introduced by Rege and Chhawchharia [13]. This nomenclature was used by them since it was Armendariz [3, Lemma 1] who initially showed that a *reduced* ring (i.e., a ring without nonzero nilpotent elements) always satisfies this condition. Armendariz rings are thus a generalization of reduced rings, and therefore, nilpotent elements play an important role in this class of rings. There are many examples of rings with nilpotent elements which are Armendariz. In fact, in [1], Anderson and Camillo proved that if $n > 2$, then $R[x]/(x^n)$ is an Armendariz ring if and only if R is reduced.

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N. K. Kim et al. [8] studied a generalization of Armendariz rings, which they called power serieswise Armendariz rings. A ring R is called power serieswise Armendariz if whenever power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for all i, j. As a generalization of power serieswise Armendariz rings, S. Hizem in [7] introduced the concept of nil power serieswise Armendariz rings and Z. K. Liu in [10] introduced the notion of S-Armendariz rings, respectively. Following S. Hizem [7], a ring R is called nil power serieswise Armendariz if whenever power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]]$ satisfy $f(x)g(x) \in nil(R)[[x]]$, then $a_i b_j \in nil(R)$ for all i, j. Let $(S, \leq)$ be a cancellative torsion-free strictly ordered monoid and let $[[R^{S,\leq}]]$ be a generalized power series ring over R. According to Z. K. Liu [10], the ring R is called *S-Armendariz* if whenever f, $g \in [[R^{S,\leq}]]$ satisfy $fg = 0$, then $f(u)g(v) = 0$ for each u, $v \in S$.

In this paper we investigate a generalization of both nil power serieswise Armendariz rings and S-Armendariz rings which we call nil generalized power serieswise Armendariz rings. We first observe the basic properties of nil generalized power serieswise Armendariz rings, constructing typical examples. We next study the relationship between the nilpotent property of R and that of the generalized power series ring $[[R^{S,\leq}]]$ whenever R is nil generalized power serieswise Armendariz.

Now let us briefly review the concept of generalized power series rings. Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is *artinian* if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S shall be denoted additively, the neutral element by 0 and $|S| \geq 2$.

Let $(S, \leq)$ be a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if s, s', $t \in S$ and $s < s'$, then $s + t < s' + t$) and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f : S \longrightarrow R$ such that $supp(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian and narrow. With pointwise addition, $[[R^{S,\leq}]]$ is an abelian additive group. For every $s \in S$ and f, $g \in [[R^{S,\leq}]]$, let $X_s(f, g) = \{(u, v) \in S \times S \mid s = u + v, f(u) \neq 0, g(v) \neq 0\}$. It follows from [15, Section 4.1] that $X_s(f, g)$ is finite. This fact allows to define the operation of convolution:

$$(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)g(v).$$

With this operation of convolution, and pointwise addition, $[[R^{S,\leq}]]$ becomes a ring (see [14, 15, 16]), which is called the ring of generalized power series. The elements of $[[R^{S,\leq}]]$ are called generalized power series with coefficients in R and exponents in S.

Let $s \in S$, $r \in R$. We define $C_r^s \in [[R^{S,\leq}]]$ as follows:

$$C_r^s(s) = r, \quad C_r^s(t) = 0 \quad (s \neq t \in S).$$

It is clear that $r \to C_r^0$ is a ring embedding of R into $[[R^{S,\leq}]]$. So we can regard R as a subring of $[[R^{S,\leq}]]$, and for any $f \in [[R^{S,\leq}]]$, $r \in R$, $fr = fC_r^0$.

Given a subset $U \subseteq R$, $[[U^{S,\leq}]]$ means the set $\{f \in [[R^{S,\leq}]] \mid f(s) \in U, s \in supp(f)\}$. In particular, $[[nil(R)^{S,\leq}]]$ stands for the set $\{f \in [[R^{S,\leq}]] \mid f(s) \in nil(R), s \in supp(f)\}$. For any $s \in S$ and any nature number n, we denote by ns the sum of n copies of s. Other concepts and notations not defined here can be found in [14, 15, 16].

2. Nil generalized power serieswise Armendariz rings

In this section, we first give the following concept, so called nil generalized power serieswise Armendariz ring, that is both a generalization of *S-Armendariz* rings and nil power serieswise Armendariz rings.

Definition 2.1. Let $(S, \leq)$ be a strictly ordered monoid. A ring R is called nil generalized power serieswise Armendariz if whenever $f, g \in [[R^{S,\leq}]]$ satisfy $fg \in [[nil(R)^{S,\leq}]]$, then $f(u)g(v) \in nil(R)$ for each $u, v \in S$.

Let $S = (\mathbb{N} \cup \{0\}, +)$, and $\leq$ is the usual order. Then $[[R^{S,\leq}]] \cong R[[x]]$. So the ring R is nil generalized power serieswise Armendariz if and only if R is nil power serieswise Armendariz. Hence a nil generalized power serieswise Armendariz ring is a generalization of a nil power serieswise Armendariz ring. Obviously, any subring of a nil generalized power serieswise Armendariz ring is also nil generalized power serieswise Armendariz.

The following proposition enable us to generate more examples of nil generalized power serieswise Armendariz rings.

Proposition 2.2. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a ring. Then the following conditions are equivalent*:

(1) *R is nil generalized power serieswise Armendariz.*

(2) *R is an NI ring.*

Proof. (1) $\Rightarrow$ (2) Suppose that $a \in nil(R)$, $r \in R$, and $0 \neq s \in S$. Define $f \in [[R^{S,\leq}]]$ via

$$f(x) = \begin{cases} 1 & \text{if} \quad x = 0, \\ r^n & \text{if} \quad x = ns,\ n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then $C_a^0(C_1^0 - C_r^s)f = C_a^0 \in [[nil(R)^{S,\leq}]]$. Hence $(C_a^0(C_1^0 - C_r^s))(0)f(s) = ar \in nil(R)$ because R is nil generalized power serieswise Armendariz. Note that $ar \in nil(R)$ implies $ra \in nil(R)$.

Now we show that $x + yz \in nil(R)$ for all $x, y, z \in nil(R)$. Since $y \in nil(R)$, we have $-y(x + yz) \in nil(R)$. Let $0 \neq s \in S$. Construct $h, g \in [[R^{S,\leq}]]$ as follows:

$$h(x) = \begin{cases} 1 & \text{if} \quad x = 0, \\ -y & \text{if} \quad x = s, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} z & \text{if} \quad x = 0, \\ x + yz & \text{if} \quad x = s, \\ 0 & \text{otherwise.} \end{cases}$$

Then $hg \in [[nil(R)^{S,\leq}]]$. Since R is nil generalized power serieswise Armendariz, $h(0)g(s) = x + yz \in nil(R)$. Then by analogy with the proof of R. Antoine [2], Lemma 3.1(d), we can show that $nil(R)$ is an ideal, and so R is an NI ring.

(2) ⇒ (1) Assume that R is an NI ring, and f, $g \in [[R^{S,\leq}]]$ are such that $fg \in [[nil(R)^{S,\leq}]]$. Then $\overline{f}\,\overline{g} = \overline{0}$, where $\overline{f}$, $\overline{g}$ are the corresponding generalized power series of f, g in $[[(R/nil(R))^{S,\leq}]]$. Observe that $R/nil(R)$ is reduced and hence *S-Armendariz* by [10]. Thus $\overline{f}(u)\overline{g}(v) = \overline{0}$ for any u, $v \in S$. Hence $f(u)g(v) \in nil(R)$ for any u, $v \in S$. Therefore R is nil generalized power serieswise Armendariz. □

Corollary 2.3. *Let $(S_1, \leq_1)$, $(S_2, \leq_2)$, …, $(S_n, \leq_n)$ be cancellative torsion-free strictly ordered monoids. Denote by $(lex \leq)$ and $(revlex \leq)$ the lexicographic order, the reverse lexicographic order, respectively, on the monoid $S_1 \times S_2 \times \cdots \times S_n$. Then the following conditions are equivalent*:

(1) *R is an NI ring.*

(2) *R is nil generalized power serieswise Armendariz for any ordered monoid $(S_i, \leq_i)$.*

(3) *R is nil generalized power serieswise Armendariz for ordered monoid $(S_1 \times S_2 \times \cdots \times S_n, (lex \leq))$.*

(4) *R is nil generalized power serieswise Armendariz for ordered monoid $(S_1 \times S_2 \times \cdots \times S_n, (revlex \leq))$.*

Proof. It is easy to see that $(S_1 \times S_2 \times \cdots \times S_n, (lex \leq))$ and $(S_1 \times S_2 \times \cdots \times S_n, (revlex \leq))$ are cancellative torsion-free strictly ordered monoids. Therefore we complete the proofs of (1) ⇔ (2), (1) ⇔ (3) and (1) ⇔ (4) by Proposition 2.2. □

A ring R is called n nil power serieswise Armendariz if $f = \sum a_{i_1,i_2,\ldots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, $g = \sum b_{j_1,j_2,\ldots,j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \in R[[x_1, x_2, \ldots, x_n]]$ satisfy $fg \in nil(R)[[x_1, x_2, \ldots, x_n]]$, then $a_{i_1,i_2,\ldots,i_n} b_{j_1,j_2,\ldots,j_n} \in nil(R)$ for all $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_n$.

Corollary 2.4. *Let R be a ring. Then the following conditions are equivalent.*

(1) *The ring R is an NI ring.*

(2) *The ring R is nil power serieswise Armendariz.*

(3) *The ring R is n nil power serieswise Armendariz.*

Proof. (1) ⇔ (2) is ciear.

(1) ⇔ (3) Note that if $S = \mathbb{N}^n$, with the product order, or the lexicographic order, or the reverse lexicographic order, then $[[R^{S,\leq}]] \cong R[[x_1, x_2, \ldots, x_n]]$ (see [15, Example 4]). Then by Corollary 2.3, we complete the proof. □

Let R be a ring and let

$$T_n(R) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \mid a_{ij} \in R \right\},$$

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\},$$

$$T(R,n) = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_i \in R \right\},$$

$$W(R) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{ij} \in R \right\},$$

and $T(R,R)$ be the trivial extension of R by R. They are all rings under usual matrix operations. Then we have the following results.

Proposition 2.5. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Then the following conditions are equivalent*:

(1) *R is nil generalized power serieswise Armendariz.*
(2) *$T_n(R)$ is nil generalized power serieswise Armendariz.*
(3) *$S_n(R)$ is nil generalized power serieswise Armendariz.*
(4) *$T(R,n)$ is nil generalized power serieswise Armendariz.*
(5) *$W(R)$ is nil generalized power serieswise Armendariz.*
(6) *$T(R,R)$ is nil generalized power serieswise Armendariz.*
(7) *$R[x]/(x^n)$ is nil generalized power serieswise Armendariz for any $n \geq 2$.*

Proof. (1) $\Rightarrow$ (2) Suppose that R is nil generalized power serieswise Armendariz. Then by Proposition 2.2, R is an NI ring. Since

$$nil(T_n(R)) = \begin{pmatrix} nil(R) & R & \cdots & R \\ 0 & nil(R) & \cdots & R \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & nil(R) \end{pmatrix},$$

it is easy to see that $T_n(R)$ is an NI ring. Then by Proposition 2.2, $T_n(R)$ is nil generalized power serieswise Armendariz.

(2) $\Rightarrow$ (1) Note that any subring of a nil generalized power serieswise Armendariz ring is also nil generalized power serieswise Armendariz. Hence (2) $\Rightarrow$ (1) is straightforward.

Similarly, we can show that (1) $\Leftrightarrow$ (3), (1) $\Leftrightarrow$ (4), (1) $\Leftrightarrow$ (5), and (1) $\Leftrightarrow$ (6).

The proof of (1) $\Leftrightarrow$ (7) follows from the fact that $R[x]/(x^n) \cong T(R,n)$ for any $n \geq 2$. $\square$

Let M be an R-R-bimodule. A $\mathbb{Z}$-bilinear map $\alpha : R \times R \longrightarrow M$ is called a *Hochschild* 2-*cocycle* if for all λ_1, λ_2, $\lambda_3 \in R$, the following equation holds true:

$$\alpha(\lambda_1\lambda_2, \lambda_3) - \alpha(\lambda_1, \lambda_2\lambda_3) = \lambda_1\alpha(\lambda_2, \lambda_3) - \alpha(\lambda_1, \lambda_2)\lambda_3.$$

Given a *Hochschild* 2-*cocycle* α, there is a ring $H_\alpha(R, M)$, called the *Hochschild extension* of R by M via α, which is $R \oplus M$ as an abelian group, and the multiplication is defined by

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2 + \alpha(r_1, r_2))$$

for all r_1, $r_2 \in R$ and all m_1, $m_2 \in M$.

This is an associative ring [5]. If $\alpha = 0$, the extension ring $H_0(R, M)$ is the trivial extension of R by M in the literature. Note that the nilpotent elements of $H_\alpha(R, M)$ is $(nil(R), M)$. Then we have the following result:

Proposition 2.6. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Then the following conditions are equivalent*:

(1) *R is nil generalized power serieswise Armendariz.*

(2) *$H_\alpha(R, M)$ is nil generalized power serieswise Armendariz*

Remark 2.7. A ring is called *symmetric* if $abc = 0$ implies $acb = 0$ for all a, b, $c \in R$. A ring R is called reversible if $ab = 0$ implies $ba = 0$ for all a, $b \in R$. Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Since commutative rings, reduced rings, symmetric rings, reversible rings, semicommutative rings and 2-*primal* rings are NI rings, by Proposition 2.2, they are nil generalized power serieswise Armendariz rings. Hence nil generalized power serieswise Armendariz rings forms a large class of rings.

Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Then for any ring R, the n by n matrix ring $M_n(R)$ is never nil generalized power serieswise Armendariz. In fact, consider $x = E_{12}$ and $y = -E_{21}$, where E_{ij} denote the (i, j)-matrix unit. Then x, $y \in nil(M_n(R))$, but $x - y \notin nil(M_n(R))$. Hence $M_n(R)$ is not an NI ring.

The next lemma is known for S-*Armendariz* rings (see [10, Proposition 3.2]).

Lemma 2.8. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S and R an S-Armendariz ring. If f_1, f_2, $\ldots$, $f_n \in [[R^{S,\leq}]]$ are such that $f_1f_2 \cdots f_n = 0$, then $f_1(u_1)f_2(u_2) \cdots f_n(u_n) = 0$ for all u_1, u_2, $\ldots$, $u_n \in S$.*

The following result shows that our definition of a nil generalized power serieswise Armendariz ring is an extension of the Zhongkui Liu's [10] S-*Armendariz* ring for the more general setting.

Proposition 2.9. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Then all S-Armendariz rings are nil generalized power serieswise Armendariz.*

Proof. Suppose that R is S-*Armendariz.* Let $a \in nil(R)$, $r \in R$ and $0 \neq s \in S$. Define $m \in [[R^{S,\leq}]]$ via

$$m(x) = \begin{cases} 1 & \text{if} \quad x = 0, \\ r^n & \text{if} \quad x = ns, \quad n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then $C_a^0(C_1^0 - C_r^s)m = C_a^0$. Suppose $a^k = 0$ for some positive integer k. Then $(C_a^0(C_1^0 - C_r^s)m)^k = (C_a^0)^k = 0$. Then by Lemma 2.8, $(C_a^0(0)(C_1^0 - C_r^s)(0)m(s))^k = (ar)^k = 0$, and so $ar \in nil(R)$, $ra \in nil(R)$.

Let a, b, $c \in nil(R)$. Without loss of generality, we may assume that a, b, c are all nonzero nilpotent elements. Let $0 \neq s \in S$. Now we claim that $a + bc \in nil(R)$. Define f, $g \in [[R^{S,\leq}]]$ via

$$f(x) = \begin{cases} 1 & \text{if} \quad x = s, \\ -b & \text{if} \quad x = 2s, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} c & \text{if} \quad x = s, \\ a + bc & \text{if} \quad x = 2s, \\ 0 & \text{otherwise.} \end{cases}$$

Let $h = fg$. Then $supp(h) = supp(fg) = \{2s, 3s, 4s\}$, $h(2s) = c \in nil(R)$, $h(3s) = a \in nil(R)$ and $h(4s) = -b(a + bc) \in nil(R)$. Let k be a positive integer such that

$$(h(2s))^k = (h(3s))^k = (h(4s))^k = c^k = a^k = (-b(a + bc))^k = 0.$$

Now we wish to claim that $(fg)^{3k} = h^{3k} = 0$. For any $w \in S$,

$$h^{3k}(w) = \sum_{(u_1,\ldots,u_{3k}) \in X_w(\underbrace{h,\ldots,h}_{3k})} h(u_1)h(u_2)\cdots h(u_{3k}),$$

where $u_i \in \{2s, 3s, 4s\}$ for all $1 \leq i \leq 3k$. Consider each

$$\begin{aligned} &(u_1, \ldots, u_{3k}) \in X_w(\underbrace{h, \ldots, h}_{3k}) \\ &= \{(u_1, \ldots, u_{3k}) \mid u_1 + \cdots + u_{3k} = w, u_i \in \{2s, 3s, 4s\}, 1 \leq i \leq 3k\}. \end{aligned}$$

It would contain at least k, u_{j_0}, where $u_{j_0} \in \{2s, 3s, 4s\}$. Suppose that

$$u_{r_1} = u_{r_2} = \cdots = u_{r_k} = u_{j_0}$$

for some

$$1 \leq r_1 < r_2 < \cdots < r_k \leq 3k.$$

For each $u_v \neq u_{r_t}$, $1 \leq t \leq k$, define $h'_v \in [[R^{S,\leq}]]$ via

$$h'_v(x) = \begin{cases} 1 & \text{if} \quad x = 0, \\ (h(u_v))^p & \text{if} \quad x = pu_v,\ p = 1, 2, \ldots, k - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(C_1^0 - C_{h(u_v)}^{u_v})h'_v = C_1^0$, and $(C_1^0 - C_{h(u_v)}^{u_v})(0)h'_v(u_v) = h(u_v)$. For convenience we write $h(u_1)h(u_2)\cdots h(u_{3k})$ as

$$\begin{aligned}&h(u_1)\cdots h(u_{r_1-1})h(u_{j_0})h(u_{r_1+1})\cdots h(u_{r_2-1})h(u_{j_0})\\&\cdots h(u_{r_k-1})h(u_{j_0})\cdots h(u_{3k}).\end{aligned}$$

By replacing each $h(u_v)$ $(u_v \neq u_{r_t}, 1 \leq t \leq k)$ by the product $(C_1^0 - C_{h(u_v)}^{u_v})h'_v$, each $h(u_{j_0})$ by $C_{h(u_{j_0})}^0$, and consider the condition that $(h(u_{j_0}))^k = 0$, we have

$$\begin{aligned}&(C_1^0 - C_{h(u_1)}^{u_1})h'_1\cdots h'_{r_1-1}C_{h(u_{j_0})}^0(C_1^0 - C_{h(u_{r_1+1})}^{u_{r_1+1}})\\&\cdots h'_{r_k-1}C_{h(u_{j_0})}^0(C_1^0 - C_{h(u_{r_k+1})}^{u_{r_k+1}})\cdots h'_{3k} = 0.\end{aligned}$$

Now since R is *S-Armendariz*, by Lemma 2.8,

$$\begin{aligned}&(C_1^0 - C_{h(u_1)}^{u_1})(0)h'_1(u_1)\cdots h'_{r_1-1}(u_{r_1-1})C_{h(u_{j_0})}^0(0)(C_1^0 - C_{h(u_{r_1+1})}^{u_{r_1+1}})(0)\cdots\\&h'_{r_k-1}(u_{r_k-1})C_{h(u_{j_0})}^0(0)(C_1^0 - C_{h(u_{r_k+1})}^{u_{r_k+1}})(0)\cdots h'_{3k}(u_{3k})\\=\;&h(u_1)h(u_2)\cdots h(u_{3k}) = 0.\end{aligned}$$

Therefore we have prove that for each

$$(u_1, u_2, \ldots, u_{3k}) \in X_w(\underbrace{h, \ldots, h}_{3k}), \quad h(u_1)h(u_2)\cdots h(u_{3k}) = 0.$$

Hence for any $w \in S$, $h^{3k}(w) = 0$, and so $h^{3k} = (fg)^{3k} = 0$. Then by Lemma 2.8, we obtain $(f(s)g(2s))^{3k} = (a+bc)^{3k} = 0$. Hence $a + bc \in nil(R)$ is proved. Then by analogy with the proof of R. Antoine [2], Lemma 3.1(d), we can show that $a - b \in nil(R)$. Hence R is an NI ring. Therefore by Proposition 2.2, R is nil generalized power serieswise Armendariz. □

The following example shows that there exists a nil generalized power serieswise Armendariz ring which is not S-Armendariz. Hence a nil generalized power serieswise Armendariz ring is not a trivial extension of an S-Armendariz ring.

Example 2.10. Let $(S, \leq)$ be a strictly totally ordered monoid satisfying the condition that $0 \leq s$ for every $s \in S$, and R a nil generalize power serieswiae Armendariz ring. Then by Proposition 2.5, $S_4(R)$ is nil generalized power serieswise Armendariz. Let $A = \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$, $B = \begin{pmatrix} 0&1&-1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$, $C = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}$, $D = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&1\\0&0&0&0 \end{pmatrix}$ be four elements in $S_4(R)$. Let $0 \neq s \in S$. Define f, $g \in [[(S_4(R))^{S,\leq}]]$ via

$$f(x) = \begin{cases} A & \text{if} \quad x = 0,\\ B & \text{if} \quad x = s,\\ 0 & \text{otherwise,}\end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} C & \text{if} \quad x = 0,\\ D & \text{if} \quad x = s,\\ 0 & \text{otherwise.}\end{cases}$$

Then $fg = 0$, but $f(0)g(s) \neq 0$. So $S_4(R)$ is not S-Armendariz.

Proposition 2.11. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and I an nil ideal of R (that is, $I \subseteq nil(R)$). Then R is nil generalized power serieswise Armendariz if and only if R/I is nil generalized power serieswise Armendariz.*

Proof. By analogy with the proof of [7, Proposition 5], we complete the proof. □

It was shown in [8, Proposition 3.10] that if I is a reduced ideal of R such that R/I is power serieswise Armendariz then R is power serieswise Armendariz. Here we have the following result:

Proposition 2.12. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and I an ideal of R. If I is semicommutative and R/I is nil generalized power serieswise Armendariz, then R is nil generalized power serieswise Armendariz.*

Proof. Let f, $g \in [[R^{S,\leq}]]$ be such that $fg \in [[nil(R)^{S,\leq}]]$. By Ribenbiom [15], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$ (that is, for all s, $t \in S$, $s \leq t$ implies $s \leq' t$). We will use transfinite induction on the strictly totally ordered set $(S, \leq')$ to show that $f(u)g(v) \in nil(R)$ for any u, $v \in S$. Let s and t denote the minimum elements of $supp(f)$ and $supp(g)$ in the $\leq'$ order, respectively. If $u \in supp(f)$ and $v \in supp(g)$ are such that $u + v = s + t$, then $s \leq' u$ and $t \leq' v$. If $s <' u$, then $s + t <' u + v = s + t$, a contradiction. Thus $u = s$. Similarly, $v = t$. Hence

$$(fg)(s+t) = \sum_{(u,v)\in X_{s+t}(f,g)} f(u)g(v) = f(s)g(t) \in nil(R)$$

because $fg \in [[nil(R)^{S,\leq}]]$.

Now suppose that $w \in S$ is such that for any u, $v \in S$ with $u + v <' w$, $f(u)g(v) \in nil(R)$. We will show that $f(u)g(v) \in nil(R)$ for any u, $v \in S$ with $u + v = w$. We write

$$X_w(f,g) = \{(u,v) \in S \times S \mid u + v = w, u \in supp(f), v \in supp(g)\}$$

as $\{(u_i, v_i) \mid i = 1, 2, \ldots, n\}$ such that

$$u_1 <' u_2 <' \cdots <' u_n.$$

Since S is cancellative, $u_1 = u_2$ and $u_1 + v_1 = u_2 + v_2 = w$ imply $v_1 = v_2$. Since $\leq'$ is a strict order, $u_1 <' u_2$ and $u_1 + v_1 = u_2 + v_2 = w$ imply $v_2 <' v_1$. Thus we have

$$v_n <' v_{n-1} <' \cdots <' v_2 <' v_1.$$

Now,

$$(fg)(w) = \sum_{(u,v)\in X_w(f,g)} f(u)g(v) = \sum_{i=1}^{n} f(u_i)g(v_i) \in nil(R).$$

For any $i \geq 2$, $u_1 + v_i <' u_i + v_i = w$, and thus, by induction hypothesis, we have $f(u_1)g(v_i) \in nil(R)$.

On the other hand, if we denote by $\overline{f}$, $\overline{g}$ the corresponding generalized power series of f and g in $[[(R/I)^{S,\leq}]]$, $\overline{f}\,\overline{g} \in [[(nil(R/I))^{S,\leq}]]$. There exists $n_{ij} \in \mathbb{N}$ such that $(f(u_i)g(v_j))^{n_{ij}} \in I$ since R/I is nil generalized power serieswise Armendariz. Then by analogy with the proof of Z. K. Liu [11], Theorem 3.6, we can show that $f(u)g(v) \in nil(R)$ for any u, $v \in S$ with $u + v = w$. Hence by transfinite induction, $f(u)g(v) \in nil(R)$ for any u, $v \in S$. Therefore R is nil generalized power serieswise Armendariz. □

Proposition 2.13. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a nil generalized power serieswise Armendariz ring. If $f_1, f_2, \ldots, f_n \in [[R^{S,\leq}]]$ are such that $f_1 f_2 \cdots f_n \in [[nil(R)^{S,\leq}]]$, then $f_1(u_1)$ $f_2(u_2) \cdots f_n(u_n) \in nil(R)$ for all u_1, u_2, $\ldots$, $u_n \in S$.*

Proof. Suppose $f_1 f_2 \cdots f_n \in [[nil(R)^{S,\leq}]]$. Then from $f_1(f_2 \cdots f_n) \in [[nil(R)^{S,\leq}]]$, it follows that $f_1(u_1)(f_2 \cdots f_n)(v) \in nil(R)$ for all u_1, $v \in S$. Thus $(C^0_{f_1(u_1)} f_2 \cdots$ $f_n)$ $(v) \in nil(R)$ for any $v \in S$, and so $C^0_{f_1(u_1)} f_2 \cdots f_n \in [[nil(R)^{S,\leq}]]$. Now from $(C^0_{f_1(u_1)} f_2)$ $(f_3 \cdots f_n) \in [[nil(R)^{S,\leq}]]$, it follows that $(C^0_{f_1(u_1)} f_2)(u_2)(f_3 \cdots$ $f_n)(w) \in nil(R)$ for all u_2, $w \in S$. Since

$$(C^0_{f_1(u_1)} f_2)(u_2) = f_1(u_1) f_2(u_2) \text{ for any } u_1,\ u_2 \in S,$$

we have

$$f_1(u_1) f_2(u_2)(f_3 \cdots f_n)(w) \in nil(R) \text{ for all } u_1,\ u_2,\ w \in S.$$

Hence

$$C^0_{(f_1(u_1) f_2(u_2))} f_3 \cdots f_n \in [[nil(R)^{S,\leq}]].$$

Continuing this manner, we see that $f_1(u_1) f_2(u_2) \cdots f_n(u_n) \in nil(R)$ for all u_1, u_2, $\ldots$, $u_n \in S$. □

Corollary 2.14. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Then the following conditions are equivalent*:

(1) *If f_1, f_2, $\ldots$, $f_n \in [[R^{S,\leq}]]$ satisfy $f_1 f_2 \cdots f_n \in [[nil(R)^{S,\leq}]]$, then $f_1(u_1)$ $f_2(u_2) \cdots f_n(u_n) \in nil(R)$ for all u_1, u_2, $\ldots$, $u_n \in S$.*

(2) *R is an NI ring.*

Corollary 2.15. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S and R a nil generalized power serieswise Armendariz ring. Then $nil([[R^{S,\leq}]]) \subseteq [[nil(R)^{S,\leq}]]$.*

Proof. It follows from Proposition 2.13. □

Recall that a ring R is said to be have bounded index of nilpotency if there exists an integer $n \geq 1$ such that $x^n = 0$ for each nilpotent element of R.

Corollary 2.16. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S and R a ring. If $[[nil(R)^{S,\leq}]] \subseteq nil([[R^{S,\leq}]])$, then R has bounded index of nilpotency.*

Proof. Otherwise, for any positive integer n, there exists $a_n \in nil(R)$ such that $a_n^n \neq 0$. Let $0 \neq s \in S$. Define $f \in [[R^{S,\leq}]]$ via

$$f(x) = \begin{cases} a_n & \text{if} \quad x = n!s, \quad n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in [[nil(R)^{S,\leq}]]$, and $supp(f) = \{s, 2!s, 3!s, \ldots, n!s, \ldots\}$. Since $[[nil(R)^{S,\leq}]] \subseteq nil([[R^{S,\leq}]])$, there exists $k \geq 2$ such that $f^k = 0$. For $n > k$, we have

$$\begin{aligned} f^k(k(n!s)) &= \sum_{(u_1,u_2,\ldots,u_k) \in X_{(kn!s)}(\underbrace{f,\ldots,f}_{k})} f(u_1)f(u_2)\cdots f(u_k) \\ &= f(n!s)f(n!s)\cdots f(n!s) = a_n^k \neq 0 \end{aligned}$$

which is impossible. □

Proposition 2.17. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a nil generalized power serieswise Armendariz ring. If $nil(R)$ is nilpotent, then $[[nil(R)^{S,\leq}]] = nil([[R^{S,\leq}]])$.*

Proof. By Corollary 2.15, we have $[[nil(R)^{S,\leq}]] \supseteq nil([[R^{S,\leq}]])$. So if suffices to show that $[[nil(R)^{S,\leq}]] \subseteq nil([[R^{S,\leq}]])$. Assume that $f \in [[nil(R)^{S,\leq}]]$. Since $nil(R)$ is nilpotent, there exists some positive integer k such that $(nil(R))^k = 0$. So for any $s \in S$,

$$f^k(s) = \sum_{(u_1,u_2,\ldots,u_k) \in X_s(\underbrace{f,\ldots,f}_{k})} f(u_1)f(u_2)\cdots f(u_k) = 0.$$

Thus $f^k = 0$, and so $f \in nil([[R^{S,\leq}]])$. Hence $[[nil(R)^{S,\leq}]] \subseteq nil([[R^{S,\leq}]])$. Therefore $[[nil(R)^{S,\leq}]] = nil([[R^{S,\leq}]])$. □

Corollary 2.18. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a nil generalized power serieswise Armendariz right noetherian ring. Then $[[nil(R)^{S,\leq}]] = nil([[R^{S,\leq}]])$.*

Proof. Since R is nil generalized power serieswise Armendariz, by Proposition 2.2, R is an NI ring. Then by the well known Levitzki's Theorem [9], $nil(R)$ is nilpotent. Hence the result follows from Proposition 2.17. □

N. K. Kim et al. have shown in [8, Proposition 3.1] that a ring R is power serieswise Armendariz if and only if $R[x]$ is power serieswise Armendariz. For nil power serieswise Armendariz rings, S. Hizem have shown in [7, Corollary 7] that if R is a semicommutative ring, then $R[x]$ is a nil power serieswise Armendariz ring. As to nil generalized power serieswise Armendariz rings, we have the following:

Proposition 2.19. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a nil generalized power serieswise Armendariz ring. If $nil(R)$ is nilpotent, then $[[R^{S,\leq}]]$ is nil generalized power serieswise Armendariz for any torsion-free cancellative monoid T and any strict order $\leq_T$ on T.*

Proof. Since R is nil generalized power serieswise Armendariz, by Proposition 2.2, R is an NI ring. Since $nil(R)$ is nilpotent, by Proposition 2.17, we have $[[nil(R)^{S,\leq}]] = nil([[R^{S,\leq}]])$, and so $[[R^{S,\leq}]]$ is an NI ring. Then by Proposition 2.2, $[[R^{S,\leq}]]$ is nil generalized power serieswise Armendariz for any torsion-free cancellative monoid T and any strict order $\leq_T$ on T. $\square$

Corollary 2.20. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a nil generalized power serieswise Armendariz right noetherian ring. Then $[[R^{S,\leq}]]$ is nil generalized power serieswise Armendariz for any torsion-free cancellative monoid T and any strict order $\leq_T$ on T.*

3. Nilpotent property of nil generalized power serieswise Armendariz rings

Let U and V be two sets of R. We use $U : V$ to represent the set $\{x \in R \mid Vx \subseteq U\}$. Then for any $U \subseteq R$, we have

$$\begin{aligned} nil(R) : U &= \{x \in R \mid Ux \subseteq nil(R)\} \\ &= \{x \in R \mid xU \subseteq nil(R)\}. \end{aligned}$$

If $nil(R)$ is an ideal, then $nil(R) : U$ is an ideal of R for any subset $U \subseteq R$, and $[[nil(R)^{S,\leq}]] : V$ is also an ideal of $[[R^{S,\leq}]]$ for any subset V of $[[R^{S,\leq}]]$. Given a generalized power series $f \in [[R^{S,\leq}]]$, let C_f denote the set $\{f(s) \mid s \in supp(f)\}$, and for a subset $V \subseteq [[R^{S,\leq}]]$, let C_V denote the set $\bigcup_{f\in V} C_f$.

Given a ring R, we define

$$NAnn_R(2^R) = \{nil(R) : U \mid U \subseteq R\}$$

and

$$NAnn_{[[R^{S,\leq}]]}(2^{[[R^{S,\leq}]]}) = \{[[nil(R)^{S,\leq}]] : V \mid V \subseteq [[R^{S,\leq}]]\}.$$

Proposition 3.1. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a nil generalized power serieswise Armendariz ring. Then*

$$\phi : NAnn_R(2^R) \longrightarrow NAnn_{[[R^{S,\leq}]]}(2^{[[R^{S,\leq}]]})$$

defined by $\phi(I) = [[I^{S,\leq}]]$ for every $I \in NAnn_R(2^R)$ is bijective.

Proof. We first prove that $[[(nil(R) : U)^{S,\leq}]] = [[nil(R)^{S,\leq}]] : U$ for any subset $U \subseteq R$. Suppose that $f \in [[(nil(R) : U)^{S,\leq}]]$. Then for any $s \in S$, $f(s) \in nil(R) : U$, and so for any $r \in U$, $f(s)r = (fC_r^0)(s) \in nil(R)$. Hence $fC_r^0 \in [[nil(R)^{S,\leq}]]$, and so $f \in [[nil(R)^{S,\leq}]] : U$. Thus $[[(nil(R) : U)^{S,\leq}]] \subseteq [[nil(R)^{S,\leq}]] : U$. Now we claim that $[[nil(R)^{S,\leq}]] : U \subseteq [[(nil(R) : U)^{S,\leq}]]$. For any generalized power series $f \in [[nil(R)^{S,\leq}]] : U$, and any $r \in U$, we have $fr = fC_r^0 \in [[nil(R)^{S,\leq}]]$. Then for any $s \in S$, $(fC_r^0)(s) = f(s)r \in nil(R)$, and so for any $s \in S$, $f(s) \in nil(R) : U$. Hence $f \in [[(nil(R) : U)^{S,\leq}]]$. Thus $[[nil(R)^{S,\leq}]] : U \subseteq [[(nil(R) : U)^{S,\leq}]]$. Hence $[[nil(R)^{S,\leq}]] : U = [[(nil(R) : U)^{S,\leq}]]$ is proved, and so ϕ is well defined.

We next claim that ϕ is injective. Let

$$I_1 = nil(R) : U_1 \in NAnn_R(2^R),$$

$$I_2 = nil(R) : U_2 \in NAnn_R(2^R),$$

and

$$nil(R) : U_1 \neq nil(R) : U_2.$$

Then $[[(nil(R) : U_1)^{S,\leq}]] \neq [[(nil(R) : U_2)^{S,\leq}]]$ is clear. Hence $\phi(I_1) \neq \phi(I_2)$. So ϕ is injective.

Finally, we show that ϕ is surjective. Let $[[nil(R)^{S,\leq}]] : V \in NAnn_{[[R^{S,\leq}]]}(2^{[[R^{S,\leq}]]})$, where $V \subseteq [[R^{S,\leq}]]$. We wish to claim that

$$[[nil(R)^{S,\leq}]] : V = [[(nil(R) : C_V)^{S,\leq}]] = \phi(nil(R) : C_V).$$

Let $f \in [[nil(R)^{S,\leq}]] : V$. Then $fg \in [[nil(R)^{S,\leq}]]$ for any $g \in V$. Since R is nil generalized power serieswise Armendariz, $f(u)g(v) \in nil(R)$ for any u, $v \in S$. Thus for any $u \in S$, $f(u)C_V \subseteq nil(R)$ and so for any $u \in S$, $f(u) \in nil(R) : C_V$. Then $f \in [[(nil(R) : C_V)^{S,\leq}]]$ and so $[[nil(R)^{S,\leq}]] : V \subseteq [[(nil(R) : C_V)^{S,\leq}]]$. Conversely, assume that $f \in [[(nil(R) : C_V)^{S,\leq}]]$. Then for any $u \in S$, $f(u)C_V \subseteq nil(R)$. Hence for any $g \in V$, it is easy to see that $fg \in [[nil(R)^{S,\leq}]]$. So $f \in [[nil(R)^{S,\leq}]] : V$. Hence $[[(nil(R) : C_V)^{S,\leq}]] \subseteq [[nil(R)^{S,\leq}]] : V$. Thus

$$[[nil(R)^{S,\leq}]] : V = [[(nil(R) : C_V)^{S,\leq}]] = \phi(nil(R) : C_V).$$

Hence ϕ is surjective. Therefore ϕ is a bijection. □

Corollary 3.2. *Let* $NAnn_{R[[x]]}(2^{R[[x]]}) = \{nil(R)[[x]] : V \mid V \subseteq R[[x]]\}$, *and let* $NAnn_{R[[x_1,\ldots,x_n]]}(2^{R[[x_1,\ldots,x_n]]}) = \{nil(R)[[x_1,\ldots,x_n]] : V \mid V \subseteq R[[x_1,\ldots,x_n]]\}$. *Then we have the following results.*

(1) *If R is nil power serieswise Armendariz, then*

$$\phi : NAnn_R(2^R) \longrightarrow NAnn_{R[[x]]}(2^{R[[x]]})$$

defined by $\phi(I) = I[[x]]$ for every $I \in NAnn_R(2^R)$ is bijective.

(2) *If R is n nil power serieswise Armendariz, then*

$$\phi : NAnn_R(2^R) \longrightarrow NAnn_{R[[x_1,\ldots,x_n]]}(2^{R[[x_1,\ldots,x_n]]})$$

defined by $\phi(I) = I[[x_1,\ldots,x_n]]$ for every $I \in NAnn_R(2^R)$ is bijective.

Proof. By Proposition 3.1, we complete the proof. □

Proposition 3.3. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a nil generalized power serieswise Armendariz ring. If for each nonempty subset $X \not\subseteq nil(R)$, $nil(R) : X$ is generated as a right ideal by a nilpotent element, then for each nonempty subset $U \subseteq [[R^{S,\leq}]]$ with $U \not\subseteq [[nil(R)^{S,\leq}]]$, $[[nil(R)^{S,\leq}]] : U$ is generated as a right ideal by a nilpotent element.*

Proof. Let U be a nonempty subset of $[[R^{S,\leq}]]$ with $U \not\subseteq [[nil(R)^{S,\leq}]]$. Suppose that $f \in [[nil(R)^{S,\leq}]] : U$. Then $fg \in [[nil(R)^{S,\leq}]]$ for each $g \in U$. Since R is nil generalized power serieswise Armendariz, $f(u)g(v) \in nil(R)$ for each u, $v \in S$. Hence for any $u \in S$, $f(u) \in nil(R) : C_U$. If $C_U \subseteq nil(R)$, then $U \subseteq [[nil(R)^{S,\leq}]]$, a contradiction. Hence there exists $p \in nil(R)$ such that $nil(R) : C_U = pR$. Now we show that $[[nil(R)^{S,\leq}]] : U = C_p^0[[R^{S,\leq}]]$. Note that for any $u \in S$, $f(u) \in nil(R) : C_U = pR$. Thus for any $u \in S$, there exists $r_u \in R$ such that $f(u) = pr_u$. Define $h \in [[R^{S,\leq}]]$ via

$$h(x) = \begin{cases} r_u & \text{if} \quad x = u \in supp(f) \quad \text{and} \quad f(u) = pr_u, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f = C_p^0 h \in C_p^0[[R^{S,\leq}]]$. Hence $[[nil(R)^{S,\leq}]] : U \subseteq C_p^0[[R^{S,\leq}]]$.

On the other hand, for each $g \in [[R^{S,\leq}]]$, $f \in U$ and each $s \in S$,

$$(C_p^0 gf)(s) = \sum_{(v,u)\in X_s(g,f)} pg(v)f(u).$$

Since R is nil generalized power serieswise Armendariz, by Proposition 2.2, R is an NI ring. Then it is easy to see that $(C_p^0 gf)(s) \in nil(R)$ for each $s \in S$. Hence $C_p^0 gf \in [[nil(R)^{S,\leq}]]$. Thus $C_p^0[[R^{S,\leq}]] \subseteq [[nil(R)^{S,\leq}]] : U$. Therefore $[[nil(R)^{S,\leq}]] : U = C_p^0[[R^{S,\leq}]]$ where $C_p^0 \in nil([[R^{S,\leq}]])$. □

Proposition 3.4. *Let $(S, \leq)$ be a torsion-free cancellative strictly ordered monoid satisfying the condition that $s \geq 0$ for all $s \in S$ and R a nil generalized power serieswise Armendariz ring. Then the following conditions are equivalent*:

(1) *For each nonempty subset $X \not\subseteq nil(R)$, $nil(R) : X$ is generated as a right ideal by a nilpotent element.*

(2) *For each nonempty subset $U \subseteq [[R^{S,\leq}]]$ with $U \not\subseteq [[nil(R)^{S,\leq}]]$, $[[nil(R)^{S,\leq}]]$ $: U$ is generated as a right ideal by a nilpotent element.*

Proof. (1) $\Longrightarrow$ (2) is immediate from Proposition 3.3.

(2) $\Longrightarrow$ (1) Let X be a nonempty subset of R with $X \not\subseteq nil(R)$. Then $X \not\subseteq [[nil(R)^{S,\leq}]]$. Then $[[nil(R)^{S,\leq}]] : X = f[[R^{S,\leq}]]$ where f is a nilpotent element of $[[R^{S,\leq}]]$. Since R is nil generalized power serieswise Armendariz, by Corollary 2.15, $f \in [[nil(R)^{S,\leq}]]$. Hence for any $s \in S$, $f(s) \in nil(R)$. In particular, $f(0) \in nil(R)$. Now we show that $nil(R) : X = f(0)R$. Since R is nil generalized power serieswise Armendariz, by Proposition 2.2, R is an NI ring. Then it is easy to see that $f(0)R \subseteq nil(R) : X$. So it suffices to show that $nil(R) : X \subseteq f(0)R$. If $m \in nil(R) : X$, then $C_m^0 \in [[nil(R)^{S,\leq}]] : X = f[[R^{S,\leq}]]$. There exists $g \in [[R^{S,\leq}]]$ such that $C_m^0 = fg$. Since $(S, \leq)$ is torsion-free cancellative strictly ordered monoid satisfying the condition that $s \geq 0$ for all $s \in S$, we have $C_m^0(0) = m = (fg)(0) = f(0)g(0) \in f(0) \cdot R$, and so $nil(R) : X \subseteq f(0) \cdot R$. Hence $nil(R) : X = f(0) \cdot R$ with $f(0) \in nil(R)$. Therefore, $nil(R) : X$ is generated as a right ideal by a nilpotent element. □

Proposition 3.5. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a nil generalized power serieswise Armendariz ring. If for each $p \notin nil(R)$, $nil(R) : p$ is generated as a right ideal by a nilpotent element, then for each $f \notin [[nil(R)^{S,\leq}]]$, $[[nil(R)^{S,\leq}]] : f$ is generated as a right ideal by a nilpotent element.*

Proof. Let $f \notin [[nil(R)^{S,\leq}]]$. Suppose that $g \in [[nil(R)^{S,\leq}]] : f$. Then $fg \in [[nil(R)^{S,\leq}]]$. Since R is nil generalized power serieswise Armendariz, $f(u)g(v) \in nil(R)$ for all u, $v \in S$. Thus $g(v) \in nil(R) : f(u)$ for all u, $v \in S$. If for all $u \in S$, $f(u) \in nil(R)$, then $f \in [[nil(R)^{S,\leq}]]$, a contradiction. Thus there exists $u \in S$ such that $f(u) \notin nil(R)$, and so there exists $q \in nil(R)$ such that $nil(R) : f(u) = qR$. Now we show that $[[nil(R)^{S,\leq}]] : f = C_q^0[[R^{S,\leq}]]$. Note that for any $v \in S$, $g(v) \in nil(R) : f(u) = qR$. So for any $v \in S$, there exists $r_v \in R$ such that $g(v) = qr_v$. Define $h \in [[R^{S,\leq}]]$ via

$$h(x) = \begin{cases} r_v & \text{if} \quad x = v \in supp(g) \quad \text{and} \quad g(v) = qr_v, \\ 0 & \text{otherwise.} \end{cases}$$

Then $g = C_q^0 h \in C_q^0[[R^{S,\leq}]]$, and so $[[nil(R)^{S,\leq}]] : f \subseteq C_q^0[[R^{S,\leq}]]$.

On the other hand, for each $h \in [[R^{S,\leq}]]$ and each $s \in S$,

$$(C_q^0 hf)(s) = \sum_{(u,v) \in X_s(h,f)} qh(u)f(v).$$

Since R is nil generalized power serieswise Armendariz, by Proposition 2.2, R is an NI ring. Then it is easy to see that $(C_q^0 hf)(s) \in nil(R)$ for all $s \in S$. Hence $C_q^0 hf \in [[nil(R)^{S,\leq}]]$. Thus $C_q^0[[R^{S,\leq}]] \subseteq [[nil(R)^{S,\leq}]] : f$. Therefore $[[nil(R)^{S,\leq}]] : f = C_q^0[[R^{S,\leq}]]$ where $C_q^0 \in nil([[R^{S,\leq}]])$. □

Proposition 3.6. *Let S be a torsion-free cancellative strictly ordered monoid satisfying the condition that $s \geq 0$ for all $s \in S$ and R a nil generalized power serieswise Armendariz ring. Then the following conditions are equivalent:*

(1) *For each $p \notin nil(R)$, $nil(R) : p$ is generated as a right ideal by a nilpotent element.*

(2) *For each $f \notin [[nil(R)^{S,\leq}]]$, $[[nil(R)^{S,\leq}]] : f$ is generated as a right ideal by a nilpotent element.*

Proof. It is similar to the proof as given in Proposition 3.4, □

Corollary 3.7. *If R is a nil power serieswise Armendariz ring, then the following conditions are equivalent:*

(1) *For each nonempty subset $X \nsubseteq nil(R)$, $nil(R) : X$ is generated as a right ideal by a nilpotent element.*

(2) *For each nonempty subset $U \subseteq R[[x]]$ with $U \nsubseteq nil(R)[[x]]$, $nil(R)[[x] : U$ is generated as a right ideal by a nilpotent element.*

Corollary 3.8. *If R is an n nil power serieswise Armendariz ring, then the following conditions are equivalent:*

(1) *For each nonempty subset* $X \nsubseteq nil(R)$, $nil(R) : X$ *is generated as a right ideal by a nilpotent element.*

(2) *For each nonempty subset* $U \subseteq R[[x_1, \dots, x_n]]$ *with* $U \nsubseteq nil(R)[[x_1, \dots, x_n]]$, $nil(R)[[x_1, \dots, x_n]] : U$ *is generated as a right ideal by a nilpotent element.*

Corollary 3.9. *If R is a nil power serieswise Armendariz ring, then the following conditions are equivalent*:

(1) *For each* $p \notin nil(R)$, $nil(R) : p$ *is generated as a right ideal by a nilpotent element.*

(2) *For each* $f \notin nil(R)[[x]]$, $nil(R)[[x]] : f$ *is generated as a right ideal by a nilpotent element.*

Corollary 3.10. *If R is an n nil power serieswise Armendariz ring, then the following conditions are equivalent*:

(1) *For each* $p \notin nil(R)$, $nil(R) : p$ *is generated as a right ideal by a nilpotent element.*

(2) *For each* $f \notin nil(R)[[x_1, \dots, x_n]]$, $nil(R)[[x_1, \dots, x_n]] : f$ *is generated as a right ideal by a nilpotent element.*

For any subset X of a ring R, $r_R(X) = \{a \in R \mid Xa = 0\}$ denotes the right annihilator of X in R. Faith [6] called a ring R right zip provided that if the right annihilator $r_R(X)$ of a subset X of R is zero, then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$. Beachy and Blair [4] showed that if R is a commutative zip ring, then the polynomial ring $R[x]$ over R is a zip ring. As a generalization of zip rings, in [13] L. Ouyang introduced the notion of weak zip rings. A ring R is a weak zip ring provided that for any subset X of R, if $nil(R) : X \subseteq nil(R)$, then there exists a finite subset $Y \subseteq X$ such that $nil(R) : Y \subseteq nil(R)$. L. Ouyang showed that if R is a semicommutative ring, then R is weak zip if and only if $R[x]$ is weak zip. In the following we investigate the weak zip property of the generalized power series ring $[[R^{S,\leq}]]$ under the condition that R is nil generalized power serieswise Armendariz.

Lemma 3.11. *Let $(S, \leq)$ be a cancellative torsion-free strictly ordered monoid and R a nil generalized power serieswise Armendariz ring. Then the following conditions are equivalent*:

(1) *R is a weak zip ring.*

(2) *For each subset* $U \subseteq [[R^{S,\leq}]]$, *if* $[[nil(R)^{S,\leq}]] : U \subseteq [[nil(R)^{S,\leq}]]$, *then there exists a finite subset* $V \subseteq U$ *such that* $[[nil(R)^{S,\leq}]] : V \subseteq [[nil(R)^{S,\leq}]]$.

Proof. (1) $\Longrightarrow$ (2) Let U be a subset of $[[R^{S,\leq}]]$ such that $[[nil(R)^{S,\leq}]] : U \subseteq [[nil(R)^{S,\leq}]]$. We first show that $nil(R) : C_U \subseteq nil(R)$. Suppose that $a \in nil(R) : C_U$. Then for any $f \in U$ and any $s \in S$, $(fC_a^0)(s) = f(s)a \in nil(R)$. Thus for any $f \in U$, $fC_a^0 \in [[nil(R)^{S,\leq}]]$, and so $C_a^0 \in [[nil(R)^{S,\leq}]] : U \subseteq [[nil(R)^{S,\leq}]]$. Hence $a = C_a^0(0) \in nil(R)$ and so $nil(R) : C_U \subseteq nil(R)$ is proved. Since R is a weak zip ring, there exists a finite subset $Y \subseteq C_U$ such that $nil(R) : C_U \subseteq nil(R)$. Without loss of generality, we may assume that $Y = \{u_1, u_2, \dots, u_n\} \subseteq C_U$. For each $u_i \in Y$, there exists some $f_{u_i} \in U$ such that $f_{u_i}(s_i) = u_i$ for some $s_i \in S$. Let V be a minimal subset of U such that $f_{u_i} \in V$ for each $u_i \in Y, 1 \leq i \leq n$. Then V is a finite subset of U and $Y \subseteq C_V$. Now we show that $[[nil(R)^{S,\leq}]] : V \subseteq [[nil(R)^{S,\leq}]]$. Suppose $h \in [[nil(R)^{S,\leq}]] : V$. Then $gh \in [[nil(R)^{S,\leq}]]$ for every $g \in V$. Then $g(u)h(v) \in nil(R)$ for each $u, v \in S$ since R is nil generalized power serieswise Armendariz, and so for any $v \in S$, $h(v) \in nil(R) : C_V \subseteq nil(R) : Y \subseteq nil(R)$. Hence $h \in [[nil(R)^{S,\leq}]]$. Therefore $[[nil(R)^{S,\leq}]] : V \subseteq [[nil(R)^{S,\leq}]]$, as desired.

(2) $\Longrightarrow$ (1)Assume that $nil(R) : X \subseteq nil(R)$ where X is a subset of R. We first show that $[[nil(R)^{S,\leq}]] : X \subseteq [[nil(R)^{S,\leq}]]$. Let $f \in [[nil(R)^{S,\leq}]] : X$. Then $xf = C_x^0 f \in [[nil(R)^{S,\leq}]]$ for each $x \in X$. Since R is nil generalized power serieswise Armendariz, $xf(s) \in nil(R)$ for each $s \in S$ and $x \in X$. Thus for each $s \in S$, $f(s) \in nil(R) : X \subseteq nil(R)$. This implies that $f \in [[nil(R)^{S,\leq}]]$ and so $[[nil(R)^{S,\leq}]] : X \subseteq [[nil(R)^{S,\leq}]]$. Then we can find a finite subset $Y \subseteq X$ such that $[[nil(R)^{S,\leq}]] : Y \subseteq [[nil(R)^{S,\leq}]]$. Now we show that $nil(R) : Y \subseteq nil(R)$. If $a \in nil(R) : Y$, then $C_a^0 \in [[nil(R)^{S,\leq}]] : Y \subseteq [[nil(R)^{S,\leq}]]$, and so $a \in nil(R)$. Hence $nil(R) : Y \subseteq nil(R)$. Therefore R is a weak zip ring. □

Proposition 3.12. *Let $(S, \leq)$ be a cancellative torsion-free strictly ordered monoid and R a nil generalized power serieswise Armendariz ring with $nil(R)$ nilpotent. Then the following conditions are equivalent*:

(1) *R is a weak zip ring.*

(2) *$[[R^{S,\leq}]]$ is a weak zip ring.*

Proof. By Proposition 2.17 and Proposition 3.9, we complete the proof. □

Corollary 3.13. *We have the following results.*

(1) *If R is a nil power serieswise Armendariz with $nil(R)$ nilpotent. Then R is a weak zip ring if and only if the power series ring $R[[x]]$ is weak zip.*

(2) *If R is an n nil power serieswise Armendariz with $nil(R)$ nilpotent. Then R is a weak zip ring if and only if the power series ring $R[[x_1, \ldots, x_n]]$ in n indeterminates is weak zip.*

Corollary 3.14. *We have the following results.*

(1) *If R is a nil power serieswise Armendariz right noetherian ring. Then R is a weak zip ring if and only if the power series ring $R[[x]]$ is weak zip.*

(2) *If R is an n nil power serieswise Armendariz right noetherian ring. Then R is a weak zip ring if and only if the power series ring $R[[x_1, \ldots, x_n]]$ in n indeterminates is weak zip.*

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