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Some Further Results on Uniqueness of Entire Functions and Fixed Points

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Abstract. In this paper, we investigate the uniqueness problem on entire functions sharing fixed points (ignoring multiplicities). Our main results improve and generalize some results due to Zhang [13], Qi-Yang [10] and Dou-Qi-Yang [1].

1. Introduction

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We assume that the reader is familiar with standard notations and fundamental results of Nevanlinna Theory as explained in [12].

We say that two meromorphic functions f and g share a small function $a(z)$ IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If f and g have the same zeros with the same multiplicities, then we say that f and g share $a(z)$ CM (counting multiplicities).

Let p be a positive integer and $a \in \mathbb{C}$. We denote by $N_p(r, \frac{1}{f-a})$ the counting function of the zeros of $f-a$ where an m-fold zero is counted m times if $m \leq p$ and p times if $m > p$. We denote by $\overline{N}_L(r, \frac{1}{f-1})$ the counting function for 1-points of both $f(z)$ and $g(z)$ about which $f(z)$ has a larger multiplicity than $g(z)$, with multiplicity not being counted. We say that a finite value z_0 is a fixed point of $f(z)$ if $f(z_0) = z_0$, and we define

$$E_f = \{z \in \mathbb{C} : f(z) = z, counting\ multiplicities\}.$$

About a famous question of Hayman [5] in 1959, Fang- Hua[3] and Yang-Hua[7] proved the following.

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Theorem A. *Let f and g be two nonconstant entire functions, and let $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

Theorem B. *Let f and g be two nonconstant entire functions, and let n and k be two positive integers with $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = -1$ or $f = tg$ for a constant t such that $t^n = 1$.*

In [2], Fang also proved the following results.

Theorem C. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n \geq 2k + 8$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, then $f = g$.*

Corresponding to the problems of entire functions that share 1 CM, many authors considered the uniqueness problems of entire functions that have fixed points, see Fang-Qiu [4], Lin-Yi [8], Zhang [13].

In order to state the results, we need the following definitions:

Definition 1. Let m^* is an integer, according to the differential polynomials $(f^n(z)(\lambda f^m(z)+\mu))^{(k)}$ and $(g^n(z)(\lambda g^m(z)+\mu))^{(k)}$ in the following Theorem D and Theorem 1, we define

$$m^* = \begin{cases} m, & \lambda \neq 0; \\ 0, & \lambda = 0. \end{cases}$$

Definition 2. Let m^{**} is an integer, according to the nonzero polynomial $P(z)$ in the following Theorem E and Theorem 2, we define

$$m^{**} = \begin{cases} m, & P(z) \neq C; \\ 0, & P(z) = C. \end{cases}$$

Recently, Qi-Yang [10] and Dou-Qi-Yang [1] proved the following results which generalize some previous results.

Theorem D. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, n, m and k be positive integers, λ and μ be constants that satisfy $|\lambda| + |\mu| \neq 0$. Suppose that $n > 2k + m^* + 4$. If $(f^n(z)(\lambda f^m(z) + \mu))^{(k)}$ and $(g^n(z)(\lambda g^m(z) + \mu))^{(k)}$ share z CM, then the following conclusions hold:*

(i) If $\lambda\mu \neq 0$, then $f^d(z) \equiv g^d(z)$, where $d = GCD(n, m)$; in particular, $f(z) \equiv g(z)$, when $d = 1$.

(ii) If $\lambda\mu = 0$, then $f = cg$ for a constant c that satisfies $c^{n+m^*} = 1$, or $k = 1$ and $f(z) = b_1 e^{bz^2}$, $g(z) = b_2 e^{-bz^2}$ for three constants b_1, b_2 and b that satisfy $4(\lambda+\mu)^2 (b_1 b_2)^{n+m^*} ((n + m^*)b)^2 = -1$.

Theorem E. *Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ or $P(z) = C$,*

$a_0, a_1, \cdots, a_{m-1}, a_m \neq 0, C \neq 0$ *are complex constant. Suppose that* $f(z)$ *and* $g(z)$ *be two transcendental entire functions, and let* n, m *and* k *be three positive integers with* $n > 2k + m^{**} + 4$. *If* $(f^n(z)P(f))^{(k)}$ *and* $(g^n(z)P(g))^{(k)}$ *share* z *CM, then the following conclusions hold:*

(i) If $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ *is not a monomial, then* $f = tg$ *for a constant* t *that satisfies* $t^d = 1$, *where* $d = (n+m, \ldots, n+m-i, \cdots, n)$, $a_{m-i} \neq 0$, *for some* $i = 0, 1, 2, \ldots, m$; *or* f *and* g *satisfy the algebraic equation* $R(f, g) \equiv 0$, *where* $R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1} w_1^{m-1} + \cdots + a_1 w_1 + a_0) - w_2^n(a_m w_2^m + a_{m-1} w_2^{m-1} + \cdots + a_1 w_2 + a_0)$;

(ii) If $P(z) = C$ *or* $P(z) = a_m z^m$, *then* $f = tg$ *for a constant* t *that satisfies* $t^{n+m^{**}} = 1$, *or* $f(z) = b_1 e^{bz^2}$, $g(z) = b_2 e^{-bz^2}$ *for three constants* b_1, b_2 *and* b *that satisfy* $4(a_m)^2(b_1 b_2)^{n+m}((n+m)b)^2 = -1$, *or* $4C^2(b_1 b_2)^n(nb)^2 = -1$.

Question: Whether the CM sharing value can be replaced by the IM sharing fixed points in the Theorem D and Theorem E? In the paper, we provide an affirmative solution by proving the following theorems.

Theorem 1. *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions, and let* n, m *and* k *be three positive integers with* $n > 5k + 4m^* + 7$, λ *and* μ *be constants that satisfy* $|\lambda| + |\mu| \neq 0$. *If* $(f^n(z)(\lambda f^m(z) + \mu))^{(k)}$ *and* $(g^n(z)(\lambda g^m(z) + \mu))^{(k)}$ *share* z *IM, then the following conclusions hold:*

(i) If $\lambda\mu \neq 0$, *then* $f^d(z) \equiv g^d(z), d = GCD(n, m)$; *especially, when* $d = 1, f(z) \equiv g(z)$;

(ii) If $\lambda\mu = 0$, *then* $f = cg$ *for a constant* c *that satisfies* $c^{n+m^*} = 1$, *or* $k = 1$ *and* $f(z) = b_1 e^{bz^2}$, $g(z) = b_2 e^{-bz^2}$ *for three constants* b_1, b_2 *and* b *that satisfy* $4(\lambda + \mu)^2(b_1 b_2)^{n+m^*}((n+m^*)b)^2 = -1$.

Theorem 2. *Let* $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ *or* $P(z) = C$, *where* $a_0, a_1, \cdots, a_{m-1}, a_m \neq 0, C \neq 0$ *are complex constant. Suppose that* $f(z)$ *and* $g(z)$ *be two transcendental entire functions, and let* n, m *and* k *be three positive integers with* $n > 5k + 4m^{**} + 7$. *If* $(f^n(z)P(f))^{(k)}$ *and* $(g^n(z)P(g))^{(k)}$ *share* z *IM, then the following conclusions hold:*

(i) If $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ *is not a monomial, thenf* $= tg$ *for a constant* t *that satisfies* $t^d = 1$, *where* $d = (n+m, \ldots, n+m-i, \cdots, n)$, $a_{m-i} \neq 0$, *for some* $i = 0, 1, 2, \ldots, m$; *or* f *and* g *satisfy the algebraic equation* $R(f, g) \equiv 0$, *where* $R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1} w_1^{m-1} + \cdots + a_1 w_1 + a_0) - w_2^n(a_m w_2^m + a_{m-1} w_2^{m-1} + \cdots + a_1 w_2 + a_0)$;

(ii) If $P(z) = C$ *or* $P(z) = a_m z^m$, *then* $f = tg$ *for a constant* t *that satisfies* $t^{n+m^{**}} = 1$, *or* $f(z) = b_1 e^{bz^2}$, $g(z) = b_2 e^{-bz^2}$ *for three constants* b_1, b_2 *and* b *that satisfy* $4(a_m)^2(b_1 b_2)^{n+m}((n+m)b)^2 = -1$, *or* $4C^2(b_1 b_2)^n(nb)^2 = -1$.

2. Some Lemmas

Lemma 1([12]). *Let* f *be a nonconstant meromorphic function, and* a_0, a_1, a_2, $\ldots$

a_n be small functions of f such that $a_n \neq 0$. Then

$$T(r,\, a_n f^n + a_{n-1} f^{n-1} + \cdots a_1 f + a_0) = nT(r,f) + S(r,f).$$

Lemma 2([9]). *Let f be a nonconstant meromorphic function, and p, k be positive integers. Then*

$$N_p(r, \frac{1}{f^{(k)}}) \leq T(r, f^{(k)}) - T(r,f) + N_{p+k}(r, \frac{1}{f}) + S(r,f), \tag{2.1}$$

$$N_p(r, \frac{1}{f^{(k)}}) \leq k\overline{N}(r,f) + N_{p+k}(r, \frac{1}{f}) + S(r,f). \tag{2.2}$$

Lemma 3([11]). *Let*

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{2.3}$$

where F and G are two nonconstant meromorphic functions. If F and G share 1 IM and $H \not\equiv 0$, then

$$\begin{aligned} T(r,F) + T(r,G) \quad \leq \quad & 2(N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + N_2(r,F) \\ & + N_2(r,G)) + 3(\overline{N}_L(r, \frac{1}{F-1} + \overline{N}_L(r, \frac{1}{G-1})) \\ & + S(r,F) + S(r,G). \end{aligned}$$

Lemma 4([12]). *Let $f(z)$ be a nonconstant meromorphic function, $a_1(z)$, $a_2(z)$ and $a_3(z)$ be three distinct small functions of $f(z)$. Then*

$$T(r,f) < \sum_{j=1}^{3} \overline{N}(r, \frac{1}{f - a_j}) + S(r,f).$$

Lemma 5([10]). *Let f and g be two nonconstant entire functions, n, m and k be three positive integers, and let $F = (f^n(z)(\lambda f^m(z) + \mu))^{(k)}$, $G = (g^n(z)(\lambda g^m(z) + \mu))^{(k)}$, where $\lambda\mu \neq 0$. If there exist two non-zero constants a_1 and a_2 such that $\overline{N}(r, \frac{1}{F-a_1}) = \overline{N}(r, \frac{1}{G})$ and $\overline{N}(r, \frac{1}{G-a_2}) = \overline{N}(r, \frac{1}{F})$, then $n \leq 2k + 2 + m$.*

Lemma 6([10]). *Suppose that F and G are given by Lemma 5. If $n > 2k + m$ and $F = G$, then $f^d(z) \equiv g^d(z), d = GCD(n,m)$.*

Lemma 7([10]). *Let f and g be two transcendental entire functions, n, m and k be three positive integers, and let $F = (f^n(z)(\lambda f^m(z) + \mu))^{(k)}$, $G = (g^n(z)(\lambda g^m(z) +$*

$\mu))^{(k)}$, where $\lambda\mu \neq 0$. If $F\ G = z^2$, then $n \leq k+2$.

Lemma 8([10]). *Let f and g be two nonconstant entire functions, n, m and k be three positive integers, and let $F = (f^n(z)(\lambda f^m(z)+\mu))^{(k)}$, $G = (g^n(z)(\lambda g^m(z)+\mu))^{(k)}$, where $|\lambda|+|\mu| \neq 0$, and $\lambda\mu = 0$. If there exist two non-zero constants a_1 and a_2 such that $\overline{N}(r, \frac{1}{F-a_1}) = \overline{N}(r, \frac{1}{G})$ and $\overline{N}(r, \frac{1}{G-a_2}) = \overline{N}(r, \frac{1}{F})$, then $n \leq 2k+2+m^*$.*

Lemma 9([10]). *Suppose that F and G are given by Lemma 8. If $n > 2k+m^*$ and $F = G$, then $f = cg$ for a constant c that satisfies $c^{n+m^*} = 1$.*

Lemma 10([6]). *Suppose that f is a nonconstant meromorphic function, $k \geq 2$ is an integer. If*

$$N(r,f) + N(r,\frac{1}{f}) + N(r,\frac{1}{f^{(k)}}) = S(r,\frac{f'}{f}),$$

then $f = e^{az+b}$, where $a \neq 0$, b are constant.

Lemma 11. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, n, m and k be positive integers, λ is a non-zero constant, and let $F = (\lambda f^{n+m}(z))'$ and $G = (\lambda g^{n+m}(z))'$. If $FG \equiv z^2$, then $f(z) = b_1 e^{bz^2}$, $g(z) = b_2 e^{-bz^2}$ for three constants b_1, b_2 and b that satisfy $4\lambda^2(b_1b_2)^{n+m^*}((n+m^*)b)^2 = -1$.*

Proof. Since

$$\lambda^2(f^{n+m})'(g^{n+m})' = z^2. \tag{2.4}$$

f and g are entire functions and $n > 5k+4m+7$, by using the arguments similar to the proof of Lemma 7 in [10], we get from (2.5) that f and g have no zeros. Let $f = e^{\alpha(z)}$, $g = e^{\beta(z)}$, where $\alpha(z)$, $\beta(z)$ are nonconstant entire functions. Set

$$h(z) = \frac{1}{f(z)g(z)}, \tag{2.5}$$

we know that $h(z) = e^{\gamma(z)}$, where $\gamma(z)$ is an entire function. We say that $\gamma(z)$ is a constant. In fact, if $\gamma(z)$ is a nonconstant entire function, then $h(z)$ is transcendental entire function. By (2.5), we obtain

$$(m+n)^2\lambda^2(f^{n+m-1})f'(g^{n+m-1})g' = z^2. \tag{2.6}$$

From (2.6) and (2.7), we get

$$(\frac{g'}{g} + \frac{1}{2}\frac{h'}{h})^2 = \frac{1}{4}(\frac{h'}{h})^2 - \frac{z^2h^{m+n}}{(m+n)^2\lambda^2}. \tag{2.7}$$

Set $\xi = \frac{g'}{g} + \frac{1}{2}\frac{h'}{h}$, then (2.8) becomes

$$\xi^2 = \frac{1}{4}(\frac{h'}{h})^2 - \frac{z^2h^{m+n}}{(m+n)^2\lambda^2}. \tag{2.8}$$

Suppose $\xi \equiv 0$, by (2.9), we have

$$h^{m+n} = \frac{(m+n)^2\lambda^2}{4z^2}(\frac{h'}{h})^2. \tag{2.9}$$

Since $h(z) = e^{\gamma(z)}$, we have by (2.10) that

$$\begin{aligned}(m+n)T(r,h) &= (m+n)m(r,h) + O(1)\\ &\leq m(r, \frac{1}{4z^2}) + 2m(r, \frac{h'}{h}) + O(1) = S(r,h).\end{aligned}$$

Therefore h is constant, which leads to a contradiction. Thus $\xi \not\equiv 0$. Differentiating (2.9), we get

$$\begin{aligned}(2.10)\quad 2\xi\xi' &= \frac{1}{2}\frac{h'}{h}(\frac{h'}{h})' - \frac{2z}{\lambda^2(m+n)^2}h^{m+n} - \frac{1}{\lambda^2(m+n)}z^2h^{m+n-1}h'\\ &= \frac{1}{2}\frac{h'}{h}(\frac{h'}{h})' - \frac{1}{\lambda^2(m+n)^2}h^{m+n-1}(2zh + (m+n)z^2h').\end{aligned}$$

Combining (2.9) and (2.11), we have

$$\frac{1}{\lambda^2(m+n)^2}h^{m+n}(2z + (m+n)z^2\frac{h'}{h} - 2z^2\frac{\xi'}{\xi}) = \frac{1}{2}\frac{h'}{h}((\frac{h'}{h})' - \frac{h'}{h}\frac{\xi'}{\xi}). \tag{2.11}$$

If $2z + (m+n)z^2\frac{h'}{h} - 2z^2\frac{\xi'}{\xi} \equiv 0$, then, we get from (2.12) that either $\frac{h'}{h} \equiv 0$ or $(\frac{h'}{h})' - \frac{h'}{h}\frac{\xi'}{\xi} \equiv 0$. If $\frac{h'}{h} \equiv 0$, then h is a constant, which leads to a contradiction. If $(\frac{h'}{h})' - \frac{h'}{h}\frac{\xi'}{\xi} \equiv 0$, we get

$$\frac{h'}{h} = \frac{\xi}{d}, \tag{2.12}$$

where $d(\neq 0)$ is a constant. Thus we deduce from (2.9) and (2.13) that

$$\frac{z^2h^{m+n}}{\lambda^2(m+n)^2} = (\frac{1}{4} - d^2)(\frac{h'}{h})^2. \tag{2.13}$$

Hence $(m+n)T(r,h) = S(r,h)$, which is a contradiction too.
Assume that $2z + (m+n)z^2\frac{h'}{h} - 2z^2\frac{\xi'}{\xi} \not\equiv 0$. Since $h = e^{\gamma(z)}$ and $\xi = \frac{g'}{g} + \frac{1}{2}\frac{h'}{h}$, by (2.9) and (2.12), we obtain

$$N(r, \frac{h'}{h}) = S(r,h), \quad N(r,\xi) = S(r,h),$$

and

$$\begin{aligned}(m+n)T(r,h) &= (m+n)m(r,h) \le m(r, \frac{1}{2z+(m+n)z^2\frac{h'}{h}-2z^2\frac{\xi'}{\xi}}) \\ &\quad +m(r, \frac{h'}{h}((\frac{h'}{h})' - \frac{h'}{h}\frac{\xi'}{\xi})) + O(1) \\ &\le m(r, \frac{h'}{h}((\frac{h'}{h})' - \frac{h'}{h}\frac{\xi'}{\xi})) + m(r, 2z+(m+n)z^2\frac{h'}{h}-2z^2\frac{\xi'}{\xi}) \\ &\quad +N(r, 2z+(m+n)z^2\frac{h'}{h}-2z^2\frac{\xi'}{\xi}) \\ &\le N(r, \frac{\xi'}{\xi}) + S(r,h) + S(r,\xi) \\ &\le T(r,\xi) + S(r,h) + S(r,\xi). \qquad (2.14)\end{aligned}$$

Observe that $h = e^{\gamma(z)}$ is a transcendental entire function, we deduce from (2.10) that

$$\begin{aligned}2T(r,\xi) &= T(r,\xi^2) + S(r,\xi) = T(r, \frac{1}{4}(\frac{h'}{h})^2 - \frac{z^2h^{m+n}}{\lambda^2}) + S(r,\xi) \\ &= N(r, \frac{1}{4}(\frac{h'}{h})^2 - \frac{z^2h^{m+n}}{\lambda^2}) + m(r, \frac{1}{4}(\frac{h'}{h})^2 - \frac{z^2h^{m+n}}{\lambda^2}) + S(r,\xi) \\ &\le (m+n)m(r,h) + N(r, (\frac{h'}{h})^2) + S(r,h) + S(r,\xi) \\ &\le (m+n)T(r,h) + S(r,h) + S(r,\xi). \qquad (2.15)\end{aligned}$$

Combining with (2.15), we obtain

$$\frac{(m+n)}{2}T(r,h) = S(r,h),$$

which leads to a contradiction. Therefore, $\gamma(z)$ is a constant, and so $h(z) = e^{\gamma(z)}$ is also a constant. By (2.6), we obtain

$$f(z)g(z) = e^{\alpha(z)}e^{\beta(z)} = C, \qquad (2.16)$$

where $C(\neq 0)$ is a constant. So we have

$$\beta(z) = -\alpha(z) + c_1, \qquad (2.17)$$

for a constant c_1. Substituting $f = e^{\alpha(z)}$, $g = e^{\beta(z)}$ into (2.8), we deduce from (2.17) and (2.18) that

$$f(z) = b_1e^{bz^2},\ g(z) = b_2e^{-bz^2},$$

where b_1, b_2 and b are three constants that satisfy $4\lambda^2(b_1b_2)^{n+m}((m+n)b)^2 = -1$. This completes the proof of Lemma 11. □

3. Proof of Theorems

Proof of Theorem 1. We distinguish two cases.

(i) $\lambda\mu \neq 0$. Set

$$F = \frac{(f^n(z)(\lambda f^m(z)+\mu))^{(k)}}{z}, \quad G = \frac{(g^n(z)(\lambda g^m(z)+\mu))^{(k)}}{z}. \tag{3.1}$$

Then F and G are transcendental meromorphic functions that share 1 IM. Let H be given by (2.3). If $H \not\equiv 0$, from Lemma 3 we know that (2.4) holds. By Lemma 1 and (2.1), we get

$$\begin{aligned} N_2(r,\frac{1}{F}) &\leq N_2(r,\frac{1}{(f^n(z)(\lambda f^m(z)+\mu))^{(k)}}) + S(r,f) \\ &\leq T(r,(f^n(z)(\lambda f^m(z)+\mu))^{(k)}) - (m+n)T(r,f) \\ &\quad + N_{k+2}(r,\frac{1}{f^n(z)(\lambda f^m(z)+\mu)}) + S(r,f) \\ &= T(r,F) - (m+n)T(r,f) \\ &\quad + N_{k+2}(r,\frac{1}{f^n(z)(\lambda f^m(z)+\mu)}) + S(r,f). \end{aligned} \tag{3.2}$$

Similarly,

$$N_2(r,\frac{1}{G}) \leq T(r,G) - (m+n)T(r,g) + N_{k+2}(r,\frac{1}{g^n(z)(\lambda g^m(z)+\mu)}) + S(r,g). \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$N_2(r,\frac{1}{F}) \leq N_{k+2}(r,\frac{1}{f^n(z)(\lambda f^m(z)+\mu)}) + S(r,f), \tag{3.4}$$

and

$$N_2(r,\frac{1}{G}) \leq N_{k+2}(r,\frac{1}{g^n(z)(\lambda g^m(z)+\mu)}) + S(r,g). \tag{3.5}$$

Moreover, from (3.2) and (3.3), we have

$$\begin{aligned} (m+n)(T(r,f)+T(r,g)) &\leq T(r,F) + T(r,G) - N_2(r,\frac{1}{F}) - N_2(r,\frac{1}{G}) \\ &\quad + N_{k+2}(r,\frac{1}{f^n(z)(\lambda f^m(z)+\mu)}) \\ &\quad + N_{k+2}(r,\frac{1}{g^n(z)(\lambda g^m(z)+\mu)}) + S(r,f) + S(r,g). \end{aligned}$$

We know that

$$\overline{N}(r,\frac{1}{F}) \leq \overline{N}(r,\frac{1}{(f^n(z)(\lambda f^m(z)+\mu))^{(k)}}) + S(r,f)$$

Combining with (2.2), we have

$$\begin{aligned}\overline{N}(r,\frac{1}{F}) &\le N_1(r,\frac{1}{(f^n(z)(\lambda f^m(z)+\mu))^{(k)}})+S(r,f)\\ &\le k\overline{N}(r,f^n(z)(\lambda f^m(z)+\mu))+N_{k+1}(r,\frac{1}{f^n(z)(\lambda f^m(z)+\mu)})+S(r,f)\\ &\le k\overline{N}(r,f(z))+N_{k+1}(r,\frac{1}{f^n(z)(\lambda f^m(z)+\mu)})+S(r,f)\\ &\le k\overline{N}(r,f(z))+N_{k+1}(r,\frac{1}{f^n(z)})+N_{k+1}(r,\frac{1}{\lambda f^m(z)+\mu})+S(r,f)\\ &\le k\overline{N}(r,f(z))+(k+1)\overline{N}(r,\frac{1}{f(z)})+N_{k+1}(r,\frac{1}{\lambda f^m(z)+\mu})\\ &\quad +S(r,f) \qquad (3.6)\end{aligned}$$

From the definition of $\overline{N}_L(r,\frac{1}{F-1})$ and (3.6),

$$\begin{aligned}\overline{N}_L(r,\frac{1}{F-1}) &\le N(r,\frac{1}{F-1})-\overline{N}(r,\frac{1}{F-1})\le N(r,\frac{F}{F'})+S(r,f)\\ &\le \overline{N}(r,\frac{F'}{F})+S(r,f)\le \overline{N}(r,\frac{1}{F})+\overline{N}(r,F)+S(r,f)\\ &\le (k+1)(\overline{N}(r,f)+\overline{N}(r,\frac{1}{f}))+N_{k+1}(r,\frac{1}{\lambda f^m(z)+\mu})\\ &\quad +S(r,f) \qquad (3.7)\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{N}_L(r,\frac{1}{G-1}) &\le (k+1)(\overline{N}(r,g)+\overline{N}(r,\frac{1}{g}))\\ &\quad +N_{k+1}(r,\frac{1}{\lambda g^m(z)+\mu})+S(r,g) \qquad (3.8)\end{aligned}$$

Combining (2.4) and (3.4)-(3.8), we get

$$\begin{aligned}&(m+n)(T(r,f)+T(r,g))\\ \le\ &2(N_{k+2}(r,\frac{1}{f^n(z)(\lambda f^m(z)+\mu)})+N_{k+2}(r,\frac{1}{g^n(z)(\lambda g^m(z)+\mu)}))\\ &+3(\overline{N}_L(r,\frac{1}{F-1})+\overline{N}_L(r,\frac{1}{G-1}))+S(r,f)+S(r,g)\\ \le\ &(2k+4)(\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g}))+2N_{k+2}(r,\frac{1}{\lambda f^m(z)+\mu})\\ &+2N_{k+2}(r,\frac{1}{\lambda g^m(z)+\mu})+3(k+1)(\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g}))\\ &+3(N_{k+1}(r,\frac{1}{\lambda f^m(z)+\mu})+N_{k+1}(r,\frac{1}{\lambda g^m(z)+\mu}))\\ &+S(r,f)+S(r,g).\end{aligned}$$

That is

$$(m+n)(T(r,f)+T(r,g)) \leq (5k+5m+7)(T(r,f)+T(r,g))+S(r,f)+S(r,g).$$

Thus,

$$(n-(5k+4m+7))(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

which leads to a contradiction as $n > 5k+4m+7$. Therefore $H \equiv 0$. Integrating twice, from (2.3) we obtain that

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.9}$$

where $A(\neq 0)$ and B are constants. From (3.9), we have

$$F = \frac{(B+1)G+(A-B-1)}{BG+(A-B)}, \quad G = \frac{(B-A)F+(A-B-1)}{BF-(B+1)}. \tag{3.10}$$

We consider the following three cases.

Case 1. $(B \neq 0, -1)$. From (3.10) we have $\overline{N}(r, \frac{1}{F-\frac{B+1}{B}}) = \overline{N}(r, G)$. From the second fundamental theorem, we have

$$T(r,F) \leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{F-\frac{B+1}{B}}) + S(r,F)$$

$$= \overline{N}(r, \frac{1}{F}) + \overline{N}(r, G) + S(r,F) \leq \overline{N}(r, \frac{1}{F}) + S(r,F). \tag{3.11}$$

From the above inequality and (3.2) applied to F, we obtain

$$\begin{aligned} T(r,F) &\leq N_1(r, \frac{1}{F}) + S(r,f) \\ &\leq T(r,F) - (m+n)T(r,f) + N_{k+1}(r, \frac{1}{f^n(z)(\lambda f^m(z)+\mu)}) + S(r,f). \end{aligned}$$

Thus

$$\begin{aligned} (m+n)T(r,f) &\leq (k+1)\overline{N}(r, \frac{1}{f}) + N_{k+1}(r, \frac{1}{\lambda f^m(z)+\mu}) + S(r,f) \\ &\leq (k+m+1)T(r,f) + S(r,f), \end{aligned}$$

which contradicts the assumption $n > 5k+4m+7$.

Case 2. $(B = 0)$. From (3.10), we obtain

$$F = \frac{G+(A-1)}{A}, \quad G = AF-(A-1). \tag{3.12}$$

If $A \neq 1$, we get by (3.12) that $\overline{N}(r, \frac{1}{F-\frac{A-1}{A}}) = \overline{N}(r, \frac{1}{G})$ and $\overline{N}(r, \frac{1}{F}) = \overline{N}(r, \frac{1}{G+(A-1)})$. By Lemma 5, we have $n \leq 2k+2+m$. This contradicts $n > 5k+4m+7$. Thus $A = 1$ and $F = G$. From Lemma 6, we have $f^d(z) \equiv g^d(z)$, where $d = GCD(n, m)$.

Case 3. $(B = -1)$. From (3.10), we get

$$F = \frac{A}{-G+(A+1)}, \quad G = \frac{(A+1)F - A}{F}. \tag{3.13}$$

If $A \neq -1$, we obtain from (3.13) that $\overline{N}(r, \frac{1}{F-\frac{A}{A+1}}) = \overline{N}(r, \frac{1}{G})$, $\overline{N}(r, F) = \overline{N}(r, \frac{1}{G-A-1})$. From the same reasoning mentioned in Case 1 and Case 2, we also get a contradiction. Thus $A = -1$. From (3.13), we have $FG = 1$, That is

$$(f^n(z)(\lambda f^m(z) + \mu))^{(k)}(g^n(z)(\lambda g^m(z) + \mu))^{(k)} = z^2,$$

from Lemma 7, this is impossible.

(ii) $\lambda\mu = 0$. Since $|\lambda| + |\mu| \neq 0$, we distinguish two cases.

Case A. $\mu = 0$, $\lambda \neq 0$. In this case, we have $F = (\lambda f^{n+m}(z))^{(k)}$ and $G = (\lambda g^{n+m}(z))^{(k)}$. Let

$$F_1 = \frac{(\lambda f^{n+m}(z))^{(k)}}{z}, \quad G_1 = \frac{(\lambda g^{n+m}(z))^{(k)}}{z}.$$

Then F_1 and G_1 share 1 IM. By the similar arguments mentioned in the proof of (i), we have $F_1 \equiv G_1$ or $F_1G_1 \equiv 1$. If $F_1 \equiv G_1$, we obtain from Lemma 9 that $f \equiv cg$, where c is a constant that satisfies $c^{n+m} = 1$. Now we assume that $F_1G_1 = 1$.

If $k = 1$, from Lemma 11 we get that$f(z) = b_1 e^{bz^2}$, $g(z) = b_2 e^{-bz^2}$ for three constants b_1, b_2 and b that satisfy $4\lambda^2(b_1b_2)^{n+m^*}((n+m^*)b)^2 = -1$.

If $k \geq 2$, then

$$\lambda^2 (f^{n+m})^{(k)}(g^{n+m})^{(k)} = z^2. \tag{3.14}$$

Since f and g are entire functions and $n > 5k+4m+7$, by using the arguments similar to the proof of Lemma 7 in [10], we know from (2.6) that f and g have no zeros. Set

$$f = e^{\alpha(z)}, \quad g = e^{\beta(z)}, \tag{3.15}$$

where $\alpha(z)$, $\beta(z)$ are nonconstant entire functions. From (3.14), we obtain

$$N(r, \frac{1}{(f^{m+n})^{(k)}}) = N(r, \frac{1}{z^2}) = O(\log r). \tag{3.16}$$

Combining with (3.15) and (3.16), we get

$$N(r, f^{m+n}) + N(r, \frac{1}{f^{m+n}}) + N(r, \frac{1}{(f^{m+n})^{(k)}}) = O(\log r).$$

From (3.15), $T(r, \frac{(f^{m+n})'}{f^{m+n}}) = T(r, (m+n)\alpha')$. If α is transcendental, we get from Lemma 10 that $f = e^{\alpha} = e^{az+b}$ for some constants $a \neq 0$ and b. This is impossible. Therefore α must be a polynomial, and β is also a polynomial. We assume that $deg(\alpha) = p$ and $deg(\beta) = p$. If $p = q = 1$, we get

$$f = e^{Az+B}, \quad g = e^{Cz+D}, \tag{3.17}$$

where A, B, C and D are constants that satisfy $AC \neq 0$. Substituting (3.17) into (3.14), we get

$$\lambda^2 (m+n)^{2k} (AC)^k e^{(m+n)(A+C)z+(m+n)(B+D)} = z^2,$$

which is impossible. Therefore $\max\{p, q\} > 1$. Without loss of generality, we assume that $p > 1$. Then $(f^{m+n})^{(k)} = Pe^{(m+n)\alpha}$, where P is a polynomial of degree $kp - k \geq k \geq 2$. From (3.14), we have $p = k = 2$ and $q = 1$. Assume that

$$f^{m+n} = e^{(m+n)(A_1 z^2 + B_1 z + C_1)}, \quad g^{m+n} = e^{(m+n)(D_1 z + E_1)},$$

where A_1, B_1, C_1, D_1, E_1 are constants such that $A_1 D_1 \neq 0$. Then we obtain

$$\begin{aligned}(3.18)\ (f^{m+n})'' &= (m+n)(4(m+n)A_1^2 z^2 + 4(m+n)A_1 B_1 z + (m+n)B_1^2 \\ &\quad +2A_1)e^{(m+n)(A_1 z^2 + B_1 z + C_1)},\end{aligned}$$

$$(3.19)\ (g^{m+n})'' = (m+n)^2 D_1^2 e^{(m+n)(D_1 z + E_1)}.$$

Substituting (3.18) and (3.19) into (3.14), we get

$$Q(z) e^{(m+n)(A_1 z^2 + (B_1 + D_1)z + C_1 + E_1)} = z^2,$$

where $Q(z)$ is a polynomial of degree 2. Since $A_1 \neq 0$, we get a contradiction.

Case B. $\lambda = 0, \mu \neq 0$. In this case, by the similar arguments mentioned in the *Case A*, f and g must satisfy $f(z) = b_1 e^{bz^2}, g(z) = b_2 e^{-bz^2}$ or $f = cg$, where b_1, b_2, b and c are constants that satisfy $4\mu^2 (b_1 b_2)^n (nb)^2 = -1$ and $c^n = 1$. This completes the proof of Theorem 1. □

Proof of Theorem 2. By using the similar arguments to those in the proof of Theorem 1, and Lemma 6–Lemma 9 in [1], we can prove Theorem 2. □

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