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# $q$-Analogue of Exponential Operators and Difference Equations

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Abstract. The present paper envisages the $q$-analogue of the exponential operators, determined by G. Dattoli and his collaborators for translation and diffusive operators which were utilized to establish analytical solutions of difference and integral equations. The generalization of their technique is expected to cover wide range of such utilization.

## 1. Introduction and Motivation

One of the most appealing results to come out of $q$-analysis is that the $q$-exponential function, defined by $D_q e_q^x = e_q^x$ , where $D_q$ is the $q$-derivative, also satisfies the same defining functional relationship for ordinary exponential functions (up to normalization), given by

$$F(x)F(y) = F(x \oplus y), \tag{1.1}$$

provided that $xy = q^{-1}yx$. (that is, $(x, y)$ belongs to the Manin quantum plane). This result was first found by Schützenberger [17] long before the non-commutative aspects of $q$-analysis were generally recognized and has been rediscovered many times subsequently e. g. in [1, 7]. It can be proved by means of $q$-combinatorics [1, 17], or by an argument based on the definition of the $q$-exponential as an eigenfunction of the $q$-derivative [7].
Besides the above well-known result, there is, in fact, an additional functional relation in the opposite order for the $q$-exponential functions, which is not so well known given by

$$F(y)F(x) = F(x + y + (1 - q^{-1})yx) \tag{1.2}$$

provided that the same condition $xy = q^{-1}yx$ holds. We first became aware of this relationship in the work of L. Faddeev and A. Yu Volkov in their study of lattice Virasoro algebra [6] who obtained a similar result in the case of a different realization of the $q$-exponential, in terms of an infinite product. Their definition

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of the $q$-exponential suffered from the drawback that it did not go over into the ordinary exponential function in the commuting limit $q \to 1$. This paper is devoted to the discussion of methods which provide the solution of the classes of $q$-difference and generalized Heat equations and we shall see that the techniques we propose offer reliable analytical tools and efficient numerical algorithms.
For completeness we quickly review Schützenberger [17] and Cigler's result [1], which will be used in our subsequent proof:

$$e_q^x e_q^y = e_q^{x\oplus y}, \quad \text{if } \ xy = q^{-1}yx, \tag{1.3}$$

where

$$e_q^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \qquad [n]_q \equiv \frac{1-q^n}{1-q}, \qquad [n]_q! \equiv [n]_q[n-1]_q \cdots [1]_q$$

and $q$-addition is defined as

$$(x\oplus y)^n \equiv \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}, \quad n = 0,1,2,\cdots, x \neq y,$$

$q$-substraction is defined as

$$x \ominus y \ = \ x \oplus -y.$$

The Nalli-Ward $q$-Taylor formula [15, p. 345], [19, p. 259],

$$f(x\oplus\lambda) \ = \ \sum_{r=0}^{\infty} \frac{\lambda^r D_q^r}{[r]_q!} f(x). \tag{1.4}$$

In 2000, Dattoli and Levi[2] discussed general methods for the solution of difference equations, which appear in physical and biological problems. Their technique play crucial role in unifying the generalized families of the difference equation.
In the present paper we have determined the $q$-analogue of the exponential operators of the type $e_q^{\lambda p(x) D_q}$ used in [2], where $\mid q \mid < 1$, $p(x)$ is a given function and $D_q = \left[\frac{d}{dx}\right]_q$. The action of the $q$-exponential operator on the generic function $f(x)$ is defined as

$$e_q^{\lambda p(x) D_q} f(x) \tag{1.5}$$

To find the change of variable, put $x \ = \ \varphi(\theta)$, let

$$p(x)\left[\frac{d}{dx}\right]_q \ = \ \left[\frac{d}{d\theta}\right]_q \qquad \text{or} \qquad \left[\frac{d\theta}{dx}\right]_q \ = \ \frac{1}{p(x)} \tag{1.6}$$

i.e.

$$\left[\frac{dx}{d\theta}\right]_q \ = \ p(x) \ \text{ or } \ \left[\frac{d\varphi(\theta)}{d\theta}\right]_q \ = \ p(\varphi(\theta)) \tag{1.7}$$

let us write $q$-antiderivative of eq.(1.6), which is also known as Similarity Factor (SF)

$$\theta \;=\; \int^x \frac{d_q\xi}{p(\xi)} \quad \text{or} \quad \varphi^{-1}(x) \;=\; \int^x \frac{d_q\xi}{p(\xi)} \tag{1.8}$$

now return to the main operator, we have

$$e_q^{\lambda p(x)\left[\frac{d}{dx}\right]_q} f(x) \;=\; e_q^{\lambda\left[\frac{d}{d\theta}\right]_q} f(\varphi(\theta))$$

the $q$-Taylor series expansion enables to write

$$e_q^{\lambda\left[\frac{d}{d\theta}\right]_q} f(\varphi(\theta)) \;=\; f(\varphi(\theta \oplus \lambda))$$

or

$$e_q^{\lambda p(x)\left[\frac{d}{dx}\right]_q} f(x) \;=\; f(\varphi(\varphi^{-1}(x) \oplus \lambda)) \tag{1.9}$$

where eq. (1.8) denotes $q$-antiderivative of the eq. (1.6) and $\varphi(\cdot)$ its inverse.
For $p(x) = 1$, the $\varphi^{-1}(x)$ is given by

$$\varphi^{-1}(x) \;=\; \int^x d_q\xi \;=\; x, \tag{1.10}$$

therefore $\varphi(x) = x$, then the operator reduces to the translation or $q$-shift operator as follows

$$e_q^{\lambda\left[\frac{d}{dx}\right]_q} f(x) = f(\varphi(\lambda \oplus x)) \;=\; f(\lambda \oplus x). \tag{1.11}$$

Next example of application of the operator (1.5), for $p(x) = \frac{x(q-1)}{\log q} = xh$, (say) we obtain

$$\varphi^{-1}(x) \;=\; \int^x \frac{1}{h\xi} d_q\xi \;=\; \ln_q(x), \tag{1.12}$$

therefore, $\varphi(x) = e_q^x$, then the operator (1.5) reduces to the $q$-dilatation operator (see [8; eq. (9.12), p.32])

$$e_q^{\lambda x h\left[\frac{d}{dx}\right]_q} f(x) = f(\varphi(\lambda \oplus \ln_q(x))) = f(e_q^{\lambda \oplus \ln_q(x)}) = f(x e_q^{\lambda}). \tag{1.13}$$

As the ordinary shift operators and their properties play a central role within the context of the theory of difference equations [10]. One can, therefore, suspect that the above $q$-exponential operators and the wealth of their properties can be exploited to develop tools which allow the solution of different forms of $q$-difference equations.

## 2. $q$-Analogue of the Generalized Difference Equations

A simple example to understand how the $q$-exponential operators can help us to solve $q$-difference equations may be illuminating. Let us consider the linear $q$-dilatation difference equation of the type

$$b_1 f(e_q^2 x) + b_2 f(e_q x) + b_3 f(x) = 0, \tag{2.1}$$

which, according to eq. (1.13), eq. (2.1) can be written in the following form

$$\left[b_1\; e_q^{2xh\left[\frac{d}{dx}\right]_q} + b_2 e_q^{xh\left[\frac{d}{dx}\right]_q} + b_3\right] f(x) = 0. \tag{2.2}$$

Suppose $f(x) = R^{\ln_q(x)}$, we have

$$e_q^{\lambda xh\left[\frac{d}{dx}\right]_q} R^{\ln_q(x)},$$

here $p(x) = \frac{x(q-1)}{\log q} = xh$, then $\varphi^{-1}(x) = \ln_q(x)$ and $\varphi(x) = e_q^x$ or

$$\varphi(\lambda \oplus \ln_q(x)) = e_q^{\lambda \oplus \ln_q(x)} = x e_q^{\lambda}$$

therefore,

$$e_q^{\lambda xh\left[\frac{d}{dx}\right]_q} R^{\ln_q x} = R^{\ln_q(x e_q^{\lambda})} = R^{\lambda} R^{\ln_q x} \tag{2.3}$$

we can associate eq. (2.2) to the characteristic equation, we have

$$[b_1 R^2 + b_2 R + b_3] R^{\ln_q(x)} = 0$$

or

$$b_1 R^2 + b_2 R + b_3 = 0, \tag{2.4}$$

whose roots $R_1$, and $R_2$ allow to write $f(x)$ in terms of the following linear combination of independent solutions:

$$f(x) = c_1 R_1^{\ln_q(x)} + c_2 R_2^{\ln_q(x)} = \sum_{\alpha=1}^{2} c_\alpha R_\alpha^{\ln_q(x)}. \tag{2.5}$$

The above example indicates that we can extend well-established methods of solutions of $q$-difference equations to other types of equations reducible to ordinary difference equations, after a proper change of variable implicit in eqs. (1.5), (1.11).

To give a further example of the flexibility of the formalism associated with exponential operators, let us consider the generalized Heat Equation of the following type

$$\begin{cases} \left[\dfrac{\partial}{\partial \lambda}\right]_q Q(x,\lambda) = \left(p(x)\left[\dfrac{\partial}{\partial x}\right]_q\right)^2 Q(x,\lambda), \\[2ex] \qquad Q(x,0) = g(x), \end{cases} \tag{2.6}$$

which can formally be solved by rewriting eq (2.6) as

$$\left[\frac{\partial}{\partial\lambda}\right]_q Q(x,\lambda) - \left(p(x)\left[\frac{\partial}{\partial x}\right]_q\right)^2 Q(x,\lambda) \;=\; 0$$

which can formally be solved by considering this as ordinary linear $q$-differential equation of order one, whose I. F. is determined as

$$e_q^{-\int\left(p(x)\left[\frac{\partial}{\partial x}\right]_q\right)^2 d_q\lambda} \;=\; e_q^{-\lambda\left(p(x)\left[\frac{\partial}{\partial x}\right]_q\right)^2}$$

therefore, we can find its general solution as

$$Q(x,\lambda)e_q^{-\lambda\left(p(x)\left[\frac{\partial}{\partial x}\right]_q\right)^2} \;=\; C,$$

where $C$ in any constant and using the given initial condition, we get

$$Q(x,0) \;=\; g(x) \;=\; C,$$

finally, we obtain the solution of the Heat equation (2.6) as

$$Q(x,\lambda) = e_q^{\lambda\left(p(x)\left[\frac{\partial}{\partial x}\right]_q\right)^2} g(x). \tag{2.7}$$

using $q$-analogue of the identity [20], we have

$$e_q^{b^2} \;=\; \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e_q^{-\xi^2+2b\xi} d_q\xi,$$

replacing $b^2$ with $\lambda\left(p(x)\left[\frac{\partial}{\partial x}\right]_q\right)^2$, we have

$$e_q^{\lambda\left(p(x)\left[\frac{\partial}{\partial x}\right]_q\right)^2} \;=\; \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e_q^{-\xi^2\oplus 2\sqrt{\lambda}\xi} d_q\xi, \tag{2.8}$$

with the use of the eq. (1.5), finally yields the solution of eq (2.6) in the form of an integral transform, which can be viewed as a generalized Gauss transform

$$e_q^{\lambda\left(p(x)\left[\frac{\partial}{\partial x}\right]_q\right)^2} g(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e_q^{-\xi^2} g(\varphi(2\xi\sqrt{\lambda}\oplus\varphi^{-1}(x)))d_q\xi.$$

or, in other words, we have

$$Q(x,\lambda) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e_q^{-\xi^2} g(\varphi(2\xi\sqrt{\lambda}\oplus\varphi^{-1}(x)))d_q\xi. \tag{2.9}$$

It is evident that the formalism associated with generalized exponential operators can be exploited in many flexible ways in finding the general solution of a large

number of problems.
Before discussing the problem in its generality, let us recall the $q$-analogue of the Euler formula (see [8], [11] )

$$\cos_q x + i\sin_q x \; = \; e_q^{ix} \;\text{ and }\; \cos_q x - i\sin_q x \; = \; e_q^{-ix}, \tag{2.10}$$

Now consider the problem of the following type

$$\sum_{\alpha=0}^{N} b_\alpha f(x\cos_q(\alpha) \oplus \sqrt{1-x^2}\sin_q(\alpha)) = 0, \tag{2.11}$$

which belongs to the families of generalized $q$-difference equation. This equation can be obtained by the action of the $q$-exponential operator on the function $f(x)$ and application of eqs. (1.3) and (2.10), we have

$$\sum_{\alpha=0}^{N} b_\alpha f(\sin_q(\sin_q^{-1} x)\cos_q(\alpha) \oplus \cos_q(\cos_q^{-1}\sqrt{1-x^2})\sin_q(\alpha)) = 0,$$
$$\sum_{\alpha=0}^{N} b_\alpha f(\sin_q(\sin_q^{-1} x)\cos_q(\alpha) \oplus \cos_q(\sin_q^{-1} x)\sin_q(\alpha)) = 0,$$
$$\sum_{\alpha=0}^{N} b_\alpha f(\sin_q(\sin_q^{-1} x \oplus \alpha)) = 0,$$
$$\sum_{\alpha=0}^{N} b_\alpha e_q^{\alpha\sqrt{1-x^2}\left[\frac{d}{dx}\right]_q} f(x) = 0.$$

According to the discussion of the previous section, the use of the $q$-exponential operator

$$\sum_{\alpha=0}^{N} b_\alpha \widehat{E_q}^{\alpha} f(x) = 0,$$

where

$$\widehat{E_q} = e_q^{\sqrt{1-x^2}\left[\frac{d}{dx}\right]_q} \tag{2.12}$$

allows to cast (2.11) in the operator form

$$\Psi(\widehat{E_q})f(x) = 0, \quad \text{where} \quad \Psi(\widehat{E_q}) = \sum_{\alpha=0}^{N} b_\alpha \widehat{E}_q^{\alpha}. \tag{2.13}$$

In this case the SF $\varphi^{-1}(x)$ is associated with (2.12) is

$$\varphi^{-1}(x) = \sin_q^{-1} x. \tag{2.14}$$

Independent solutions of (2.11) can be constructed in terms of the function $R^{\sin_q^{-1}(x)}$, which satisfies the identity

$$\widehat{E_q}^{\alpha} R^{\sin_q^{-1}(x)} = R^{\alpha} R^{\sin_q^{-1}(x)}, \tag{2.15}$$

the general solution of (2.11) can finally be written as

$$f(x) = \sum_{\alpha=0}^{N} c_{\alpha} R_{\alpha}^{\sin_q^{-1}(x)}. \tag{2.16}$$

Similarly, consider the following example, we have

$$\sum_{\alpha=0}^{N} b_{\alpha} f(x \cos_q(\alpha) \ominus \sqrt{1-x^2} \sin_q(\alpha)) = 0, \tag{2.11$'$}$$

which also belongs to the families of generalized $q$-difference equation.

$$\widehat{E_q} = e_q^{-\sqrt{1-x^2}\left[\frac{d}{dx}\right]_q} \tag{2.12$'$}$$

allows to cast (2.11′) in the operator form (2.13). In this case the SF associated with (2.12′) is

$$\varphi^{-1}(x) = \cos_q^{-1} x. \tag{2.14$'$}$$

Independent solutions of (2.11′) can be therefore constructed in terms of the function $R^{\cos^{-1}(x)}$, which satisfies the identity

$$\widehat{E_q}^{\alpha} R^{\cos_q^{-1}(x)} = R^{\alpha} R^{\cos_q^{-1}(x)}. \tag{2.15$'$}$$

The general solution of (2.11′) can finally be written as

$$f(x) = \sum_{\alpha=0}^{N} c_{\alpha} R_{\alpha}^{\cos_q^{-1}(x)}. \tag{2.16$'$}$$

Where $R_{\alpha}$ are the roots of the characteristic equation

$$\Psi(R) = 0. \tag{2.17}$$

From the above discussion it is now clear that, whenever one deals with equations of the type

$$\sum_{\alpha=0}^{N} b_{\alpha} f(\varphi(\alpha \oplus \varphi^{-1}(x))) = 0, \tag{2.18}$$

one can associate it with the generalized exponential operator

$$\widehat{E_q}^{\alpha} = e_q^{\alpha p(x)\left[\frac{d}{dx}\right]_q}, \tag{2.19}$$

which allows to cast (2.18) in the operator form (2.13) and we get the relevant solution in the form

$$f(x) = \sum_{\alpha=0}^{N} c_\alpha R_\alpha^{\int^x \frac{d\xi}{p(\xi)}}. \tag{2.20}$$

Moreover, let us consider another interesting example, we have

$$\sum_{\alpha=0}^{N} b_\alpha f\left(\frac{x}{1-\alpha x}\right) = 0, \tag{2.21}$$

by making use of the shift operator $e_q^{\frac{x^2}{q}\left[\frac{d}{dx}\right]_q}$, which allows to cast (2.21) in the operator form (2.13) i.e.

$$\sum_{\alpha=0}^{N} b_\alpha f\left(-\frac{1}{\alpha - \frac{1}{x}}\right) = 0,$$

$$\sum_{\alpha=0}^{N} b_\alpha e_q^{\alpha \frac{x^2}{q}\left[\frac{d}{dx}\right]_q} f(x) = 0, \qquad \text{or} \qquad \sum_{\alpha=0}^{N} b_\alpha \widehat{E_q}^{\alpha} f(x) = 0,$$

where $\widehat{E_q} = e_q^{\frac{x^2}{q}\left[\frac{d}{dx}\right]_q}$, $\Psi(\widehat{E_q})f(x) = 0$ and $\Psi(E_q) = \sum_{\alpha=0}^{N} b_\alpha \widehat{E_q}^{\alpha}$. In this case the SF associated with (2.11) is $\varphi^{-1}(x) = -\frac{1}{x}$, Its solution can thus be written as

$$f(x) = \sum_{\alpha=1}^{N} c_\alpha R_\alpha^{-\frac{1}{x}}. \tag{2.22}$$

The validity of the above solutions is limited to the case in which $R_\alpha$ is not a multiple root of the characteristic equation; this point will be discussed in the concluding section.

In the tunes of Dattoli et al. ([2]; p.656(26)), let us introduce the following $q$-analogue of the operational identities:

$$\begin{cases} \widehat{E_q}^{\pm\alpha} b^{\int^x \frac{d_q\xi}{p(\xi)}} = b^{\pm\alpha} b^{\int^x \frac{d_q\xi}{p(\xi)}}, \\ \\ \widehat{E_q}^{\pm\alpha} (b^{\int^x \frac{d_q\xi}{p(\xi)}} \phi(x)) = b^{\int^x \frac{d_q\xi}{p(\xi)}} (b\widehat{E_q})^{\pm\alpha} \phi(x) \quad . \end{cases} \tag{2.23}$$

valid for exponential operators of the form (2.19).

We note that according to the first of (2.23) the non-homogeneous equation

$$\Psi(\widehat{E_q})f(x) = Cb^{\int^x \frac{d_q\xi}{p(\xi)}} \tag{2.24}$$

where $C$ is a constant and $b$ is not a root of the characteristic equation, admits the particular solution

$$f(x) = \frac{Cb^{\int^x \frac{d_q\xi}{p(\xi)}}}{\Psi(b)}. \tag{2.25}$$

In the slightly more complicated case

$$\Psi(\widehat{E_q})f(x) = Cb^{\int^x \frac{d_q\xi}{p(\xi)}}\phi(x), \tag{2.26}$$

the second of (2.23) yields

$$f(x) = Cb^{\int^x \frac{d_q\xi}{p(\xi)}}\frac{1}{\Psi(b\widehat{E_q})}\phi(x). \tag{2.27}$$

Further comments shall be discussed in the concluding section.

## 3. $q$-Anologue of Generalized Shift Operators and Jackson Derivatives

The linear equations involving discrete power of the $q$-exponential operators have been considered in the previous section. Here the examples are discussed in which the arbitrary exponents are taken. The introductory example is

$$\frac{f(e_q^{\lambda}x) - f(x)}{\lambda} = g(x), \tag{3.1}$$

The use of the dilatation operator allows to cast eq. (2.27) in the form of a the Jackson derivative [9], namely

$$\frac{e_q^{\lambda h\left[\frac{d}{dx}\right]_q} - 1}{\lambda} f(x) = g(x) \tag{3.2}$$

where $h = \frac{q-1}{\ln q}$, $f(x)$ is unknown, $\lambda \in C$, and $g(x)$ is an analytical function. The operator on the left hand side can formally be inverted and by writing the differentiation variable in terms of the inverse of the SF we find

$$f(e_q^{\xi}) = \frac{\lambda}{e_q^{\lambda h\left[\frac{d}{d\xi}\right]_q} - 1} g(e_q^{\xi}). \tag{3.3}$$

The operator on the right hand side of (3.3) can be expanded as

$$\frac{\lambda}{e_q^{\lambda h\left[\frac{d}{d\xi}\right]_q} - 1} = \frac{\lambda}{\lambda h\left[\frac{d}{d\xi}\right]_q + \frac{1}{[2]_q!}\left(\lambda h\left[\frac{d}{d\xi}\right]_q\right)^2 + \frac{1}{[3]_q!}\left(\lambda h\left[\frac{d}{d\xi}\right]_q\right)^3 + \cdots}$$

$$= \frac{1}{h\left[\frac{d}{d\xi}\right]_q\left[1+\frac{1}{[2]_q!}\left(\lambda h\left[\frac{d}{d\xi}\right]_q\right)+\frac{1}{[3]_q!}\left(\lambda h\left[\frac{d}{d\xi}\right]_q\right)^2+\cdots\right]}$$

$$= \frac{1}{h\left[\frac{d}{d\xi}\right]_q}\left[1+\left(\frac{1}{[2]_q!}\lambda h\left[\frac{d}{d\xi}\right]_q+\frac{1}{[3]_q!}\left(\lambda h\left[\frac{d}{d\xi}\right]_q\right)^2+\cdots\right)\right]^{-1}$$

$$= \frac{1}{h\left[\frac{d}{d\xi}\right]_q}\left[1-\left(\frac{1}{[2]_q!}\lambda h\left[\frac{d}{d\xi}\right]_q+\frac{1}{[3]_q!}\left(\lambda h\left[\frac{d}{d\xi}\right]_q\right)^2+\cdots\right)\right.$$

$$\left.+\left(\frac{1}{[2]_q!}\lambda h\left[\frac{d}{d\xi}\right]_q+\frac{1}{[3]_q!}\left(\lambda h\left[\frac{d}{d\xi}\right]_q\right)^2+\cdots\right)^2+\cdots\right]$$

$$= \frac{1}{h\left[\frac{d}{d\xi}\right]_q}\left[1-\frac{1}{[2]_q}\lambda h\left[\frac{d}{d\xi}\right]_q+\left(\frac{1}{([2]_q!)^2}-\frac{1}{[3]_q!}\right)\left(\lambda h\left[\frac{d}{d\xi}\right]_q\right)^2\right.$$

$$\left.-\left(\frac{1}{[4]_q!}-\frac{[2]_q}{([2]_q)^2[3]_q}+\frac{1}{([2]_q!)^3}\right)\left(\lambda h\left[\frac{d}{d\xi}\right]_q\right)^3+\cdots\right]$$

$$= \frac{1}{h\left[\frac{d}{d\xi}\right]_q}\left[1-\frac{1}{[2]_q}\lambda h\left[\frac{d}{d\xi}\right]_q+\frac{q^2}{[2]_q[3]_q!}\left(\lambda h\left[\frac{d}{d\xi}\right]_q\right)^2\right.$$

$$\left.-\frac{q^3(q-1)}{[2]_q[3]_q![4]_q}\left(\lambda h\left[\frac{d}{d\xi}\right]_q\right)^3+\cdots\right]$$

$$= \frac{1}{h\left[\frac{d}{d\xi}\right]_q}\left[B_{0,q}+\frac{B_{1,q}}{[1]_q!}\lambda h\left[\frac{d}{d\xi}\right]_q+\frac{B_{2,q}}{[2]_q!}\left(\lambda h\left[\frac{d}{d\xi}\right]_q\right)^2+\cdots\right]$$

or

$$\frac{\lambda}{e_q^{\lambda h\left[\frac{d}{d\xi}\right]_q}-1} = \frac{1}{h\left[\frac{d}{d\xi}\right]_q}\sum_{n=0}^{\infty}\frac{B_{n,q}}{[n]_q!}(\lambda h)^n\left[\frac{d}{d\xi}\right]_q^n, \tag{3.4}$$

where

$$B_{0,q}=1, B_{1,q}=-\frac{1}{[2]_q}, B_{2,q}=\frac{q^2}{[3]_q!}, B_{3,q}=-\frac{q^3(q-1)}{[2]_q[4]_q}, \cdots \tag{3.5}$$

are Nalli-Ward-Alsalam(NWA) $q$-Bernoulli numbers (see [5]; p. 27 eqs. (146), (148)) and $\left[\frac{d}{d\xi}\right]_q^{-1}$ is the $q$-antiderivative of the $q$-derivative operator with respect to $\xi$.

Since $g(x)$ has a $q$-Taylor expansion $(g(x) = \sum_{m=0}^{\infty} b_m x^m)$, we get from (3.3) and (3.4)

$$\begin{aligned}
&f(e_q^{\xi}) \\
&= \frac{1}{h\left[\frac{d}{d\xi}\right]_q} \sum_{n=0}^{\infty} \frac{B_{n,q}}{[n]_q!}(\lambda h)^n \left[\frac{d}{d\xi}\right]_q^n \left(\sum_{m=0}^{\infty} b_m e_q^{m\xi}\right) \\
&= \frac{1}{h\left[\frac{d}{d\xi}\right]_q} \left[B_{0,q} + \frac{B_{1,q}}{[1]_q!}\lambda h \left[\frac{d}{d\xi}\right]_q + \frac{B_{2,q}}{[2]_q!}\left(\lambda h \left[\frac{d}{d\xi}\right]_q\right)^2 \right. \\
&\qquad \left. + \frac{B_{3,q}}{[3]_q!}\left(\lambda h \left[\frac{d}{d\xi}\right]_q\right)^3 + \cdots\right] \sum_{m=0}^{\infty} b_m e_q^{m\xi} \\
&= \frac{1}{h\left[\frac{d}{d\xi}\right]_q} \left[1 - \frac{1}{[2]_q}\lambda h \left[\frac{d}{d\xi}\right]_q + \frac{q^2}{[2]_q![3]_q!}\left(\lambda h \left[\frac{d}{d\xi}\right]_q\right)^2 \right. \\
&\qquad \left. - \frac{q^3(q-1)}{[2]_q[3]_q![4]_q}\left(\lambda h \left[\frac{d}{d\xi}\right]_q\right)^3 + \cdots\right] \sum_{m=0}^{\infty} b_m e_q^{m\xi} \\
&= \frac{1}{h\left[\frac{d}{d\xi}\right]_q} \left[b_0 + \sum_{m=1}^{\infty} b_m e_q^{m\xi} - \frac{1}{[2]_q}\lambda h \sum_{m=0}^{\infty} b_m [m]_q e_q^{m\xi} \right. \\
&\qquad \left. + \frac{q^2}{[2]_q![3]_q!}(\lambda h)^2 \sum_{m=0}^{\infty} b_m [m]_q^2 e_q^{m\xi} - \frac{q^3(q-1)}{[2]_q[3]_q![4]_q}(\lambda h)^3 \sum_{m=0}^{\infty} b_m [m]_q^3 e_q^{m\xi} + \cdots\right] \\
&= \frac{1}{h}\left[b_0\xi + \sum_{m=1}^{\infty} \frac{b_m}{[m]_q} e_q^{m\xi} - \frac{1}{[2]_q}\lambda h \sum_{m=0}^{\infty} b_m e_q^{m\xi} \right. \\
&\qquad \left. + \frac{q^2}{[2]_q![3]_q!}(\lambda h)^2 \sum_{m=0}^{\infty} b_m [m]_q e_q^{m\xi} - \frac{q^3(q-1)}{[2]_q[3]_q![4]_q}(\lambda h)^3 \sum_{m=0}^{\infty} b_m [m]_q^2 e_q^{m\xi} + \cdots\right] \\
&= \frac{b_0\xi}{h} + b_1 e_q^{\xi}\frac{1}{h}\left[1 + \left(\frac{\lambda h}{[2]_q!} + \frac{(\lambda h)^2}{[3]_q!} + \cdots\right)\right]^{-1} \\
&\qquad + \frac{b_2}{[2]_q} e_q^{2\xi}\frac{1}{h}\left[1 + \left(\frac{[2]_q\lambda h}{[2]_q!} + \frac{([2]_q\lambda h)^2}{[3]_q!} + \cdots\right)\right]^{-1} + \cdots \\
&= \frac{b_0\xi}{h} + \frac{b_1 e_q^{\xi}}{h(1 + \frac{\lambda h}{[2]_q!} + \frac{(\lambda h)^2}{[3]_q!} + \cdots)} + \frac{b_2 e_q^{2\xi}}{[2]_q h[1 + (\frac{[2]_q\lambda h}{[2]_q!} + \frac{([2]_q\lambda h)^2}{[3]_q!} + \cdots)]} + \cdots \\
&= \frac{b_0\xi}{h} + \frac{\lambda b_1 e_q^{\xi}}{\frac{\lambda h}{[1]_q!} + \frac{(\lambda h)^2}{[2]_q!} + \frac{(\lambda h)^3}{[3]_q!} + \cdots} + \frac{\lambda b_2 e^{2\xi}}{\frac{[2]_q\lambda h}{[1]_q!} + \frac{([2]_q\lambda h)^2}{[2]_q!} + \frac{([2]_q\lambda h)^3}{[3]_q!} + \cdots} + \cdots \\
&= \frac{b_0\xi}{h} + \frac{\lambda b_1 e_q^{\xi}}{e_q^{[1]_q h\lambda} - 1} + \frac{\lambda b_2 e_q^{2\xi}}{e_q^{[2]_q h\lambda} - 1} + \frac{\lambda b_3 e_q^{3\xi}}{e_q^{[3]_q \lambda h} - 1} + \cdots
\end{aligned}$$

or

$$f(e_q^{\xi}) = \sum_{m=1}^{\infty} b_m \frac{\lambda e_q^{m\xi}}{e_q^{\lambda h[m]_q} - 1} + \frac{b_0 \xi}{h}, \tag{3.6}$$

going back to the original variable, we get

$$f(x) = \sum_{m=1}^{\infty} b_m \frac{\lambda x^m}{e_q^{\lambda h[m]_q} - 1} + \frac{b_0 \ln_q(x)}{h}. \tag{3.7}$$

The right hand side of eq. (3.6) provides the solution of our problem.

Take another example $g(x) = \sin_q(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{[2m+1]_q!}$, we find

$$\begin{aligned}
&f(e_q^{\xi}) \\
&= \frac{1}{h\left[\frac{d}{d\xi}\right]_q} \sum_{n=0}^{\infty} \frac{B_{n,q}(\lambda h)^n}{[n]_q!} \left[\frac{d}{d\xi}\right]_q^n \sum_{m=0}^{\infty} \frac{(-1)^m e_q^{(2m+1)\xi}}{[2m+1]_q!} \\
&= \frac{1}{h\left[\frac{d}{d\xi}\right]_q} \left[B_{0,q} + \frac{B_{1,q}}{[1]_q!}\lambda h \left[\frac{d}{d\xi}\right]_q + \frac{B_{2,q}}{[2]_q!}\left(\lambda h \left[\frac{d}{d\xi}\right]_q\right)^2 + \cdots\right] \sum_{m=0}^{\infty} \frac{(-1)^m e_q^{(2m+1)\xi}}{[2m+1]_q!} \\
&= \frac{1}{h\left[\frac{d}{d\xi}\right]_q} \left[1 - \frac{1}{[2]_q}\lambda \left[\frac{d}{d\xi}\right]_q + \frac{q^2}{[2]_q[3]_q!}\left(\lambda \left[\frac{d}{d\xi}\right]_q\right)^2 - \cdots\right] \sum_{m=0}^{\infty} \frac{(-1)^m e_q^{(2m+1)\xi}}{[2m+1]_q!} \\
&= \frac{1}{h} \sum_{m=0}^{\infty} \frac{(-1)^m}{[2m+1]_q[2m+1]_q!} e_q^{(2m+1)\xi} - \frac{1}{h[2]_q}\lambda \sum_{m=0}^{\infty} \frac{(-1)^m}{[2m+1]_q!} e_q^{(2m+1)\xi} \\
&\quad + \frac{q^2}{[2]_q[3]_q!}\frac{1}{h}\lambda^2 \sum_{m=0}^{\infty} \frac{(-1)^m [2m+1]_q}{[2m+1]_q!} e_q^{(2m+1)\xi} \\
&\quad - \frac{q^3(q-1)}{[2]_q[4_q][3]_q!}\frac{1}{h}\lambda^3 \sum_{m=0}^{\infty} \frac{(-1)^m [2m+1]_q^2}{[2m+1]_q!} e_q^{(2m+1)\xi} \cdots \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m e_q^{(2m+1)\xi}}{h[2m+1]_q[2m+1]_q!} \left[1 - \frac{\lambda[2m+1]_q}{[2]_q} + \left(\frac{1}{([2]_q!)^2} - \frac{1}{[3]_q!}\right)\lambda^2 [2m+1]_q^2 \right. \\
&\quad \left. - \left(\frac{1}{[4]_q!} - \frac{[2]_q}{([2]_q)^2[3]_q} + \frac{1}{([2]_q!)^3}\right)\lambda^3 [2m+1]_q^3 + \cdots\right] \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m e_q^{(2m+1)\xi}}{h[2m+1]_q[2m+1]_q!} \left[1 + \frac{\lambda[2m+1]_q}{[2]_q!} + \frac{\lambda^2[2m+1]_q^2}{[3]_q!} + \cdots\right]^{-1}
\end{aligned}$$

or

$$f(e_q^{\xi}) = \sum \frac{\lambda}{e_q^{\lambda h[2m+1]_q} - 1} \frac{(-1)^m}{[2m+1]_q!} e_q^{(2m+1)\xi}$$

or

$$f(x) = \sum \frac{\lambda}{e_q^{\lambda h[2m+1]_q} - 1} \frac{(-1)^m}{[2m+1]_q!} x^{2m+1}. \tag{3.8}$$

This is essentially the series defining $g(x)$, provided that $b_m$ is replaced by $\dfrac{b_m \lambda}{e_q^{\lambda h[m]_q} - 1}$. If, e.g., we take $g(x) = \cos_q(x)$, we find

$$f(x) = \sum \frac{\lambda}{e_q^{\lambda h[2m]_q} - 1} \frac{(-1)^m}{[2m]_q!} x^{2m}, \tag{3.8$'$}$$

and for $g(x) = e_q^{x^p}$, we get

$$f(x) = \sum_{m=1}^{\infty} \frac{\lambda}{e_q^{\lambda h[pm]_q} - 1} \frac{x^{pm}}{[m]_q!} + \frac{\ln_q(x)}{h}. \tag{3.9}$$

We, therefore, conclude that the primitive of the Jackson derivative can be constructed according to the above mentioned results.
This method can also be generalized and the concept of Jackson derivative extended to other forms of exponential operators. In this case we consider equation of the type

$$\frac{f(x)\cos_q(\lambda) \oplus \sqrt{1-x^2}\sin_q(\lambda)}{\lambda} \;=\; g(x), \tag{3.10}$$

with the assistance of eq. (2.12), we write the eq. (3.10) as follows

$$\frac{e_q^{\lambda\sqrt{1-x^2}\left[\frac{d}{dx}\right]_q} f(x) - f(x)}{\lambda} \;=\; g(x)$$

or

$$\frac{e_q^{\lambda\sqrt{1-x^2}\left[\frac{d}{dx}\right]_q} - 1}{\lambda} f(x) \;=\; g(x), \tag{3.11}$$

by assuming $g(x)$ is an odd function there taking $x \;=\; \sin_q \xi$, we have

$$\left[\frac{d}{d\xi}\right]_q \;=\; \left[\frac{d}{dx}\right]_q \left[\frac{dx}{d\xi}\right]_q \;=\; \cos_q \xi \left[\frac{d}{dx}\right]_q$$

$$\frac{e_q^{\lambda\left[\frac{d}{d\xi}\right]_q} - 1}{\lambda} f(\sin_q \xi) \;=\; g(\sin_q \xi),$$

or

$$f(\sin_q \xi) \;=\; \frac{\lambda}{e_q^{\lambda\left[\frac{d}{d\xi}\right]_q} - 1} g(\sin_q \xi), \tag{3.12}$$

let us find out the expansion of the first factor of the right hand side of the eq. (3.12) with the help of the eq. (3.4), we have

$$f(\sin_q \xi) = \left[\frac{d}{d\xi}\right]_q^{-1} \sum_{n=0}^{\infty} \frac{B_{n,q}}{[n]_q!}\lambda^n \left[\frac{d}{d\xi}\right]_q^n g(\sin_q \xi), \tag{3.13}$$

since $g(x)$ in an odd and analytic function, then $g(\sin_q \xi)$ can be expanded by $q$-Taylor expansion such as $g(\sin_q \xi) = \sum_{m=0}^{\infty} b_{2m+1}(\sin_q \xi)^{2m+1}$, we have from (3.13)

$$\begin{aligned}
&f(\sin_q \xi)\\
&= \left[\frac{d}{d\xi}\right]_q^{-1} \sum_{n=0}^{\infty} \frac{B_{n,q}}{[n]_q!}\lambda^n \left[\frac{d}{d\xi}\right]_q^n \sum_{m=0}^{\infty} b_{2m+1}(\sin_q \xi)^{2m+1}\\
&= \left[\frac{d}{d\xi}\right]_q^{-1} \sum_{n=0}^{\infty} \frac{B_{n,q}}{[n]_q!}\lambda^n \left[\frac{d}{d\xi}\right]_q^n \sum_{m=0}^{\infty} b_{2m+1}\left[\frac{e_q^{i\xi} - e_q^{-i\xi}}{2i}\right]^{2m+1}\\
&= \left[\frac{d}{d\xi}\right]_q^{-1} \sum_{n=0}^{\infty} \frac{B_{n,q}}{[n]_q!}\lambda^n \left[\frac{d}{d\xi}\right]_q^n \sum_{m=0}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}}(-e_q^{-i\xi})^{2m+1}[1 - e_q^{2i\xi}]^{2m+1}\\
&= \left[\frac{d}{d\xi}\right]_q^{-1} \left[B_{0,q} + \frac{B_{1,q}\lambda}{[1]_q!}\left[\frac{d}{d\xi}\right]_q + \frac{B_{2,q}\lambda^2}{[2]_q!}\left[\frac{d}{d\xi}\right]_q^2 + \frac{B_{3,q}\lambda^3}{[3]_q!}\left[\frac{d}{d\xi}\right]_q^3 \cdots\right]\\
&\cdot \sum_{m=0}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}}(-e_q^{-i\xi})^{2m+1} \sum_{s=0}^{2m+1} \left(\begin{array}{c} 2m+1 \\ s \end{array}\right)_q (-1)^{2m+1-s} e_q^{[2(2m+1-s)]i\xi}
\end{aligned}$$

substituting the values of Bernoulli's numbers from eq. (3.5), we have

$$\begin{aligned}
&f(\sin_q \xi) \qquad (3.14)\\
&= \left[\frac{d}{d\xi}\right]_q^{-1} \left[1 - \frac{\lambda}{[2]_q}\left[\frac{d}{d\xi}\right]_q + \frac{q^2\lambda^2}{[2]_q[3]_q!}\left[\frac{d}{d\xi}\right]_q^2 - \frac{q^3(q-1)\lambda^3}{[2]_q[4]_q[3]_q!}\left[\frac{d}{d\xi}\right]_q^3 \cdots\right]\\
&\cdot \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \sum_{s=0}^{2m+1} \left(\begin{array}{c} 2m+1 \\ s \end{array}\right)_q (-1)^s e_q^{i[2(m-s)+1]\xi}\\
&= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\frac{d}{d\xi}\right]_q^{-1} \left[\sum_{s=0}^{2m+1} \left(\begin{array}{c} 2m+1 \\ s \end{array}\right)_q (-1)^s e_q^{i[2(m-s)+1]\xi}\right.\\
&\quad - \frac{\lambda}{[2]_q}\left[\frac{d}{d\xi}\right]_q \cdot \sum_{s=0}^{2m+1} \left(\begin{array}{c} 2m+1 \\ s \end{array}\right)_q (-1)^s e_q^{i[2(m-s)+1]\xi}\\
&\quad + \frac{q^2\lambda^2}{[2]_q[3]_q!}\left[\frac{d}{d\xi}\right]_q^2 \sum_{s=0}^{2m+1} \left(\begin{array}{c} 2m+1 \\ s \end{array}\right)_q\\
&\quad \left.(-1)^s e_q^{i[2(m-s)+1]\xi} - \cdots\right]
\end{aligned}$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\frac{d}{d\xi}\right]_q^{-1} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s}_q (-1)^s e_q^{i[2(m-s)+1]\xi}\right.$$

$$-\frac{\lambda}{[2]_q} \sum_{s=0}^{2m+1} \binom{2m+1}{s}_q (-1)^s \{i[2(m-s)+1]_q\} e_q^{i[2(m-s)+1]\xi}$$

$$\left. +\frac{q^2\lambda^2}{[2]_q[3]_q!} \sum_{s=0}^{2m+1} \binom{2m+1}{s}_q (-1)^s \{i[2(m-s)+1]_q\}^2 e_q^{i[2(m-s)+1]\xi} - \cdots\right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s}_q \frac{(-1)^s e_q^{i[2(m-s)+1]\xi}}{i[2(m-s)+1]_q}\right.$$

$$-\frac{\lambda}{[2]_q} \sum_{s=0}^{2m+1} \binom{2m+1}{s}_q (-1)^s e_q^{i[2(m-s)+1]\xi}$$

$$\left. +\frac{q^2\lambda^2}{[2]_q[3]_q!} \sum_{s=0}^{2m+1} \binom{2m+1}{s}_q (-1)^s \{i[2(m-s)+1]_q\} e_q^{i[2(m-s)+1]\xi} - \cdots\right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s}_q \frac{(-1)^s e_q^{i[2(m-s)+1]\xi}}{i[2(m-s)+1]_q}\right.$$

$$\left\{1 - \frac{\lambda}{[2]_q} i[2(m-s)+1]_q + \left(\frac{1}{([2]_q)^2} - \frac{1}{[3]_q!}\right)\lambda^2 \{i[2(m-s)+1]_q\}^2\right.$$

$$\left.\left. -\left(\frac{1}{[4]_q!} - \frac{[2]_q}{([2]_q)^2[3]_q} + \frac{1}{([2]_q)^3}\right)\lambda^3 \{i[2(m-s)+1]_q\}^3 + \cdots\right\}\right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s}_q\right.$$

$$\left. \cdot \frac{(-1)^s e_q^{i[2(m-s)+1]\xi}}{\{i[2(m-s)+1]_q\}\left\{1 + \frac{1}{[2]_q!}\{i[2(m-s)+1]_q\lambda\} + \frac{1}{[3]_q!}\{i[2(m-s)+1]_q\lambda\}^2 + \cdots\right\}}\right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s}_q\right.$$

$$\left. \cdot \frac{(-1)^s \lambda (e_q^{i\xi})^{2(m-s)+1}}{\{i[2(m-s)+1]_q\lambda\} + \frac{1}{[2]_q!}\{i[2(m-s)+1]_q\lambda\}^2 + \frac{1}{[3]_q!}\{i[2(m-s)+1]_q\lambda\}^3 + \cdots}\right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s}_q \frac{(-1)^s \lambda (\cos_q \xi + i \sin_q \xi)^{2(m-s)+1}}{e_q^{i[2(m-s)+1]_q\lambda} - 1}\right]$$

or

$$f(\sin_q \xi) = \sum_{m=1}^{\infty} \frac{b_{2m+1}\lambda}{2^{2m+1}}(-i)^{2m+1} \cdot \left[\sum_{s=0}^{2m+1} \left( \begin{array}{c} 2m+1 \\ s \end{array} \right)_q \frac{(-1)^s(\sqrt{1-\sin_q \xi^2}+i\sin_q \xi)^{2(m-s)+1}}{e_q^{i[2(m-s)+1]_q\lambda}-1}\right]$$

finally,

$$f(x) = \sum_{m=1}^{\infty} \frac{b_{2m+1}\lambda}{2^{2m+1}}(-i)^{2m+1} \cdot \left[\sum_{s=0}^{2m+1} \left( \begin{array}{c} 2m+1 \\ s \end{array} \right)_q \frac{(-1)^s(\sqrt{1-x^2}+ix)^{2(m-s)+1}}{e_q^{i[2(m-s)+1]_q\lambda}-1}\right] \tag{3.15}$$

It is interesting to note that, in this case too, the criterion to evaluate the primitive of the Jackson derivative, associated with the operator (2.12), can easily be inferred.

Let us note that the procedure we have discussed can also be extended to the cases involving the generalized Gauss transform. In fact the solution of

$$\frac{e_q^{\lambda\left(x\left[\frac{d}{dx}\right]_q\right)^2}-1}{\lambda} f(x) = g(x) \tag{3.16}$$

or in other words, we have

$$\frac{\lambda}{e_q^{\lambda\left(x\left[\frac{d}{dx}\right]_q\right)^2}-1} g(x) = f(x), \tag{3.17}$$

let us suppose $x = e_q^{\xi}$, then $\left[\frac{d}{d\xi}\right]_q = \left[\frac{d}{dx}\right]_q \left[\frac{dx}{d\xi}\right]_q = e_q^{\xi}\left[\frac{d}{dx}\right]_q = x\left[\frac{d}{dx}\right]_q$, now from eq. (3.17), we have

$$\frac{\lambda}{e_q^{\lambda\left[\frac{d}{dx}\right]_q^2}-1} g(e_q^{\xi}) = f(e_q^{\xi}), \tag{3.18}$$

further, after following the steps as we followed in getting the result (3.4), we obtain the expansion of first factor of l.h.s. as

$$\frac{\lambda}{e_q^{\lambda\left[\frac{d}{dx}\right]_q^2}-1} = \left[\frac{d}{d\xi}\right]_q^{-2} \sum_{n=0}^{\infty} \frac{B_{n,q}}{[n]_q!}\lambda^n \left[\frac{d}{d\xi}\right]_q^{2n}, \tag{3.19}$$

now by the virtue of the analyticity of $g(x)$, we expand $g(x)$, by $q$-Taylor series, i.e.

$$f(e_q^{\xi}) = \left[\frac{d}{d\xi}\right]_q^{-2} \sum_{n=0}^{\infty} \frac{B_{n,q}}{[n]_q!}\lambda^n \left[\frac{d}{d\xi}\right]_q^{2n} \sum_{m=0}^{\infty} b_m e_q^{mx},$$

proceeding of the steps as proceeded in finding the result (3.7), we obtain

$$f(e_q^{\xi}) = \sum_{m=1}^{\infty} b_m \frac{\lambda}{e_q^{\lambda[m]_q^2} - 1} e_q^{m\xi} + b_0 \xi$$

or in other words, if we take $b_0 \;=\; 0$, then

$$f(x) = \sum_{m=1}^{\infty} b_m \frac{\lambda}{e_q^{\lambda[m]_q^2} - 1} x^m. \tag{3.20}$$

Further comments on the results of this section will be discussed in the forthcoming concluding section.

## 4. The Remarks

In the previous section we have considered $q$-linear difference equations, a (trivial) non-linear example, $q$-analogue to Riccati equation, is given blow

$$f(xe_q) - f(x) + b^{\ln(x)} f(xe_q) f(x) = 0, \tag{4.1}$$

which can be solved using the auxiliary function $g(x) = \frac{1}{f(x)}$and thus getting

$$\frac{1}{f(x)} - \frac{1}{f(xe_q)} + b^{\ln_q(x)} \;=\; 0,$$

or

$$g(xe_q) - g(x) = b^{\ln_q(x)}, \tag{4.2}$$

from the operator (1.5), we have

$$e_q^{hx\left[\frac{d}{dx}\right]_q} g(x) - g(x) \;=\; b^{\ln_q(x)}, \tag{4.3}$$

or in other words we write the above eq. (4.3)as

$$\begin{aligned} g(x) &= \frac{b^{\ln_q(x)}}{e_q^{hx\left[\frac{d}{dx}\right]_q} - 1} \\ &= -\left[1 - e_q^{hx\left[\frac{d}{dx}\right]}\right]^{-1} b^{\ln_q(x)} \\ &= -\left[1 + e_q^{hx\left[\frac{d}{dx}\right]_q} + e_q^{2hx\left[\frac{d}{dx}\right]_q} + e_q^{3hx\left[\frac{d}{dx}\right]_q} + \cdots\right] b^{\ln_q(x)} \\ &= -\left[b^{\ln_q(x)} + e_q^{hx\left[\frac{d}{dx}\right]_q} b^{\ln_q(x)} + e_q^{2hx\left[\frac{d}{dx}\right]_q} b^{\ln_q(x)} + e_q^{3hx\left[\frac{d}{dx}\right]_q} b^{\ln_q(x)} + \cdots\right] \end{aligned}$$

$$= -\left[b^{\ln_q(x)} + bb^{\ln_q(x)} + b^2 b^{\ln_q(x)} + b^3 b^{\ln_q(x)} + \cdots\right]$$
$$= -\left[1 + b + b^2 + b^3 + \cdots\right] b^{\ln_q(x)} \;=\; \frac{b^{\ln_q(x)}}{b-1}$$

thus finding as a particular solution

$$f(x) \;=\; \frac{1}{g(x)} \;=\; \frac{b-1}{b^{\ln_q(x)}}. \tag{4.4}$$

Moreover, in general, equations of the type

$$\sum_{\alpha=0}^{N} b_\alpha f(\varphi(\varphi^{-1}(x) \oplus \alpha)) = \epsilon[f(x)]^n, \tag{4.5}$$

standard perturbation methods can be used. At the lowest order in $\epsilon(f \cong f_0 + \epsilon f_1)$ we find

$$\sum_{\alpha=0}^{N} b_\alpha f_0(\varphi(\alpha \oplus \varphi^{-1}(x))) = 0 \tag{4.6}$$

and

$$\sum_{\alpha=0}^{N} b_\alpha f_1(\varphi(\alpha \oplus \varphi^{-1}(x))) = R^{n \int^x \frac{d_q\xi}{p(\xi)}}, \tag{4.7}$$

where $R$ is one of the roots of the characteristic equation associated with (4.5). The first-order contribution $f_1$ can therefore be evaluated by using eq. (2.15), which should be modified as follows:

$$f(x) = \frac{C\left(\int^x \frac{d_q\xi}{p(\xi)}\right) b^{\int^x \frac{d_q\xi}{p(\xi)} - 1}}{\Psi'(b)}, \quad \Psi'(b) = \left[\left[\frac{d}{dR}\right]_q \Psi(R)\right]_{R=b}, \tag{4.8}$$

if $b$ is a simple root of the characteristic equation.
Let us now go back to the problem of treating $q$-exponential operators of the type

$$\widehat{E}^m_{q,\lambda} = e_q^{\lambda\left(p(x)\left[\frac{d}{dx}\right]_q\right)^m}. \tag{4.9}$$

We have seen that for m $= 2$ and $\lambda > 0$ they can be viewed as generalized Gauss transform. Before discussing the problem more deeply, we recall the following important relation [2]:

$$\begin{cases} e_q^{\lambda\left[\frac{d}{dx}\right]_q^m} x^n = H_n^{(m)}(x,\lambda;q), \\ \\ H_n^{(m)}(x,\lambda;q) = [n]_q! \displaystyle\sum_{r=0}^{\left[\frac{n}{m}\right]} \frac{\lambda^r x^{n-mr}}{[r]_q![n-mr]_q!}. \end{cases} \tag{4.10}$$

Which holds for negative or positive $\lambda$ and $H_n^{(m)}(x,\lambda;q)$ are $q$-analogue of Kampé de Feriét polynomials, and satisfy the identity

$$\left[\frac{\partial}{\partial\lambda}\right]_q H_n^{(m)}(x,\lambda;q) = \left[\frac{\partial}{\partial x}\right]_q^m H_n^{(m)}(x,\lambda;q). \tag{4.11}$$

According to eq. (4.8) we also find

$$e_q^{\lambda\left[\frac{d}{dx}\right]_q^m} g(x) = e_q^{\lambda\left[\frac{d}{dx}\right]_q^m} \sum_{n=0}^{\infty} b_n x^n \;=\; \sum_{n=0}^{\infty} b_n H_n^{(m)}(x,\lambda;q). \tag{4.12}$$

It is therefore easy to realize that

$$E_{q,\lambda}^m x = \sum_{n=0}^{\infty} \phi_n H_n^{(m)}(F(x),\lambda;q), \tag{4.13}$$

where we have assumed that the function $F^{-1}(\cdot)$ can be expanded in the power series

$$F^{-1}(\zeta) = \sum_{n=0}^{\infty} \phi_n \zeta^n. \tag{4.14}$$

It is clear that eq. (4.12) can be further handled to extend the action of the operators (4.8) to any function $g(x)$. It is worth considering the possibility of extending the definition of operators (4.8) to the case of not necessarily integer $m$. In the case of $m = \frac{1}{2}$ eq. (4.9) should be replaced by

$$\begin{cases} e_q^{\lambda\left[\frac{d}{dx}\right]_q^{\frac{1}{2}}} x^n = H_n^{(\frac{1}{2})}(x,\lambda;q), \\ H_n^{(\frac{1}{2})}(x,\lambda;q) = [n]_q! \displaystyle\sum_{r=0}^{2n} \frac{\lambda^r x^{n-\frac{r}{2}}}{[r]_q!\Gamma_q(n-\frac{r}{2}+1)}. \end{cases} \tag{4.15}$$

It is evident that in this case $H_n^{(\frac{1}{2})}(x,\lambda;q)$ is a relation analogous to (4.10) holds, namely

$$\left[\frac{\partial^2}{\partial\lambda^2}\right]_q H_n^{(\frac{1}{2})}(x,\lambda;q) \;=\; \left[\frac{\partial}{\partial x}\right]_q H_n^{(\frac{1}{2})}(x,\lambda;q). \tag{4.16}$$

or involving semi derivatives [13]

$$\left[\frac{\partial}{\partial\lambda}\right]_q H_n^{(\frac{1}{2})}(x,\lambda;q) \;=\; \left[\frac{\partial}{\partial x}\right]_q^{\frac{1}{2}} H_n^{(\frac{1}{2})}(x,\lambda;q). \tag{4.16$'$}$$

This definition can be extended to any $m = \frac{1}{p}$, for some integer $p$.

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