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Sequence Space $m(M, \phi)^F$ of Fuzzy Real Numbers Defined by Orlicz Functions with Fuzzy Metric

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Abstract. The sequence space $m(M, \phi)^F$ of fuzzy real numbers is introduced. Some properties of this sequence space like solidness, symmetricity, convergence-free etc. are studied. We obtain some inclusion relations involving this sequence space.

1. Introduction

The concept of *fuzzy set theory* was introduced by L.A. Zadeh in the year 1965. Later on different classes of sequences of fuzzy numbers have been investigated by Esi [2], Nuray and Savas [6], Syau [9], Tripathy and Baruah ([13], [14], [15]), Tripathy and Borgohain [16], Tripathy and Dutta ([17], [18]), Tripathy and Sarma [20] and many others.

An *Orlicz function* is a function $M : [0, \infty) \to [0, \infty)$, which is *continuous, non-decreasing* and *convex* with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

If the *convexity* of the Orlicz function is replaced by $M(x + y) \le M(x) + M(y)$, then this function is called as *modulus function*.

Remark. An Orlicz function satisfies the inequality $M(\lambda x) \le \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Sargent [8] introduced the crisp set sequence space $m(\phi)$ and studied some properties of this space. Later on it was studied from the sequence space point of view and some matrix classes were characterized with one member as $m(\phi)$ by Rath and Tripathy [7], Tripathy [10] and others. In this article we introduce the space $m(M, \phi)^F$ of fuzzy real numbers defined by Orlicz function.

Throughout the article $w^F, \ell^F, \ell^F_\infty$ represent the classes of *all, absolutely*

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summable and *bounded* sequences of fuzzy real numbers respectively.

2. Definitions and Background

Definition 2.1. A *fuzzy real number* X is a fuzzy set on R i.e. a mapping $X : R \to I(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

Definition 2.2. A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

Definition 2.3. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

Definition 2.4. A fuzzy real number X is said to be *upper semi-continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a+\varepsilon))$, for all $a \in I$ is open in the usual topology of R.

The class of all *upper semi-continuous, normal, convex* fuzzy real numbers is denoted by $R(I)$.

Definition 2.5. For $X \in R(I)$, the α-level set X^α, for $0 < \alpha \leq 1$ is defined by, $X^\alpha = \{t \in R : X(t) \geq \alpha\}$. The 0-level set of X i.e. X^0 is the closure of strong 0-cut, i.e. $cl\{t \in R : X(t) > 0\}$.

Definition 2.6. The *absolute value* of $X \in R(I)$ is defined by,

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{for } t \geq 0 \text{ ;} \\ 0 \quad \text{otherwise.} \end{cases}$$

Definition 2.7. For $r \in R$ and $\overline{r} \in R(I)$ is defined as,

$$\overline{r}(t) = \begin{cases} 1 & \text{if } t = r \text{ ;} \\ 0 & \text{if } t \neq r. \end{cases}$$

Definition 2.8. The *additive* and *multiplicative identities* of $R(I)$ are denoted by $\overline{0}$ and $\overline{1}$.

Definition 2.9. Let D be the set of all closed bounded intervals $X = [X^L, X^R]$.

Define $d : D \times D \to R$ by $d(X,Y) = \max\{|X^L - Y^L|, |X^R - Y^R|\}$. Then clearly (D, d) is a *complete metric space*.

Define $\overline{d} : R(I) \times R(I) \to R$ by $\overline{d}(X,Y) = \sup\limits_{0<\alpha\leq 1} d(X^\alpha, Y^\alpha)$, for $X, Y \in R(I)$. Then it is well known that $(R(I), \overline{d})$ is a *complete metric space*.

Definition 2.10. A sequence $X = (X_k)$ of fuzzy real numbers is said to *converge* to the fuzzy number X_0, if for every $\varepsilon > 0$, there exists $k_0 \in N$ such that $\overline{d}(X_k, X_0) < \varepsilon$, for all $k \geq k_0$.

Definition 2.11. A sequence space E is said to be *solid* if $(Y_n) \in E$, whenever $(X_n) \in E$ and $|Y_n| \le |X_n|$, for all $n \in N$.

Definition 2.12. Let $X = (X_n)$ be a sequence, then $S(X)$ denotes the set of all permutations of the elements of (X_n) i.e. $S(X) = \{(X_{\pi(n)}) : \pi$ is a permutation of $N\}$. A sequence space E is said to be *symmetric* if $S(X) \subset E$ for all $X \in E$.

Definition 2.13. A sequence space E is said to be *convergence-free* if $(Y_n) \in E$ whenever $(X_n) \in E$ and $X_n = \bar{0}$ implies $Y_n = \bar{0}$.

Definition 2.14. A sequence space E is said to be *monotone* if E contains the canonical pre-images of all its step spaces.

Lemma 2.1. *A sequence space E is solid implies that E is monotone.*

Definition 2.15. Let $\wp_s$ be the class of all subsets of N those do not contain more than s number of elements. Throughout (ϕ_n) is a non-decreasing sequence of positive real numbers such that $n\phi_{n+1} \le (n+1)\phi_n$ for all $n \in N$.

The space $m(\phi)$ introduced by Sargent [8] is defined by,

$$m(\phi) = \left\{ (x_k) \in w : \|x\|_{m(\phi)} = \sup_{s \ge 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| < \infty \right\}.$$

Afterwards different types of generalizations of the classes of sequences $m(\phi)$ was introduced and investigated by Rath and Tripathy [7], Tripathy ([10], [11]) and many others.

Definition 2.16. Lindenstrauss and Tzafriri [5] used the notion of Orlicz function and introduced the sequence space:

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right), \text{ for some } \rho > 0 \right\}$$

The space ℓ_M with the norm,

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\},$$

becomes a *Banach space*, which is called an *Orlicz sequence space*. The space ℓ_M is closely related to the space ℓ_p, which is an Orlicz sequence space with $M(x) = x^p$, for $1 \le p \le \infty$.

In the later stage different classes of Orlicz sequence spaces were introduced and studied by Altin, Et and Tripathy [1], Esi [2], Tripathy, Altin and Et [12], Tripathy and Mahanta [19], Tripathy and Sarma ([20], [21]) and many others.

Definition 2.17. Let $d_F : R(I) \times R(I) \to R(I)$ be the *fuzzy metric*. Let the mappings $L, M : [0,1] \times [0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments and satisfy, $L[0,0] = 0$ and $M[1,1] = 1$. i.e. $L = \min\{p,q\}$ and $M = \max\{p,q\}$, where $p, q \in [0,1]$.

Let $\lambda : R(I) \times R(I) \to R$ such that $\lambda(X,Y) = \sup\limits_{0<\alpha\leq 1} \lambda_\alpha(X^\alpha, Y^\alpha)$, where $\lambda_\alpha : R \times R \to R$ and $\lambda_\alpha(X^\alpha, Y^\alpha) = \min\{|X_1^\alpha - Y_1^\alpha|, |X_2^\alpha - Y_2^\alpha|\}$.

Similarly, let $\rho : R(I) \times R(I) \to R$ be such that $\rho(X,Y) = \sup\limits_{0<\alpha\leq 1} \rho_\alpha(X^\alpha, Y^\alpha)$, where $\rho_\alpha : R \times R \to R$ and $\rho_\alpha(X^\alpha, Y^\alpha) = \max\{|X_1^\alpha - Y_1^\alpha|, |X_2^\alpha - Y_2^\alpha|\}$.

Since the distance between two fuzzy numbers is again a fuzzy number, so the α- level set of this distance d_F between the fuzzy real numbers X and Y is denoted by,

$$[d(X,Y)]_\alpha = [\lambda_\alpha(X^\alpha, Y^\alpha), \rho_\alpha(X^\alpha, Y^\alpha)], 0 < \alpha \leq 1.$$

The quadruple $(R(I), d_F, M, N)$ is called a *fuzzy metric space* and d_F is a fuzzy metric, if,

1. $d_F(X,Y) = \overline{0}$ if and only if $X = Y$.
2. $d_F(X,Y) = d_F(Y,X)$, for all $X, Y \in R(I)$.
3. For all $X, Y, Z \in R(I)$,
   (*i*) $d_F(X,Y)(s+t) \geq L(d_F(X,Z)(s), d_F(Z,Y)(t))$, whenever $s \leq \lambda_1(X,Z)$, $t \leq \lambda_1(Z,Y)$ and $s + t \leq \lambda_1(X,Y)$.
   (*ii*) $d_F(X,Y)(s+t) \leq R(d_F(X,Z)(s), d_F(Z,Y)(t))$, whenever $s \geq \lambda_1(X,Z)$, $t \geq \lambda_1(Z,Y)$ and $s + t \geq \lambda_1(X,Y)$.

Using the concept of Orlicz function and fuzzy metric, we introduce the following sequence spaces,

$$m(M,\phi)^F = \left\{ (X_k) \in w^F : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{\lambda(X_k, \overline{0})}{r}\right); \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{\rho(X_k, \overline{0})}{r}\right) \right\},$$

for all $r > 0$

3. Main Results

Theorem 3.1. *The sequence space $m(M,\phi)^F$ is a metric space with the metric defined by,*

$$\overline{d}(X,Y)_M = \inf\left\{ r > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{\lambda(X_k, \overline{0})}{r}\right) \leq 1; \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{\rho(X_k, \overline{0})}{r}\right) \leq 1 \right\},$$

for $X, Y \in m(M, \phi)^F$

Proof. Let $X, Y, Z \in m(M, \phi)^F$.
(*i*) $\overline{d}(X, Y)_M = 0$.
This implies,

$$\lambda(X_k, Y_k) = 0 \text{ and } \rho(X_k, Y_k) = 0, \text{ for all } k \in N. (\text{ Since } M(0) = 0).$$

Which implies,

$$\sup_{0<\alpha\leq 1} \lambda_\alpha(X_k^\alpha, Y_k^\alpha) = 0 \Rightarrow \lambda_\alpha(X_k^\alpha, Y_k^\alpha) = 0, \text{ for all } \alpha \in (0, 1].$$

$$\Rightarrow \min\{|X_{k1}^\alpha - Y_{k1}^\alpha|, |X_{k2}^\alpha - Y_{k2}^\alpha|\} = 0, \text{ for all } \alpha \in (0, 1]. \tag{3.1}$$

$$\sup_{0<\alpha\leq 1} \rho_\alpha(X_k^\alpha, Y_k^\alpha) = 0 \Rightarrow \rho_\alpha(X_k^\alpha, Y_k^\alpha) = 0, \text{ for all } \alpha \in (0, 1].$$

$$\Rightarrow \max\{|X_{k1}^\alpha - Y_{k1}^\alpha|, |X_{k2}^\alpha - Y_{k2}^\alpha|\} = 0, \text{ for all } \alpha \in (0, 1]. \tag{3.2}$$

From (3.1) and (3.2), it follows that, $X_k = Y_k \Rightarrow X = Y$.

Conversely, assume that, $X = Y$. Then, using the definition of λ and ρ, we get,

$$\lambda_\alpha(X_k^\alpha, Y_k^\alpha) = 0 \text{ and } \rho_\alpha(X_k^\alpha, Y_k^\alpha) = 0, \text{ for all } k \in N, \alpha \in (0, 1].$$

Which implies,

$$\sup_{0<\alpha\leq 1} \lambda_\alpha(X_k^\alpha, Y_k^\alpha) = 0 \text{ and } \sup_{0<\alpha\leq 1} \rho_\alpha(X_k^\alpha, Y_k^\alpha) = 0, \text{for all } k \in N.$$

It follows that, $\lambda(X_k, Y_k) = 0$ and $\rho(X_k, Y_k) = 0$.

Using the continuity of M, we get, $\overline{d}(X, Y)_M = 0$. Which shows that, $\overline{d}(X, Y)_M = 0$ if and only if $X = Y$.

(*ii*) $\overline{d}(X, Y)_M$

$$= \inf\left\{r > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{\lambda(X_k, Y_k)}{r}\right) \leq 1; \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{\rho(X_k, Y_k)}{r}\right) \leq 1\right\}.$$

From the definition of λ, it follows,

$$\begin{aligned}
\lambda(X_k, Y_k) &= \sup_{0<\alpha\leq 1} \lambda_\alpha(X_k^\alpha, Y_k^\alpha) \\
&= \sup_{0<\alpha\leq 1} [\min\{|X_{k1}^\alpha, Y_{k1}^\alpha|, |X_{k2}^\alpha, Y_{k2}^\alpha|\}] \\
&= \sup_{0<\alpha\leq 1} [\min\{|Y_{k1}^\alpha, X_{k1}^\alpha|, |Y_{k2}^\alpha, X_{k2}^\alpha|\}] \\
&= \sup_{0<\alpha\leq 1} \lambda_\alpha(Y_k^\alpha, X_k^\alpha) \\
&= \lambda(Y_k, X_k).
\end{aligned}$$

Proceeding in the same way, we get, $\rho(X_k, Y_k) = \rho(Y_k, X_k)$. Thus we get,

$$\inf\left\{r > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k, Y_k)}{r}\right) \leq 1; \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\rho(X_k, Y_k)}{r}\right) \leq 1\right\}$$

$$= \inf\left\{r > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(Y_k, X_k)}{r}\right) \leq 1; \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\rho(Y_k, X_k)}{r}\right) \leq 1\right\}$$

$= \overline{d}(Y, X)_M$. Hence, $\overline{d}(X, Y)_M = \overline{d}(Y, X)_M$.

(iii) Let $r_1, r_2 > 0$ such that,

$$\sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k, Z_k)}{r_1}\right) \leq 1.$$

$$\sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(Z_k, Y_k)}{r_2}\right) \leq 1.$$

Let $r = r_1 + r_2$, then we have,

$$\begin{aligned}
&\sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k, Y_k)}{r}\right)\\
&\quad\leq \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k, Z_k)}{r_1 + r_2} + \frac{\lambda(Z_k, Y_k)}{r_1 + r_2}\right)\\
&\quad\leq \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{r_1}{r_1 + r_2}\left(\frac{\lambda(X_k, Z_k)}{r_1}\right) + \frac{r_2}{r_1 + r_2}\left(\frac{\lambda(Z_k, Y_k)}{r_2}\right)\right)\\
&\quad\leq \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\frac{r_1}{r_1 + r_2}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k, Z_k)}{r_1}\right)\\
&\qquad\qquad + \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\frac{r_2}{r_1 + r_2}\sum_{k\in\sigma} M\left(\frac{\lambda(Z_k, Y_k)}{r_2}\right)\\
&\quad\leq 1.
\end{aligned}$$

Since r's are non-negative, so taking the infimum of such r's, we get,

$$\begin{aligned}
&\inf\left\{r > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k, Y_k)}{r}\right) \leq 1\right\}\\
&\leq \inf\left\{r_1 > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k, Z_k)}{r_1}\right) \leq 1\right\}\\
&\qquad + \inf\left\{r_2 > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(Z_k, Y_k)}{r_2}\right) \leq 1\right\}
\end{aligned}$$

Proceeding in the same way, we get,

$$\inf\left\{r > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\rho(X_k, Y_k)}{r}\right) \leq 1\right\}$$

$$\leq \inf\left\{r_1 > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\rho(X_k, Z_k)}{r_1}\right) \leq 1\right\}$$
$$+ \inf\left\{r_2 > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\rho(Z_k, Y_k)}{r_2}\right) \leq 1\right\}$$

Thus we have,

$$\inf\left\{r > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k, Y_k)}{r}\right) \leq 1; \right.$$
$$\left. \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\rho(X_k, Y_k)}{r}\right) \leq 1\right\}$$
$$\leq \inf\left\{r_1 > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k, Z_k)}{r_1}\right) \leq 1; \right.$$
$$\left. \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\rho(X_k, Z_k)}{r_1}\right) \leq 1\right\}$$
$$+\inf\left\{r_2 > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(Z_k, Y_k)}{r_2}\right) \leq 1; \right.$$
$$\left. \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\rho(Z_k, Y_k)}{r_2}\right) \leq 1\right\}$$
$$\Rightarrow \overline{d}(X, Y)_M \leq \overline{d}(X, Z)_M + \overline{d}(Z, Y)_M.$$

This proves that $m(M, \phi)^F$ is a metric space. □

Theorem 3.2. *The sequence space $m(M, \phi)^F$ is a complete metric space with the metric defined by,*

$$\overline{d}(X, Y)_M = inf\left\{r > 0 : \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k, Y_k)}{r}\right) \leq 1; \right.$$
$$\left. \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\rho(X_k, Y_k)}{r}\right) \leq 1\right\},$$

for $X, Y \in m(M, \phi)^F$

Proof. Let $(X^{(i)})$ be a Cauchy sequence in $m(M, \phi)^F$ such that, $X^{(i)} = (X_n^{(i)})_{n=1}^{\infty}$. Let $\varepsilon > 0$ be given. For a fixed $x_0 > 0$, choose $p > 0$ such that $M\left(\frac{px_0}{2}\right) \geq 1$. Then there exists a positive integer $n_0 = n_0(\varepsilon)$ such that,

$$\overline{d}(X^{(i)}, X^{(j)})_M < \frac{\varepsilon}{px_0}, \text{ for all } i, j \geq n_0.$$

By the definition of $\overline{d}_M$, we get;

$$\inf\left\{r>0: \sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\lambda(X_k^{(i)},X_k^{(j)})}{r}\right)\leq 1;\right. \quad (3.3)$$

$$\left.\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\rho(X_k^{(i)},X_k^{(j)})}{r}\right)\leq 1\right\}<\varepsilon$$

for all $i,j\geq n_0$. Which implies,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\lambda(X_k^{(i)},X_k^{(j)})}{r}\right)\leq 1 \quad (3.4)$$

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\rho(X_k^{(i)},X_k^{(j)})}{r}\right)\leq 1 \quad (3.5)$$

From (3.4) we get,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\lambda(X_k^{(i)},X_k^{(j)})}{r}\right)\leq 1$$

On taking $s=1$ and varying σ over $\wp_s$, we get,

$$\sum_{k\in\sigma}M\left(\frac{\lambda(X_k^{(i)},X_k^{(j)})}{r}\right)\leq \phi_1, \text{ for all } i,j\geq n_0.$$

$$\Rightarrow M\left(\frac{\lambda(X_k^{(i)},X_k^{(j)})}{\overline{d}(X^{(i)},X^{(j)})}\right)\leq \phi_1\leq M\left(\frac{px_0}{2}\right).$$

Using the continuity of M, we get,

$$\lambda_\alpha(X_k^{(i)},X_k^{(j)})\leq \frac{px_0}{2}.\frac{\varepsilon}{px_0}=\frac{\varepsilon}{2},$$

i.e $(X_k^{(i)})$ is a Cauchy sequence of $R(I)$. Since $R(I)$ is complete, so it follows that, $(X_k^{(i)})$ is also convergent.

Let, $\lim_i X_k^{(i)}=X_k,$ for each $k\in N$. We have to prove that,

$$\lim_i X^{(i)}=X \text{ and } X\in m(M,\phi)^F.$$

Since M is continuous, so on taking $j\to\infty$ and fixing i, we get from (3.4);

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\lambda(X_k^{(i)},X_k)}{r}\right)\leq 1, \text{ for some } r>0 \text{ and } i\geq n_0.$$

Proceeding in the same way, we get from (3.5):

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\rho(X_k^{(i)},X_k)}{r}\right)\leq 1,\ \text{for some } r>0 \text{ and } i\geq n_0.$$

Now on taking the infimum of such r's together, we get from (3.3):

$$\inf\left\{r>0:\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\lambda(X_k^{(i)},X_k)}{r}\right);\right.$$
$$\left.\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\rho(X_k^{(i)},X_k)}{r}\right)\leq 1\right\}<\varepsilon,$$

for some $r>0$ and $i\geq n_0$. Which shows, $\overline{d}(X^{(i)},X)_M<\varepsilon$, for all $i\geq n_0$. i.e. $\lim_i X^{(i)}=X$.

Now, to show that $X\in m(M,\phi)^F$. We have,

$$\begin{aligned}\overline{d}(X,\overline{\theta})_M &\leq \overline{d}(X,X^{(i)})_M+\overline{d}(X^{(i)},\overline{\theta})_M\\ &< \varepsilon+M,\ \text{for all } i\geq n_0(\varepsilon).\end{aligned}$$

i.e. $\overline{d}(X,\overline{\theta})_M$ is finite. Which implies $x\in m(M,\phi)^F$. Hence $m(M,\phi)^F$ is a complete metric space. This completes the proof of the theorem. Proofs are similar for other spaces also. □

Theorem 3.3. *The sequence space $m(M,\phi)^F$ is solid.*

Proof. Let $(X_k)\in m(M,\phi)^F$. Then we have, for some $r>0$,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\lambda_\alpha(X_k,\overline{0})}{r}\right)<\infty;\ \sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\rho_\alpha(X_k,\overline{0})}{r}\right)<\infty.$$

Let (Y_k) be a sequence of fuzzy numbers with,

$$[d(Y_k,\overline{0})]_\alpha=[\lambda_\alpha(Y_k^\alpha,0),\rho_\alpha(Y_k^\alpha,o)],\ \text{for } 0<\alpha\leq 1,$$

Such that, $\lambda(Y_k,\overline{0})\leq\lambda(X_k,\overline{0})$ and $\rho(Y_k,\overline{0})\leq\rho(X_k,\overline{0})$.

Since M is non-decreasing continuous function, so we get, for some $r>0$,

$$M\left(\frac{\lambda(Y_k,\overline{0})}{r}\right)\leq M\left(\frac{\lambda(X_k,\overline{0})}{r}\right)\ \text{and}\ M\left(\frac{\rho(Y_k,\overline{0})}{r}\right)\leq M\left(\frac{\rho(X_k,\overline{0})}{r}\right).$$

Which implies,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}M\left(\frac{\lambda(Y_k,\overline{0})}{r}\right)\leq\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}M\left(\frac{\lambda(X_k,\overline{0})}{r}\right)<\infty,\quad\text{for some } r>0.$$

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}M\left(\frac{\rho(Y_k,\overline{0})}{r}\right)\leq \sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}M\left(\frac{\rho(X_k,\overline{0})}{r}\right)<\infty,\quad \text{for some } r>0.$$

Which implies,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}M\left(\frac{\lambda(Y_k,\overline{0})}{r}\right)<\infty,\quad \text{for some } r>0.$$

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}M\left(\frac{\rho(Y_k,\overline{0})}{r}\right)<\infty,\quad \text{for some } r>0.$$

Which shows, $(Y_k)\in m(M,\phi)^F$. Hence, $m(M,\phi)^F$ is solid. This completes the proof. □

Theorem 3.4. *The sequence space $m(M,\phi)^F$ is symmetric.*

Proof. Let $(X_k)\in m(M,\phi)^F$ and (Y_k) be a rearrangement of (X_k), such that,

$$X_k=Y_{m_k},\ \text{for each } k\in N.$$

Then, we have, $\lambda(X_k,\overline{0})=\lambda(Y_{m_k},\overline{0})$ and $\rho(X_k,\overline{0})=\rho(Y_{m_k},\overline{0})$.

Using the continuity of M, we get,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}M\left(\frac{\lambda(X_k,\overline{0})}{r}\right)=\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}M\left(\frac{\lambda(Y_{m_k},\overline{0})}{r}\right),\quad \text{for some } r>0.$$

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}M\left(\frac{\rho(X_k,\overline{0})}{r}\right)=\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}M\left(\frac{\rho(Y_{m_k},\overline{0})}{r}\right),\quad \text{for some } r>0.$$

Which implies,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}M\left(\frac{\lambda(Y_{m_k},\overline{0})}{r}\right)<\infty \text{ and } \sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}M\left(\frac{\rho(Y_{m_k},\overline{0})}{r}\right)<\infty,$$

for some $r>0$. Which shows, $(Y_k)\in m(M,\phi)^F$. Hence $m(M,\phi)^F$ is symmetric. This completes the proof. □

Proposition 3.1. *The sequence space $m(M,\phi)^F$ is not convergence-free.*

Proof. The result follows from the following example.

Example 3.1. Consider the sequence (X_k) defined as follows:

$$X_k(t)=\begin{cases}1+kt, & \text{for } t\in[-\frac{1}{k},0]\\ 1-kt, & \text{for } t\in[0,\frac{1}{k}]\\ 0 & \text{otherwise}\end{cases}$$

Then we have, for some $r>0$,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\lambda(X_k,\overline{0})}{r}\right)<\infty$$

and

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\rho(X_k,\overline{0})}{r}\right)<\infty$$

Which shows, $(X_k)\in m(M,\phi)^F$.

Now, let us take another sequence (Y_k) such that,

$$Y_k(t)=\begin{cases}1+\frac{t}{k^2}, & \text{for } t\in[-k^2,0]\\ 1-\frac{t}{k^2}, & \text{for } t\in[0,k^2]\end{cases}$$

But,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\lambda(Y_k,\overline{0})}{r}\right)=\infty \text{ and } \sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\rho(Y_k,\overline{0})}{r}\right)=\infty$$

Thus, $(Y_k)\notin m(M,\phi)^F$. Thus $m(M,\phi)^F$ is not convergence-free. This completes the proof. □

Proposition 3.2. *$m(M,\phi)^F\subseteq m(M,\phi,p)^F$, for $1\leq p<\infty$.*

Proof. Let $X\in m(M,\phi)^F$, then we have, for some $r>0$,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\lambda(X_k,\overline{0})}{r}\right)=K(<\infty)$$

Hence, for each fixed s, we have,

$$\sum_{k\in\sigma}M\left(\frac{\lambda(X_k,\overline{0})}{r}\right)\leq K\phi_s,\ \text{for } \sigma\in\wp_s.$$

$$\Rightarrow\left[\sum_{k\in\sigma}\left(M\left(\frac{\lambda(X_k,\overline{0})}{r}\right)\right)^p\right]^{\frac{1}{p}}\leq K\phi_s,\ \text{for } \sigma\in\wp_s.$$

$$\Rightarrow\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\left[\sum_{k\in\sigma}\left(M\left(\frac{\lambda(X_k,\overline{0})}{r}\right)\right)^p\right]^{\frac{1}{p}}\leq K.$$

$$\Rightarrow\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\left[\sum_{k\in\sigma}\left(M\left(\frac{\lambda(X_k,\overline{0})}{r}\right)\right)^p\right]^{\frac{1}{p}}<\infty.$$

Proceeding in the same way, we get,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\left[\sum_{k\in\sigma}\left(M\left(\frac{\rho(X_k,\overline{0})}{r}\right)\right)^p\right]^{\frac{1}{p}}<\infty.$$

Which implies $X\in m(M,\phi,p)^F$, for $1\leq p<\infty$.

This completes the proof. □

Proposition 3.3. *$m(M,\phi)^F \subseteq m(M,\psi)^F$ if and only if $\sup\limits_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$, for $0 < p < \infty$.*

Proof. Suppose, $\sup\limits_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) = K(<\infty)$, then we have, $\phi_s \leq K\psi_s$.

Now, if $(X_k) \in m(M,\phi)^F$, then,

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k,\overline{0})}{r}\right) < \infty.$$

$$\Rightarrow \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{K\psi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k,\overline{0})}{r}\right) \leq \infty.$$

$$\Rightarrow (X_k) \in m(M,\psi)^F.$$

Hence, $m(M,\phi)^F \subseteq m(M,\psi)^F$.

Conversely, suppose that $m(M,\phi)^F \subseteq m(M,\psi)^F$. To show that, $\sup\limits_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) = \sup\limits_{s\geq 1}(\eta_s) < \infty$.

Suppose, $\sup\limits_{s\geq 1}(\eta_s) = \infty$. Then there exists a subsequence (η_{s_i}) of (η_s) such that,

$$\lim_{i\to\infty}(\eta_{s_i}) = \infty.$$

Then for $(X_k) \in m(M,\phi)^F$, we have,

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\psi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k,\overline{0})}{r}\right) \geq \sup_{s_i\geq 1,\sigma\in\wp_{s_i}} \frac{\eta_{s_i}}{\phi_{s_i}}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k,\overline{0})}{r}\right) = \infty.$$

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\psi_s}\sum_{k\in\sigma} M\left(\frac{\lambda(X_k,\overline{0})}{r}\right) = \infty.$$

Proceeding in the same way, we get,

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\psi_s}\sum_{k\in\sigma} M\left(\frac{\rho(X_k,\overline{0})}{r}\right) = \infty.$$

Which implies that $(X_k) \notin m(M,\psi)^F$, a contradiction. This completes the proof. □

Corollary 3.1. *$m(M,\phi)^F = m(M,\psi)^F$ if and only if $\sup\limits_{s\geq 1}(\eta_s) < \infty$ and $\sup\limits_{s\geq 1}(\eta_s^{-1}) < \infty$, where $\eta_s = \frac{\phi_s}{\psi_s}$.*

Theorem 3.5. *$\ell_p(M)^F \subseteq m(M,\phi,p)^F \subseteq \ell_\infty(M)^F$, for $1 \le p < \infty$.*

Proof. On taking $M(x) = x^p$, for $1 \le p < \infty$ and $\phi_n = 1$, for all $n \in N$. We get, $m(M,\phi,p)^F = \ell_p(M)^F$. So, the first inclusion is clear.

Next, suppose that, $(X_k) \in m(M,\phi,p)^F$ that implies that,

$$\sup_{s\ge 1, \sigma \in \wp_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left(M\left(\frac{\lambda(X_k,\overline{0})}{r}\right)\right)^p\right)^{\frac{1}{p}} = K(<\infty).$$

For, $s = 1, M\left(\frac{\lambda(X_k,\overline{0})}{r}\right) \le K\phi_1, k \in \sigma$. Which implies that, $\sup_{k\ge1} M\left(\frac{\lambda(X_k,\overline{0})}{r}\right) < \infty$.

Following the same way, we get,

$$\sup_{k\ge1} M\left(\frac{\lambda(X_k,\overline{0})}{r}\right) < \infty.$$

Which implies,$(X_k) \in \ell_\infty(M)^F$. This completes the proof. □

Putting $\psi_n = 1$, for all $n \in N$, in Corollary 3.1, we get,

Proposition 3.4. *$m(M,\phi,p)^F = \ell_p(M)^F$ if and only if $\sup_{s\ge1}(\phi_s) < \infty$ and $\sup_{s\ge1}(\phi_s^{-1}) < \infty$.*

Putting $\psi_n = n$, for all $n \in N$, in Corollary 3.1, we get,

Corollary 3.2. *$m(M,\phi,p)^F = \ell_\infty(M)^F$ if $\lim_{s\to\infty}\left(\frac{\phi_s}{s}\right) > 0$, for $0 < p < \infty$.*

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