KYUNGPOOK Math. J. 53(2013), 407-417
http://dx.doi.org/10.5666/KMJ.2013.53.3.407

t-Prüfer Modules

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Abstract. In this article, we characterize t-Prüfer modules in the class of faithful multiplication modules. As a corollary, we also characterize Krull modules. Several properties of a t-invertible submodule of a faithful multiplication module are given.

1. Introduction

Let R be a commutative ring with identity and M be a unital R-module. M is said to be *faithful* if $ann_R(M) = 0$. M is called a *multiplication module* if each submodule N of M has the form IM for some ideal I of R, equivalently, for any submodule N of M, $N = (N :_R M)M$. M is called a *cancellation module* if for all ideals I and J of R, $IM \subseteq JM$ implies $I \subseteq J$. It was shown in [5, Proposition 2.2] that if R is an integral domain and M is a faithful multiplication R-module, then M is finitely generated. Thus it follows from [12, Theorem 3.1] that a faithful multiplication module M over an integral domain is a cancellation module. Hence we have that $I(N :_R M) = (IN :_R M)$ for all submodules N of M and all ideals I of R. It was also shown in [13, Lemma 2.1] that if M is a faithful multiplication R-module over integral domain R, then M is torsion-free.

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† This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0011996).
Received December 8, 2012; accepted April 19, 2013.
2010 Mathematics Subject Classification: 13A15, 13F05, 13C13.
Key words and phrases: t-Prufer module, faithful multiplication module, Krull module, t-invertible submodule.

Let R be an integral domain with quotient field K and M be a unital R-module. Let N be a nonzero submodule of M and let $N^{-1} = (M :_K N) = \{x \in K \mid xN \subseteq M\}$. Then N^{-1} is an R-submodule of K and $NN^{-1} \subseteq M$. Following Naoum and Al-Alwan, in [1], N is said to be *invertible* in M if $NN^{-1} = M$. Clearly M is invertible in M, and it is proved in [1, Remark 3.8] that R is an integral domain if and only if every nonzero cyclic submodule of the faithful multiplication R-module M is invertible in M (i.e., M is a D_1-module).

Let M be a faithful multiplication module over an integral domain R and let N be a submodules of M. In [4], M. Ali defined $N_v = (N^{-1})^{-1} = R :_K (M :_K N)$ and showed that $N^{-1} = (N :_R M)^{-1}$, and hence $N_v = (N :_R M)_v$ (in particular, $M_v = R$), and then introduced the concept of a divisorial submodule or v-submodule of M as follows: N is a *v-submodule* if $N = N_vM$. N is called a *v-submodule of finite type* if $N = L_vM$ for some finitely generated submodule L of M. It follows that N is a v-submodule of M if and only if $(N :_R M)$ is a v-ideal of R. If L and N are submodules of a multiplication module M with $L \subseteq N$, then $L_v \subseteq N_v$. It was also shown that for any submodule N of a multiplication R-module, $N \subseteq N_vM$. If N is a submodule of M and I is an ideal of R such that either I_v is invertible or N_v is invertible, then as it was remarked in [4, p.144] it is easily seen that $(IN)_v = (IN :_R M)_v = (I(N :_R M))_v = I_v(N :_R M)_v = I_vN_v$. In [16], we introduced the concept of t-invertible submodule of a multiplication module and gave some characterizations of faithful multiplication Krull modules, Mori modules and π-modules. Also, we defined $N_t = \bigcup \{L_v \mid L$ is a finitely generated submodule of M contained in $N\}$. It is easily seen that N_t is an R-submodule of K, $N_t \subseteq N_v$ and $(NN^{-1})_t \subseteq R$. N is said to be *t-(resp. v-)invertible submodule of M* if $(NN^{-1})_t = R$ (resp. $(NN^{-1})_v = R$). It is clear that every invertible submodule of M is t-invertible and every t-invertible submodule is v-invertible. By [6, Proposition 2.2], N is finitely generated if and only if $(N :_R M)$ is finitely generated. Using this result, we have that $(N :_R M)_t = N_t$ and $N \subseteq N_tM$. So we call N a *t-submodule* of M if $N = N_tM$. N is called a *t-submodule of finite type* if $N = L_tM$ for some finitely generated submodule L of M. Clearly every v-submodule is a t-submodule. If I is a t-ideal of R, then IM is a t-submodule of M. Also if N is finitely generated, then N is a v-submodule of M if and only if N is a t-submodule of M.

In section 2 we characterize faithful multiplication t-Prüfer modules (i.e., every nonzero finitely generated submodule is t-invertible). In section 3 we give several properties of a t-invertible submodule of a faithful multiplication module.

2. t-Prüfer Modules

We begin with this section by giving a characterization of when a submodule of the module is $*$-submodule of finite type.

Lemma 2.1. *Let M be a faithful multiplication module over an integral domain R, I an ideal of R, N a submodules of M, and let $* = v$ or t. Then*

(1) *N is a $*$-submodule of finite type if and only if $(N :_R M)$ is a $*$-ideal of finite*

type.

(2) *I is a $*$-ideal of finite type if and only if IM is a $*$-submodule of finite type.*

Proof. Since the assertion for the case $* = v$ is similar to that of $* = t$, we only consider the case $* = t$. (1) If N is a t-submodule of finite type, then $N = L_tM$ for some finitely generated submodule L of M and $(N :_R M)$ is a t-ideal by [16, Proposition 2.2]. Since $N_t = (N :_R M)_t$, we have that $(N :_R M)_tM = (L :_R M)_tM$. Hence $(N :_R M) = (L :_R M)_t$ and $(L :_R M)$ is a finitely generated ideal of R. Conversely, assume that $(N :_R M)$ is a t-ideal of finite type. Then by [16, Proposition 2.2], $(N :_R M) = I_t$ for some finitely generated ideal I of R and N is a t-submodule. Hence $N = (N :_R M)M = I_tM = (IM :_R M)_tM = (IM)_tM$ and IM is a finitely generated submodule of M. $\square$

Proposition 2.2. *The following conditions are equivalent for a faithful multiplication module M over an integral domain R.*

(1) *Each t-submodule of M is of finite type.*

(2) *M satisfies the ascending chain condition on t-submodules.*

(3) *M satisfies the ascending chain condition on v-submodules.*

(4) *For each nonzero submodule N of M, there exists a finitely generated submodule L of M contained in N such that $N_t = L_t$.*

Proof. The equivalences of (1), (2) and (3) follow from Lemma 2.1 and [20, Theorem 1.1]. To complete the proof, we show that (1) $\Leftrightarrow$ (4).

(1) $\Rightarrow$ (4). By Lemma 2.1, each t-ideal of R is of finite type. Then for each nonzero ideal I, there exists a finitely generated ideal J of R contained in I such that $I_t = J_t$ [20, Theorem 1.1]. Let N be a nonzero submodule of M. Since M is a multiplication module, $(N :_R M)$ is a nonzero ideal of R. Then $(N :_R M)_t = B_t$ for some finitely generated ideal $B \subseteq (N :_R M)$ of R. Thus $(BM)_t = (BM :_R M)_t = B_t = N_t$ and $BM \subseteq N$ is a finitely generated submodule of M.

(4) $\Rightarrow$ (1). Let N be any t-submodule of M. Then by hypothesis, there exists a finitely generated submodule L of M contained in N such that $N_t = L_t$. Thus $N = N_tM = L_tM$, and so N is of finite type. $\square$

Recall from [3, p. 25] that an R-module M is called a *cyclic submodule module (CSM)* if every submodule of M is cyclic. It was shown in [3] that a faithful multiplication module over an integral domain R is a CSM if and only if R is a principal ideal domain. In [16], we call an R-module M a *v-Prüfer module* if $(NN^{-1})_v = R$ for every nonzero finitely generated submodule N of M and M a *Krull module* if M_P is a CSM R_P-module for each $P \in X^{(1)}(R)$, which is the set of height-one prime ideals of R, $M = \bigcap_{P \in X^{(1)}(R)} M_P$, and each nonzero $x \in M$ is primitive in all but a finite number of M_P. It was shown in [16, Theorem 2.1(1) and Theorem 2.8] that a faithful multiplication module over an integral domain R is a

v-Prüfer (resp., Krull) module if and only if R is a v-Prüfer (resp., Krull) domain. The following result is an analogue to Cohen's theorem for v-Prüfer modules.

Theorem 2.3. *Let R be an integral domain and M a faithful multiplication and v-Prüfer module. If every prime t-submodule of M is of finite type, then every t-submodule of M is of finite type.*

Proof. By [16, Theorem 2.5], M is a v-Prüfer module if and only if R is a v-Prüfer domain. Let P be any prime t-ideal of R. Then PM is a prime t-submodule of M. Since PM is a t-submodule of finite type, we have $PM = Q_tM$ for some finitely generated submodule Q of M. Then $P = (Q :_R M)_t$ and $(Q :_R M)$ is a finitely generated ideal of R. Hence P is a t-ideal of finite type. By [17, Theorem 2.3], every t-ideal of R is of finite type. Let N be a t-submodule of M. Then $(N :_R M)$ is a t-ideal of finite type, and so $(N :_R M) = B_t$ for some finitely generated ideal B of R. Hence $N = (N :_R M)M = B_tM = (BM :_R M)_tM = (BM)_tM$ and BM is a finitely generated submodule of M. □

Several characterizations of Krull modules were given in [16]. In the following, we characterize Krull modules in terms of v-Prüfer modules, which generalizes [17, Theorem 2.4].

Corollary 2.4. *Let R be an integral domain and M a faithful multiplication and v-Prüfer module. Then the following are equivalent.*

(1) *The ascending chain condition on t-submodules holds.*

(2) *Each t-submodule of M is of finite type (and hence each v-submodule of M is of finite type)*

(3) *Each prime t-submodule is of finite type.*

(4) *M is a Krull module.*

Proof. Note first that R is a v-Prüfer domain ([16, Theorem 2.5(1)]).

(1) $\Leftrightarrow$ (2) This follows from Lemma 2.1 and [17, Theorem 2.4].

(2) $\Leftrightarrow$ (3) This follows from Lemma 2.1, [17, Theorem 2.4] and Theorem 2.3.

(2) $\Rightarrow$ (4) Let I be any t-ideal of R. Since IM is a t-submodule, IM is of finite type. Then I is of finite type by Lemma 2.1. Thus by again [17, Theorem 2.4] R is a Krull domain. It follows from [16, Theorem 2.8] that M is a Krull module.

(4) $\Rightarrow$ (2) By [16, Theorem 2.8], R is a Krull domain. Then every t-ideal of R is of finite type. Let N be a t-submodule of M. Then $(N :_R M)$ is a t-ideal of finite type by Lemma 2.1. Thus N is of finite type. □

We recall that an integral domain R is a *t-Prüfer domain* (or *Prüfer v-multiplication domain* (for short *PvMD*)) if $(II^{-1})_t = R$ for every nonzero finitely generated ideal I of R. Now we extend this concept to the module case: An R-module M is a *t-Prüfer module* if $(NN^{-1})_t = R$ for every nonzero finitely generated submodule N of M.

Lemma 2.5. *Let M be a faithful multiplication module over an integral domain R. Then M is a t-Prüfer module if and only if R is a t-Prüfer domain.*

Proof. Let I be a nonzero finitely generated ideal of R. Since M is a faithful multiplication module, IM is a nonzero finitely generated submodule of M. So IM is t-invertible. By [16, Lemma 2.1], I is t-invertible. Thus R is a t-Prüfer domain. Conversely, let N be a nonzero finitely generated submodule of M. Then $(N :_R M)$ is a nonzero finitely generated ideal of R, and so $(N :_R M)$ is t-invertible. Thus N is t-invertible by [16, Proposition 2.2]. Hence M is a t-Prüfer module. □

Recall from [3] that an R-module M is called a *valuation module* if for all $m, n \in M$, either $Rm \subseteq Rn$ or $Rn \subseteq Rm$, or equivalently, for all submodules L, N of M, either $L \subseteq N$ or $N \subseteq L$. It was shown that a faithful multiplication module over an integral domain R is a valuation module if and only if R is a valuation domain.

Theorem 2.6. *Let M be a faithful multiplication module over an integral domain R. Then the following are equivalent.*

(1) *M is a t-Prüfer module.*

(2) *For every maximal t-ideal P of R, M_P is a valuation module.*

(3) *Every v-submodule of finite type is t-invertible.*

Proof. (1) $\Leftrightarrow$ (2) If M is a t-Prüfer module, then R is a t-Prüfer domain by Lemma 2.5. Then R_P is a valuation domain for every maximal t-ideal P of R. Since M_P is a faithful multiplication R_P-module, M_P is a valuation module by [5, Proposition 2.2]. Conversely, if M_P is a valuation module, then by [5, Proposition 2.2] R_P is a valuation domain. Thus R is a t-Prüfer domain. Hence M is a t-Prüfer module by Lemma 2.5.

(1) $\Leftrightarrow$ (3) Suppose that M is a t-Prüfer module and let N be a v-submodule of finite type. By Lemma 2.1, $(N :_R M)$ is a v-ideal of finite type. Then $(N :_R M) = B_v$ for some finitely generated ideal $B \subseteq (N :_R M)$. Since BM is finitely generated, we have BM is t-invertible, and so B is t-invertible by [16, Lemma 2.1]. Then $(N :_R M)$ is t-invertible. By [16, Proposition 2.2], N is t-invertible. Conversely, let N be a nonzero finitely generated submodule of M. Since M is a faithful multiplication module, $(N :_R M)$ is finitely generated. Then $I = (N :_R M)_v$ is a v-ideal of finite type. By Lemma 2.1, IM is a v-submodule of finite type. So IM is t-invertible. Then $(N :_R M)$ is t-invertible, and so N is t-invertible. Hence M is a t-Prüfer module. □

Let M be a faithful multiplication module over an integral domain R. By [16, Lemma 2.1 and Proposition 2.2], every nonzero ideal of R is v-invertible if and only if every nonzero submodule of M is v-invertible. Recall from [16] that M is called an *essential module* if $M = \bigcap_{P\in\Lambda} M_P$, where $\Lambda \subseteq \mathrm{Spec}(R)$, and each localization M_P of M at $P \in \Lambda$ is a valuation module.

Theorem 2.7. *Let M be a faithful multiplication module over an integral domain R. Suppose that for each nonzero submodule N of M, N_v is of finite type. Then the following are equivalent.*

(1) *Every nonzero submodule of M is v-invertible.*

(2) *M is a v-Prüfer module.*

(3) *M is a t-Prüfer module.*

(4) *M is an essential module.*

Proof. (4) $\Rightarrow$ (1) $\Rightarrow$ (2) are trivial.

(2) $\Rightarrow$ (3) This follows from [16, Theorem 2.5] and Lemma 2.5

(3) $\Rightarrow$ (4) Let I be a finitely generated ideal of R. Then IM is a finitely generated submodule. By hypothesis, IM is t-invertible. Then I is t-invertible, and so R is a t-Prüfer domain. Then R is an essential domain. Hence M is an essential module [16, Theorem 2.5]. □

Recall from [16] that M is called a *Mori module* if M satisfies the ascending chain condition on v-submodules of M. If M is a faithful multiplication and Mori module, then for each nonzero submodule N of M, N_v is of finite type by [16, Theorem 2.7].

Corollary 2.8. *Let R be an integral domain and M a faithful multiplication and Mori module. Then the following are equivalent.*

(1) *Every nonzero submodule of M is v-invertible.*

(2) *M is a Krull module.*

(3) *M is a v-Prüfer module.*

(4) *M is a t-Prüfer module.*

(5) *M is an essential module.*

3. t-invertible Submodules

In this section we give several properties of a t-invertible submodule of a faithful multiplication module.

Proposition 3.1. *Let M be a faithful multiplication module over an integral domain R. If every nonzero prime submodule of M contains a t-invertible prime submodule, then there exists a nonempty collection Λ of minimal prime ideals of R such that $M = \bigcap_{P\in\Lambda} M_P$ and each M_P, $P \in \Lambda$, is a CSM R_P-module.*

Proof. Let I be any nonzero prime ideal of R. By hypothesis, there exists a t-invertible prime submodule N of M such that $N \subseteq IM$. Then I contains a t-invertible prime ideal $(N :_R M)$ of R. By [15, Lemma 3.3], there exists a nonempty collection Λ of minimal prime ideals of R such that $R = \bigcap_{P\in\Lambda} R_P$ and each R_P,

$P \in \Lambda$, is a PID. By [5, Proposition 2.2], M_P is a CSM R_P-module. Now we show that $M = \bigcap_{P\in\Lambda} M_P$. Clearly $M \subseteq \bigcap_{P\in\Lambda} M_P$. Let $0 \neq x \in \bigcap_{P\in\Lambda} M_P$. By [16, Lemma 2.4], $(M :_R Rx) \nsubseteq P$ for each $P \in \Lambda$. Since $(M :_R Rx)$ is a t-ideal and every proper t-ideal is contained in a maximal t-ideal of R, $(M :_R Rx) = R$. Hence $x \in M$. □

Let M be a faithful multiplication module over an integral domain R. Then M is finitely generated and torsion-free. Also, for every prime ideal P of R, $M_P \cong R_P$ as an R_P-module. It follows that $N_P \cong (N :_R M)_P$, where N is a submodule of M [3].

Lemma 3.2. *Let M be a faithful multiplication module over an integral domain R. Suppose that N is a finitely presented submodule of M. Then*

(1) $(N :_R M)_P \cong (N_P : M_P)$ *for every prime ideal P of R.*

(2) $(N :_R M)_P$ *is a principal ideal of R_P if and only if N_P is a cyclic R_P-module for every prime ideal P of R.*

Proof. (1) Since M and N are torsion-fee, we have $(N :_R M)_P \cong \mathrm{Hom}(M, N)_P$. Also, $\mathrm{Hom}(M, N)_P \cong \mathrm{Hom}(M_P, N_P)$ since N is finitely presented. Therefore $(N :_R M)_P \cong (N_P : M_P)$. (2) follows from (1). □

Recall that an integral domain R is called a *coherent domain* if every finitely generated ideal of R is finitely presented. Then it is well known that R is a coherent domain if and only if every finitely generated torsion-free R-module is finitely presented. Therefore, if R is a coherent domain and N is finitely generated, then Lemma 3.2 holds.

The following result generalizes [17, Lemma 1.5 and Corollary 1.6].

Proposition 3.3. *Let M be a faithful multiplication module over an integral domain R, N a finitely generated submodule of M. If N is finitely presented, then N is t-invertible if and only N_P is a cyclic R_P-module for every maximal t-ideal P of R.*

Proof. Since N is a finitely generated submodule of M, we have $(N :_R M)$ is a finitely generated ideal of R. If N is t-invertible, then by [16, Proposition 2.2], $(N :_R M)$ is a t-invertible ideal of R. By [17, Lemma 1.5], $(N :_R M)_P$ is principal for every maximal t-ideal P of R. Hence N_P is a cyclic R_P-module. The proof of the converse is similar. □

Corollary 3.4. *Let M be a faithful multiplication module over an integral domain R, N a v-submodule of finite type. If N is finitely presented, then N is t-invertible if and only N_P is a cyclic R_P-module for every maximal t-ideal P of R.*

Proof. By Lemma 2.1, N is a v-submodule of finite type if and only if $(N :_R M)$ is a v-ideal of finite type. Then by [17, Corollary 1.6], N is t-invertible, if and only if $(N :_R M)_P$ is principal for every maximal t-ideal P of R, if and only if N_P is a cyclic R_P-module. □

Let M be a faithful multiplication module over an integral domain R. If N is a t-invertible prime t-submodule of M, then N is a maximal t-submodule of M.

The following result generalizes [17, Theorem 1 and Corollary 2].

Proposition 3.5. *Let M be a faithful multiplication module over an integral domain R and let N be a nonzero submodule of M. If every prime t-submodule minimal over N is of finite type, then there are only finitely many t-submodules minimal over N.*

Proof. Let I be a prime t-ideal minimal over $(N :_R M)$. Then IM is a prime t-submodule minimal over N. By hypothesis, IM is of finite type, and so by Lemma 2.1, I is of finite type. Then there are only finitely many prime t-ideals $\{P_1, \cdots, P_n\}$ minimal over $(N :_R M)$ by [11, Theorem 1]. Hence $P_1M, \cdots, P_nM$ are only finitely many prime t-submodules minimal over N. □

Corollary 3.6. *Let M be a faithful multiplication module over an integral domain R and let N be a nonzero submodule of M. If every prime t-submodule minimal over N is t-invertible, then N is contained in only finitely many minimal prime t-submodules.*

Proof. This follows from the fact that every t-invertible t-submodule is of finite type. □

Lemma 3.7. *Let M be a faithful multiplication module over an integral domain R, J a finitely generated ideal of R and N a proper v-submodule of M. If $(JN)_v = J_vN_v$, then $(IN)_t = I_tN_t$ for every ideal I of R.*

Proof. Recall that $(IN)_t = \bigcup\{L_v \mid L$ is a finitely generated submodule of M contained in $IN\}$. For each L in the definition, $L \subseteq JN$ for some finitely generated ideal $J \subseteq I$. Then $L_v \subseteq (JN)_v = J_vN_v \subseteq I_tN_t$. Thus $(IN)_t \subseteq I_tN_t$. The reverse inclusion is trivial. □

Corollary 3.8. *Let R be a Dedekind domain and let M be a faithful multiplication R-module, I an ideal of R, and N a proper submodule of M. Then $(IN)_t = I_tN_t$.*

Following [14], a *TV-domain* is a domain in which every t-ideal is a v-ideal. Noetherian domains and Krull domains are TV-domains, cf. [14, p. 291].

Corollary 3.9. *Let R be a TV-domain and let M be a faithful multiplication R-module, J a finitely generated ideal of R, and N a proper submodule of M. If $(JN)_v = J_vN_v$, then $(IN)_t = I_tN_t$ for every ideal I of R.*

Corollary 3.10. *Let M be a faithful multiplication module over an integral domain R, J a finitely generated ideal of R and N a proper v-submodule of M. If $(JN)_v = J_vN_v$, then IN is a t-submodule of M for every t-ideal I of R.*

The following result generalizes [17, Corollary 4 and Corollary 5].

Theorem 3.11. *Let M be a faithful multiplication module over an integral domain R, J a finitely generated ideal of R and N a proper v-submodule of M. If $(JN)_v = J_vN_v$, then the following are equivalent.*

(1) *$N = (P_1^{a_1} \cdots P_n^{a_n})_t N^*$, where $P_1, \cdots, P_n$ are t-invertible prime t-ideals of R and N^* is a t-invertible prime t-submodule of M.*

(2) *Every prime t-submodule minimal over N is t-invertible.*

(3) *Every prime t-submodule containing N is t-invertible.*

(4) *There is a finite set of t-invertible prime t-ideals $\{P_1, \cdots, P_n\}$ and a t-invertible prime t-submodule N^* of M such that for every t-submodule L containing N, $L = (P_1^{a_1} \cdots P_n^{a_n})_t N^*$.*

(5) *Every t-submodule containing N is t-invertible.*

Moreover, in the case where $(1) - (5)$ hold, every t-submodule L containing N is a v-submodule of finite type and the set of t-submodules containing N is finite.

Proof. Obviously $(2) \Rightarrow (3)$ and $(4) \Rightarrow (1)$. [16, Lemma 2.1] shows that $(4) \Rightarrow (5)$. Clearly $(5) \Rightarrow (3)$.

Now we show that $(1) \Rightarrow (2)$. Suppose that $N = (P_1^{a_1} \cdots P_n^{a_n})_t N^*$, where $P_1, \cdots, P_n$ are t-invertible prime t-ideals of R and N^* is a t-invertible prime t-submodule of M. Then $(N :_R M) = ((P_1^{a_1} \cdots P_n^{a_n})_t N^* :_R M) = (P_1^{a_1} \cdots P_n^{a_n})_t (N^* :_R M) = (P_1^{a_1} \cdots P_n^{a_n}(N^* : M))_t$ for each nonnegative inter a_i, since $(N :_R M)$ and $(N^* :_R M)$ are t-ideals of R. By [11, Corollary 4], every prime t-ideal minimal over $(N :_R M)$ is t-invertible. Hence every prime t-submodule minimal over N is t-invertible.

$(3) \Rightarrow (4)$ Suppose that every prime t-submodules containing N is t-invertible. By Corollary 3.6, the set of prime t-submodules over N is finite, say $\{L_1, \cdots, L_n\}$. Then $\{Q_1, \cdots, Q_n\}$ is the set of prime t-ideals minimal over $(N :_R M)$, where $Q_i = (L_i :_R M)$. Since each L_i is t-invertible, each $Q_i = (L_i : M)$ is t-invertible by [16, Proposition 2.2]. Then there is a finite set of t-invertible prime t-ideals $\{P_1, \cdots, P_n\}$ such that every t-ideal containing $(N :_R M)$ is a t-product of powers of the P_i's. If L is a t-submodule containing N, then $(L :_R M) = (P_1^{a_1} \cdots P_n^{a_n})_t$ for each nonnegative inter a_i. Since $L = (L :_R M)M$, we have that $L_t = (P_1^{a_1} \cdots P_n^{a_n} M)_t = (P_1^{a_1} \cdots P_n^{a_n-1} P_n M)_t = (IN^*)_t$, where $I = (P_1^{a_1} \cdots P_n^{a_n-1})_t$ and $N^* = P_n M$. By Lemma 2.7, $L_t = (IN^*)_t = I_t(N^*)_t$. Hence $L = L_t M = I_t(N^*)_t M = I_t N^* = (P_1^{a_1} \cdots P_n^{a_n-1})_t N^*$.

Suppose that $(1) - (5)$ hold for N and let L be a t-submodules containing N. Then there is a finite set of t-invertible prime t-ideals $\{Q_1, \cdots, Q_n\}$ and a t-invertible prime t-submodule N^* of M such that $L = (Q_1^{a_1} \cdots Q_n^{a_n})_t N^*$. Hence L itself is t-invertible by [16, Lemma 2.1]. Then $(L :_R M)$ is a t-invertible t-ideal, and so $(L :_R M)$ is a v-ideal. Therefore $L = (L :_R M)M$ is a v-submodule. Finally, since the set of t-submodules containing N is $\{ (P_1^{b_1} \cdots P_n^{b_n})_t N^* \mid 0 \leq b_i \leq a_i \}$, where $N = (P_1^{a_1} \cdots P_n^{a_n})_t N^*$, the set of t-submodules containing N is finite. □

Proposition 3.12. *Let M be a faithful multiplication module over an integral domain R, N a nonzero submodule of M. If $N = P_1 \cdots P_n N^*$, where each P_i is an invertible prime ideal of R and N^* is an invertible prime submodule of M, then every v-submodule containing N is invertible.*

Proof. Let L be a v-submodule containing N. Then $(L :_R M)$ is a v-ideal containing $(N :_R M)$. Since $(N :_R M) = P_1 \cdots P_n(N^* :_R M)$ and $(N^* :_R M)$ is invertible, $(L :_R M)$ is invertible by [17, Corollary 5]. Hence L is invertible. □

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