J Electr Eng Technol Vol. 8, No. 6: 1579-1589, 2013
http://dx.doi.org/10.5370/JEET.2013.8.6.1579

ISSN(Print) 1975-0102
ISSN(Online) 2093-7423

# Four Representative Applications of the Energy Shaping Method for Controlled Lagrangian Systems

**Wai Man Ng*, Dong Eui Chang** and Seong-Ho Song†**

**Abstract** – We provide a step-by-step, easy-to-follow procedure for the method of controlled Lagrangian systems. We apply this procedure to solve the energy shaping problem for four benchmark examples: the inertial wheel pendulum, an inverted pendulum on a cart, the system of ball and beam and the Furuta pendulum.

**Keywords**: Controlled lagrangian system, Energy shaping, Step-by-Step design procedure, Benchmark examples

## 1. Introduction

The energy shaping method is a way to stabilize a mechanical system by altering its energy function by feedback so that the equilibrium point of interest becomes a non-degenerate minimum of the altered energy function. It has the advantage that it provides a constructive procedure for generating stabilizing control laws and yields large regions of convergence. This method is sometimes called the method of controlled Lagrangians in the Lagrangian approach and it has been actively developed [1-9]. As a result, the criteria and the matching conditions for energy shaping for nonlinear mechanical systems with one degree of underactuation and linear mechanical systems with an arbitrary degree of underactuation are now well understood [7, 8]. However, fully worked-out examples using these results are lacking. In this paper, we illustrate how to apply the method of controlled Lagrangians with four benchmark examples: the inertial wheel pendulum, an inverted pendulum on a cart, the system of ball and beam and the Furuta pendulum, through a step-by-step, easy-to-follow procedure.

## 2. Preliminaries

We first review the basic scenario for the energy shaping problem. Given a mechanical system, the configuration space is denoted as $Q$ with $q$ and $\dot{q}$, the position and velocity vectors respectively. We focus on controlled Lagrangian systems, i.e. mechanical system whose law of motion is governed by the Lagrangian of the form:

$$L(q,\dot{q}) = \frac{1}{2} m(\dot{q},\dot{q}) - V(q),$$

where $m = m_q$ is the positive-definite symmetric mass matrix while $\frac{1}{2}m(\dot{q},\dot{q})$ and $V(q)$ are the kinetic and potential energy of the system, respectively. A controlled Lagrangian system can be described by a triple $(L,F,W)$, where $L$ is the Lagrangian, $F$ is the external force, and $W$ is the control bundle along which the control force acts on the system.

In what follows, we call $n = \dim Q$ the degree of freedom, $n_2 = \dim W$ the degree of actuation and $n_1 = n - n_2$ the degree of underactuation. We will use Greek alphabetical indices and Roman alphabetical indices over different ranges:

$$\begin{aligned} &\alpha, \beta, \gamma, \ldots = 1, \ldots, n_1; \\ &a, b, c, \ldots = n_1 + 1, \ldots, n; \\ &i, j, k, \ldots = 1, \ldots, n, \end{aligned}$$

unless stated otherwise. By adopting the Einstein summation convention, the equations of motion in local coordinates are given by

$$m_{\alpha j}\ddot{q}^j + [jk,\alpha]\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^\alpha} = F_\alpha$$
$$m_{aj}\ddot{q}^j + [jk,a]\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^a} = F_a + u_a$$

Where

$$[ij,l] = \frac{1}{2}\left(\frac{\partial m_{lj}}{\partial q^j} + \frac{\partial m_{jl}}{\partial q^i} - \frac{\partial m_{ij}}{\partial q^l}\right)$$

are the Christoffel symbols of the first kind of $m$. Here, we

† Corresponding Author: Dept. of Electronics Engineering, Hallym University, Korea. (ssh@hallym.ac.kr)
* Dept. of Applied Mathematics, University of Waterloo, Canada. (waiman@ymail.com)
** Dept. of Applied Mathematics, University of Waterloo, Canada. (dechang@math.uwaterloo.ca)
Received: March 26, 2013; Accepted: July 12, 2013

have assumed

$$W = Span\left\{dq^a \mid a = n_1+1,\cdots,n\right\} \tag{1}$$

Two controlled Lagrangian systems $(L,F,W)$ and $(\hat{L},\hat{F},\hat{W})$, where $L(q,\dot{q}) = \frac{1}{2}m(\dot{q},\dot{q}) - V(q)$ and $\hat{L}(q,\dot{q}) = \frac{1}{2}\hat{m}(\dot{q},\ \dot{q}) - \hat{V}(q)$, are feedback equivalent if for any control $u \in W$, there exists $\hat{u} \in \hat{W}$ such that the closed loop dynamics are the same, and conversely.

In this paper, we follow the setting as in [7]: Given a controlled Lagrangian system $(L, F=0, W)$ with no external force, we try to find a feedback equivalent system $(\hat{L},\hat{F},\hat{W})$ in which $\hat{F}$ is a gyroscopic force dependent on velocity of degree two, i.e. the *k*-th component $\hat{F}_k$ of the force $\hat{F}$ is given by

$$\hat{F}_k = \hat{C}_{ijk}\dot{q}^i\dot{q}^j \tag{2}$$

where $\hat{C}_{ijk}$ satisfy the following conditions:

$$\hat{C}_{ijk} = \hat{C}_{jik},\ \hat{C}_{ijk} + \hat{C}_{jki} + \hat{C}_{kij} = 0\,. \tag{3}$$

In other words, this external force $\hat{F}$ does no work on the feedback equivalent system.

It can be proved [7] that the existence of a feedback equivalent system $(\hat{L},\hat{F},\hat{W})$ for a given controlled Lagrangian system $(L,0,W)$ is related to the existence of solutions for a system of PDEs that are known as matching conditions:

**Theorem 1** [7]: $(L,0,W)$ is feedback equivalent to $(\hat{L},\hat{F},\hat{W})$ with a gyroscopic force $\hat{F}$ of degree 2 if and only if there exists a non-degenerate mass matrix $\hat{m}$ and a potential function $\hat{V}$ such that the following equations are satisfied:

$$\frac{\partial V}{\partial q^\alpha} - \hat{T}_{\alpha i}m^{ij}\frac{\partial V}{\partial q^j} = 0, \tag{4}$$

$$\hat{J}_{\alpha\beta\gamma} + \hat{J}_{\beta\gamma\alpha} + \hat{J}_{\gamma\alpha\beta} = 0, \tag{5}$$

where $m_{ij}$ (resp. $m^{ij}$) is the (i,j)-entry of m (resp. $m^{-1}$), $\hat{T}$ is an $n\times n$ symmetric matrix defined by

$$\hat{T} = m\hat{m}^{-1}m \tag{6}$$

and

$$\hat{J}_{\alpha\beta\gamma} = \frac{1}{2}\hat{T}_{\gamma s}m^{sk}\left(\frac{\partial \hat{T}_{\alpha\beta}}{\partial q^k} - \Gamma^r_{\beta k}\hat{T}_{\alpha r} - \Gamma^r_{\alpha k}\hat{T}_{\beta r}\right),$$

where $\Gamma^r_{jk} = m^{ri}[jk,i]$ are the Christoffel symbols of the second kind of m.

We introduce $\hat{T}$ since only $\hat{T}_{\alpha k}$, but not $\hat{T}_{ab}$, appear in the matching PDEs; if we used $\hat{m}$ then all entries of $\hat{m}$ would appear in the matching PDEs. Suppose that we have obtained a feasible solution $\hat{T}$ (and hence $\hat{m}$, since $\hat{m} = m\hat{T}m^{-1}$ by (6)) and $\hat{V}$ for the matching conditions. Then, we can obtain the Lagrangian $\hat{L}$ for the feedback equivalent system. Also, the corresponding control bundle $\hat{W}$ is given by

$$\hat{W} = \hat{m}m^{-1}W.$$

Hence, what is left is to compute the gyroscopic force $\hat{F}$. Following [8], we introduce

$$\hat{S}_{ijk} = m_{ip}m_{jq}\hat{m}^{pl}\hat{m}^{qs}(m_{kr}\hat{m}^{rt}\widehat{[ls,t]} - [ls,k]), \tag{7}$$

$$\hat{A}_{ijk} = m_{ip}m_{jq}m_{kr}\hat{m}^{pl}\hat{m}^{qs}\hat{m}^{rt}C_{lst}, \tag{8}$$

where $\widehat{[ls,t]}$ are the Christoffel symbols of the first kind of $\hat{m}$. Once $\hat{m}$ is determined, we can compute $\hat{S}_{ijk}$. Then, determine $\hat{A}_{ijk}$ in terms of $\hat{S}_{ijk}$ using the following scheme:

(a) $\hat{A}_{ij\alpha} = \hat{S}_{ij\alpha}$,

(b) $\hat{A}_{\beta\gamma a} = -\hat{S}_{a\beta\gamma} - \hat{S}_{\gamma a\beta}$,

(c) $\hat{A}_{\gamma ab} = \hat{A}_{b\gamma a} = -\frac{1}{2}\hat{S}_{ab\gamma}$,

(d) Finally, choose any $\hat{A}_{abc}$ such that $\hat{A}_{abc} + \hat{A}_{bca} + \hat{A}_{cab} = 0$. For simplicity, we can take $\hat{A}_{abc} = 0$.

Notice that under this scheme $\hat{A}_{ijk}$ satisfy the properties in (3). Once $\hat{A}_{ijk}$ are determined, we can obtain the gyroscopic force terms $\hat{C}_{ijk}$ by (8), or equivalently,

$$\hat{C}_{ijk} = \hat{m}_{xi}\hat{m}_{yj}\hat{m}_{zk}m^{xr}m^{ys}m^{zt}\hat{A}_{rst} \tag{9}$$

- ***Procedure for solving energy shaping problems***

We can now summarize the general procedure for getting a nonlinear control force for a given controlled Lagrangian system with degree of underactuation equal to $n_1 \geq 1$. (This procedure is from [9]):

S1. Check if the linearization of the given controlled Lagrangian is controllable or its uncontrollable subsystem is oscillatory. [1] If neither holds, then

[1] A linear system $\dot{x} = Ax$ is oscillatory if A is diagonalizable and all eigenvalues of A are nonzero and purely imaginary.

stop;otherwise, proceed to the next step. [7]

S2. Get a solution for $\hat{V}$ and the $(\alpha, i)$ entries $\hat{T}_{\alpha i}$ of $\hat{T}$ which solve the matching PDEs (4) and (5), keeping in mind that the $n_1 \times n_1$ matrix $[T_{\alpha\beta}]$ is positive definite around $q = 0$ and $\hat{V}$ has a non-degenerate minimum at $q = 0$. In particular, $\hat{T}_{11}$ should be positive around $q = 0$ when the degree of underactuation $n_1$ is one.

S3. Choose the rest of the entries $\hat{T}_{ab}$ of $\hat{T}$ so that $\hat{T}$ is positive definite, at least at q = 0. In particular, when the degree of freedom $n$ is two, one should choose $\hat{T}_{22} > (\hat{T}_{12})^2 / \hat{T}_{11}$.

S4. Obtain the mass matrix $\hat{m}$ of the feedback equivalent system, through the equation: $\hat{m} = m\hat{T}^{-1}m$ .

S5. Compute the gyroscopic force $\hat{F}$ in (2) by computing $\hat{S}_{ijk}$ , $\hat{A}_{ijk}$ and then $\hat{C}_{ijk}$ by (7), (9) and steps (a) – (d) between (8) and (9).

S6. Compute the control bundle $\hat{W}$ , which is given by

$$\hat{W} = Span\left\{ \left. \begin{bmatrix} m^{ai}\hat{m}_{i1} \\ \cdots \\ m^{ai}\hat{m}_{in} \end{bmatrix} \right| a = n_1 + 1, \cdots, n \right\}.$$

S7. Choose a dissipative, $\hat{W}$ -valued linear symmetric control force $\hat{u}$. [2] In particular, for systems with degree of underactuation equal to $n_1$ , one may choose

$$\hat{u} = -K^T DK\dot{q}, \tag{10}$$

where $D$ is any $(n - n_1) \times (n - n_1)$ symmetric positive definite matrix and $K$ is the $(n - n_1) \times n$ matrix defined by

$$K = \begin{bmatrix} m^{n_1+1i}\, \hat{m}_{i1} \cdots m^{n_1+1i}\, \hat{m}_{in} \\ \vdots \quad \ddots \quad \vdots \\ m^{n_i}\hat{m}_{i1} \quad \cdots \quad m^{n_i}\hat{m}_{in} \end{bmatrix}.$$

S8. Compute the corresponding control force u:

$$u_a = [jk, a]\dot{q}^j \dot{q}^k + \frac{\partial V}{\partial q^a} - m_{ar}\widehat{m}^{rs}\left( \widehat{[jk, s]}\dot{q}^j \dot{q}^k + \frac{\partial V}{\partial q^s} - \widehat{C}_{jks}\dot{q}^j \dot{q}^k - \widehat{u}_s \right) \tag{11}$$

where $a = n_1 + 1, \cdots, n$ . Note that $u_\alpha$ for $\alpha = 1, \cdots, n_1$ is zero by (1).

Notice that in the above procedure, we require $\hat{F}$ to be gyroscopic and $\hat{u}$ dissipative. This implies that for every $\dot{q}$ , $< \hat{F}, \dot{q} > = 0$ and $< \hat{u}, \dot{q} > \leq 0$ . Hence the time derivative of the total energy $\hat{E}$ of the feedback equivalent system satisfies

$$\frac{d\hat{E}}{dt} = \left\langle \hat{F} + \hat{u}, \dot{q} \right\rangle = 0 + \left\langle \hat{u}, \dot{q} \right\rangle \leq 0.$$

As a result, Lyapunov stability of the equilibrium $(q, \dot{q}) = (0, 0)$ is guaranteed.

The crucial part of solving an energy shaping problem is to obtain a solution for the matching PDEs. For degree of underactuation equal to one, the matching conditions in Theorem 1 reduce to two PDEs, one for $\hat{V}$ and the other for $\hat{T}$ :

$$\frac{\partial V}{\partial q^1} - \hat{T}_{i1}m^{ij}\frac{\partial V}{\partial q^j} = 0, \tag{12}$$

$$\hat{T}_{1s}m^{sk}\left( \frac{\partial \hat{T}_{11}}{\partial q^k} - 2\Gamma^{\gamma}_{1k}\hat{T}_{1\gamma} \right) = 0. \tag{13}$$

It turns out that for this class of mechanical systems, the conditions for energy shaping are related to the linearization of the given system, summarized as follows:

**Theorem 2** ([7]). Given $(L, 0, W)$ with one degree of underactuation, let $(L^l, 0, W^l)$ be its linearized system at equilibrium $(q, \dot{q}) = (0, 0)$ . Then there exists a feedback equivalent $(\hat{L}, \hat{F}, \hat{W})$ with $\hat{F}$ gyroscopic of degree 2 and $\hat{V}$ having a non-degenerate minimum at (0, 0) if and only if the uncontrollable dynamics, if any, of $(L^l, 0, W^l)$ is oscillatory. In addition if $(L^l, 0, W^l)$ is controllable, then $(\hat{L}, \hat{F}, \hat{W})$ can be exponentially stabilized by any linear symmetric dissipative feedback onto $\hat{W}$ .

Notice that for systems with higher degrees of underactuation, "energy-shapability" of the linearization is only necessary, but not sufficient for that of the original nonlinear system. Hence, the existence of solution for the matching PDEs requires further study.

## 3. Example 1: Inertial Wheel Pendulum

We follow the setting in [10], as shown in Fig. 1. The configuration space is

$$Q = S^1 \times S^1 = \{(q^1, q^2) \mid q^1, q^2 \in (-\pi, \pi]\} .$$

The moments of inertia of the rod and the wheel are $I_1$ and $I_2$ respectively, and the distance of the center of mass of the rod from the unactuated joint (not shown in the figure) is $\ell_{c1}$. The control force $u$ is the torque applied to the inertial wheel. Let $A = m_1\ell_{c1}^2 + m_2\ell_1^2 + I_1 + I_2$ and $I_l = I_2$, then $A > I$ and the Lagrangian is given by

$$L = \frac{1}{2}(A(\dot{q}^1)^2 + 2I\dot{q}^1\dot{q}^2 + I(\dot{q}^2)^2) - m_0 g\cos q^1,$$

[2] The linear symmetric force means a force $F$ of the form $F(q, \dot{q}) = S(q)\dot{q}$ where $S(q)$ is a symmetric matrix-valued function of $q$.

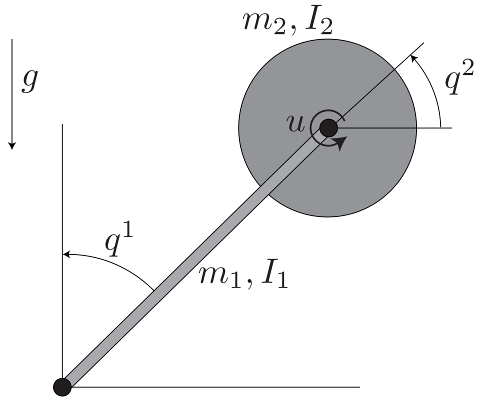



**Fig. 1.** The inertial wheel pendulum

where $m_0 = m_1\ell_{c1} + m_2\ell_1$ and $g$ is the gravitational constant. The equilibrium $(q,\dot{q}) = (0,0)$ is unstable. The linearization of this system at $(q,\dot{q}) = (0,0)$ is controllable. Hence by Theorem 2 we can apply energy shaping to stabilize the equilibrium. The corresponding matching PDEs are

$$(\hat{T}_{11} - \hat{T}_{12})\frac{\partial \hat{T}_{11}}{\partial q^1} + \left(-\hat{T}_{11} + \frac{A\hat{T}_{12}}{I}\right)\frac{\partial \hat{T}_{11}}{\partial q^2} = 0, \tag{14}$$

$$(\hat{T}_{11} - \hat{T}_{12})\frac{\partial \hat{V}}{\partial q^1} + \left(-\hat{T}_{11} + \frac{A\hat{T}_{12}}{I}\right)\frac{\partial \hat{V}}{\partial q^2} + m_0(A-I)\sin q^1 = 0. \tag{15}$$

We first solve (14). If we choose $\hat{T}_{11} = 1$ and $\hat{T}_{12} = b_0$ where $b_0 \in \Re$, then (14) is satisfied. Substituting this pair of $\hat{T}$ entries into (15), one can solve for $\hat{V}$ which reads

$$\hat{V} = -\frac{m_0(A-I)\cos q^1}{b_0 - 1} + f\left(\frac{(Ab_0 - I)q^1}{(b_0-1)I} + q^2\right)$$

for any smooth function *f*. Now, for simplicity, we choose a particular set of parameters, say, $b_0 = 2$ and $f(x) = x^2$ so that the potential energy becomes

$$\hat{V} = -m_0(A-I)\cos q^1 + \frac{1}{I^2}\left((2A-I)q^1 + Iq^2\right)^2.$$

Since $A > I$, the critical points of $\hat{V}$ are $q = 0$ and $(q^1, q^2) = (\pi, -\frac{2A-I}{I}\pi)$. Notice that $q = 0$ is the only minimum point for $\hat{V}$. The total energy function has a nondegenerate minimum at $(q,\dot{q}) = (0,0)$ provided that the $\hat{T}$ matrix (and hence $\hat{m}$) is positive definite. Note that $\hat{T}_{22}$ is still free for which we just choose 8. The resulting positive definite $\hat{T}$ is then given by

$$\hat{T} = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix},$$

from which we can calculate $\hat{m}$:

$$\hat{m} = \begin{bmatrix} 2A^2 - AI + \frac{I^2}{4} & \frac{3}{2}AI - \frac{I^2}{4} \\ \frac{3}{2}AI - \frac{I^2}{4} & \frac{5}{4}I^2 \end{bmatrix}.$$

Since $\hat{m}$ is a constant matrix, $\hat{S}_{ijk} = 0$ and $\hat{A}_{ijk} = 0$. As a result, all $\hat{C}_{ijk}$ terms are zeros.

We now choose the following dissipative control force for the feedback equivalent system, according to (10) with $D = 1$:

$$\hat{u} = -\begin{bmatrix} \frac{2A-I}{I} \\ 1 \end{bmatrix}\begin{bmatrix} \frac{2A-I}{I} & 1 \end{bmatrix}\begin{bmatrix} \dot{q}^1 \\ \dot{q}^2 \end{bmatrix}.$$

The corresponding control law *u*, obtained by (11), reads as follows:

$$u = 4\left(\frac{2A-I}{I^2}(2q^1 + \dot{q}^1) + \frac{2q^2 + \dot{q}^2}{I} + \frac{3}{2}m_0 \sin q^1\right).$$

By Theorem 2, local exponential stability is guaranteed around the equilibrium $(q,\dot{q}) = (0,0)$. To find the region of attraction, however, one needs to apply the LaSalle invariance principle. First, notice that with $\hat{u}$ defined as in (10), the time derivative of the total energy function is given by

$$\frac{d\hat{E}}{dt} = \langle \hat{u}, \dot{q} \rangle = -\left(\frac{2A-I}{I}\dot{q}^1 + \dot{q}^2\right)^2 \leq 0.$$

We now choose an $r > 0$ so that the set

$$\Omega_r = \left\{(q,\dot{q}) \in Q \,\middle|\, \hat{E}(q,\dot{q}) \leq r\right\}$$

is compact and does not include $(q^1, q^2, \dot{q}^1, \dot{q}^2) = (\pi, -\frac{2A-I}{I}\pi, 0, 0)$. Note that $\Omega_r$ is positively invariant since $d\hat{E}/dt$ is non-positive. We then need to consider all $(q,\dot{q})$ such that $d\hat{E}/dt$ is zero, i.e.

$$\dot{q}^2 = -\frac{2A-I}{I}\dot{q}^1.$$

Let $S$ be the set of points at which $d\hat{E}/dt = 0$ in $\Omega_r$, viz

$$S = \left\{(q,\dot{q}) \in Q_r \,\middle|\, \dot{q}^2 = -\dot{q}^1(2A-I)/I\right\}.$$

Define $M$ to be the largest invariant subset of $S$. Let $(q(t),\dot{q}(t))$ be a trajectory in $M$. In what follows, the argument t is suppressed for the sake of brevity. Then the trajectory should also satisfy

$$q^2 = -q^1(2A-I)/I + C, \tag{16}$$

for some fixed $C$. Substituting (16) into the equations of motion for the feedback equivalent system $(\hat{L},\hat{F},\hat{W})$, we have the following systems of differential equations:

$$\begin{aligned} &IA(A-I)\ddot{q}^1 + I(I-A)m_0 \sin q^1 - C(2A-I) = 0, \\ &I(A-I)\ddot{q}^1 + 2C = 0. \end{aligned}$$

Eliminating $\ddot{q}^1$ in the above equations, we can immediately see that $q^1$ must be a constant, say $C_1$, then by (16) $q^2 = -\frac{2A-I}{I}C_1 + C$. Substituting $(q^1,q^2) = (C_1, -\frac{2A-I}{I}C_1 + C)$ into the equations of motion, we can show that $C = 0$ and $C_1 = 0$ or $\pi$ both of which imply $\dot{q}^1 = \dot{q}^2 = 0$. Since $(\pi, -\frac{2A-I}{I}\pi, 0, 0) \notin \Omega_r$, we conclude the largest invariant subset $M$ of $S$ is {(0, 0, 0, 0)} only. Hence, by LaSalle invariance principle, asymptotic stability is achieved in $\Omega_r$. Furthermore, since $\Omega_r$ is compact, we have exponential stability in $\Omega_r$ by Lemma 1 in the Appendix.

## 4. Example 2: Inverted Pendulum on a Cart

In this system in Fig. 2, we assume the rod has negligible mass in order to simplify our model. The configuration space is

$$Q = \left\{(q^1,q^2) \,\middle|\, q^1 \in (-\pi/2,\pi/2), q^2 \in \Re\right\}.$$

which considers the pendulum only above the horizontal line. The Lagrangian is given by

$$\begin{aligned} L(q,\dot{q}) &= \frac{1}{2}m_1\ell^2(\dot{q}^1)^2 + \frac{1}{2}(m_1+m_2)(\dot{q}^2)^2 \\ &\quad + m_1\ell\dot{q}^1\dot{q}^2\cos q^1 - m_1 g\ell\cos q^1 \end{aligned}$$

where $g$ is the gravitational constant. The potential energy $V(q) = m_1 g\ell\cos q^1$ does not attain a minimum at $q$ = 0, and hence the equilibrium point $(q,\dot{q}) = (0,0)$ is unstable. The linearization of this system at (0, 0) is

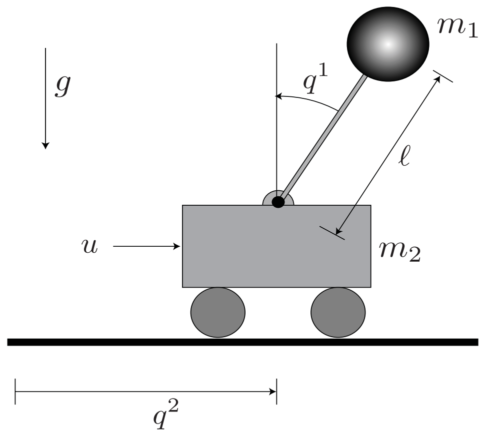



**Fig. 2.** An inverted pendulum on a running cart

controllable. Hence, by Theorem 2 we can use the energy shaping method to stabilize this system around the equilibrium.

The matching conditions are

$$\begin{aligned} &\left(-\frac{m_1+m_2}{m_1\ell^2}\hat{T}_{11} + \frac{\cos q^1}{\ell}\hat{T}_{12}\right) \\ &\left(\frac{\partial\hat{T}_{11}}{\partial q^1} + \frac{2m_1\hat{T}_{11}\cos q^1\sin q^1 - 2\hat{T}_{12}m_1\ell\sin q^1}{-(m_1+m_2)+m_1\cos^2 q^1}\right) \\ &+\left(\frac{\cos q^1}{\ell}\hat{T}_{11} - \hat{T}_{12}\right)\frac{\partial\hat{T}_{11}}{\partial q^2} = 0, \\ &\left(-\frac{m_1+m_2}{m_1\ell^2}\hat{T}_{11} + \frac{\cos q^1}{\ell}\hat{T}_{12}\right)\frac{\partial\hat{V}}{\partial q^1} \\ &+\left(\frac{\cos q^1}{\ell}\hat{T}_{11} - \hat{T}_{12}\right)\frac{\partial\hat{V}}{\partial q^2} \\ &+m_1 g\ell\sin q^1(-(m_1+m_2)+m_1\cos^2 q^1) = 0. \end{aligned}$$

We now try to obtain closed-form solutions for $\hat{T}$ and $\hat{V}$. First, we may start with the following possible choice:

$$\hat{T}_{11} = A_0 + A_1\cos^2 q^1,$$

where $A_0$ and $A_1 \neq 0$ are constants to be determined. Putting this ansatz into the first matching condition, we can obtain

$$\hat{T}_{12} = \frac{A_0 m_1 + A_1(m_1+m_2)}{m_1\ell}\cos q^1, \tag{17}$$

Or

$$\hat{T}_{12} = \frac{(m_1+m_2)(A_0 + A_1\cos^2 q^1)}{m_1\ell\cos q^1}.$$

Notice that the second solution for $\hat{T}_{12}$ will lead to a potential energy function $\hat{V}$ whose Hessian is not positive definite at $q$ = 0. Hence, we should resort to the first solution of $\hat{T}_{12}$ as in (17) and solve the matching condition for $\hat{V}$ to obtain

$$\hat{V} = \frac{1}{A_0} m_1^2 g \ell^3 \cos q^1 + f\left( q^2 + \frac{A_1 \ell}{A_0} \sin q^1 \right),$$

where $f = f(x)$ is any smooth function. Since we require $\hat{V}$ to have a nondegenerate minimum at $q$ = 0, we may set $f(x) = x^2$ so that

$$\hat{V} = \frac{1}{A_0} m_1^2 g \ell^3 \cos q^1 + \left( q^2 + \frac{A_1 \ell}{A_0} \sin q^1 \right)^2.$$

To have positive definiteness of $\hat{T}$ at least around $q$ = 0, we may impose $A_1 > 0 > A_0$ so that $\hat{T}_{11} > 0$ at least around $q$ = 0 and take any $\hat{T}_{22}$ such that $\det \hat{T} > 0$. In particular, if we take $A_1 = 2$ and $A_0 = -\varepsilon$, where $\varepsilon$ is fixed and $\varepsilon \in (0,2)$, then

$$\hat{T}_{22} = \frac{\cos^2 q^1 (2(m_1 + m_2) - \varepsilon m_1)^2 + 1}{m_1^2 \ell^2 (2\cos^2 q^1 - \varepsilon)},$$

which makes $\det \hat{T} = 1/(m_1^2 \ell^2) > 0$ for all $q$. In short, we have the following $\hat{T}$ matrix and potential energy $\hat{V}$:

$$\hat{T} = \begin{bmatrix} 2\cos^2 q^1 - \varepsilon & \dfrac{2(m_1 + m_2) - \varepsilon m_1}{m_1 \ell} \cos q^1 \\ \dfrac{2(m_1 + m_2) - \varepsilon m_1}{m_1 \ell} \cos q^1 & \dfrac{\cos^2 q^1 (2(m_1 + m_2) - \varepsilon m_1)^2 + 1}{m_1^2 \ell^2 (2\cos^2 q^1 - \varepsilon)} \end{bmatrix},$$

$$\hat{V} = -\frac{1}{\varepsilon} m_1^2 g \ell^3 \cos q^1 + \left( \frac{\varepsilon q^2 + 2\ell \sin q^1}{\varepsilon} \right)^2.$$

Define a subset $\Re_\varepsilon$ of $Q$ as follows:

$$\Re_\varepsilon = \left( -\cos^{-1} \sqrt{\frac{\varepsilon}{2}}, \cos^{-1} \sqrt{\frac{\varepsilon}{2}} \right) \times \Re.$$

Then $\hat{T}$ is positive definite over $\Re_\varepsilon$. Furthermore, (0, 0, 0, 0) is the only critical point of $\hat{V}$ within the region $\Re_\varepsilon$. The resulting mass matrix is $\hat{m} = [\hat{m}_{ij}]$ where

$$\hat{m}_{11} = \frac{m_1^2 \ell^4}{2\cos^2 q^1 - \varepsilon} [4\cos^2 q^1 (m_1 + m_2)^2 + 4m_1^2 \cos^6 q^1 - 8\cos^4 q^1 (m_1^2 + m_1 m_2) + 1],$$

$$\hat{m}_{12} = \frac{m_1^2 \ell^3 \cos q^1}{2\cos^2 q^1 - \varepsilon} [-4\varepsilon \cos^2 q^1 (m_1^2 + m_1 m_2) + 1 + 2\varepsilon (m_1 + m_2)^2 + 2\varepsilon m_1^2 \cos^4 q^1],$$

$$\hat{m}_{22} = \frac{m_1^2 \ell^2}{2\cos^2 q^1 - \varepsilon} [\varepsilon^2 m_1^2 \cos^4 q^1 + \varepsilon^2 (m_1 + m_2)^2 + \cos^2 q^1 (1 - 2\varepsilon^2 m_1^2 - 2\varepsilon^2 m_1 m_2)].$$

Following the procedure, we can compute the gyroscopic force $\hat{F} = [\hat{F}_1, \hat{F}_2]^T$ for the shaped system:

$$\hat{F}_1 = \frac{m_1^2 \ell^2 \cos q^1 \sin q^1}{(2\cos^2 q^1 - \varepsilon)^2} \dot{q}^2 Expr(q, \dot{q}),$$

$$\hat{F}_2 = \frac{m_1^2 \ell^2 \cos q^1 \sin q^1}{(2\cos^2 q^1 - \varepsilon)^2} \dot{q}^1 Expr(q, \dot{q}),$$

where

$$Expr(q, \dot{q}) = (2\ell \dot{q}^1 + \varepsilon \dot{q}^2) \{ 2\varepsilon (m_1^2 \cos^2 q^1 - (m_1 + m_2)^2) + 2\varepsilon^2 m_1 (m_1 \sin^2 q^1 + m_2) - 1 \}.$$

The control bundle $\hat{W}$ is equal to

$$\hat{W} = Span\left\{ \begin{bmatrix} \dfrac{2\ell}{\varepsilon} \cos q^1 & 1 \end{bmatrix}^T \right\}.$$

We now choose a control force $\hat{u}$ as in (10):

$$\hat{u} = -\begin{bmatrix} \dfrac{2\ell}{\varepsilon} \cos q^1 \\ 1 \end{bmatrix} \begin{bmatrix} \dfrac{2\ell}{\varepsilon} \cos q^1 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}^1 \\ \dot{q}^2 \end{bmatrix}.$$

One can then compute $u$ by (11). Note that by Theorem 2, local exponential stability is guaranteed around $(q, \dot{q}) = (0, 0)$. To compute the region of attraction, one applies the LaSalle invariance principle. As in the case of inertial wheel pendulum, we start by choosing r > 0 so that the set

$$\Omega_r = \left\{ (q, \dot{q}) \in \Re_\varepsilon \times \Re^2 \,\middle|\, \hat{E}(q, \dot{q}) \le r \right\}$$

is compact. Then we define the set

$$S = \left\{ (q, \dot{q}) \in \Omega_r \,\middle|\, \frac{d\hat{E}}{dt} = 0 \right\} = \left\{ (q, \dot{q}) \in \Omega_r \,\middle|\, 2\ell \dot{q}^1 \cos q^1 + \varepsilon \dot{q}^2 = 0 \right\}.$$

We note that the total energy function $\hat{E}$ has a zero time derivative if and only if $\dot{q}^2 = -(\frac{2\ell}{\varepsilon} \cos q^1) \dot{q}^1$ from which we have

$$q^2 = -\frac{2\ell}{\varepsilon}\sin q^1 + C, \quad (18)$$

where $C$ is a constant. Let $M$ be the largest invariant subset of $S$ and consider an arbitrary trajectory $(q(t),\dot{q}(t))$ in $M$. This trajectory should satisfy the equations of motion of the feedback equivalent system together with (18). Substituting (18) into those equations of motion, we have

$$\sin q^1(\dot{q}^1)^2 = \frac{2C(2\cos^2 q^1 - \varepsilon) + \frac{m_1 g\ell^2}{2}\sin 2q^1}{m_1^2\ell^3}, \quad (19)$$

$$\ddot{q}^1 = \frac{4C\cos q^1 + m_1 g\ell^2 \sin q^1}{m_1^2\ell^3}. \quad (20)$$

Multiplying (20) by $\cos q^1$ and subtracting it from (19), one can obtain

$$\sin q^1(\dot{q}^1)^2 - \cos q^1\ddot{q}^1 = -\frac{2C\varepsilon}{m_1^2\ell^3}.$$

Then by integration twice with respect to $t$, we have

$$\sin q^1 = \frac{C\varepsilon}{m_1^2\ell^3}t^2 + C_1 t + C_2,$$

where $C_1, C_2$ are constant. Now, since $\sin q^1$ is always bounded, the above equation holds only if $C = C_1 = 0$, implying that $q^1$ must be a constant. As $C = 0$ and $\dot{q}^1 = 0$, (20) implies $\sin q^1 = 0$, i.e. $q^1 = 0$ or $\pi$. When $q^1 = 0$, so is $q^2$. In other words, $M = \{(0,0,0,0)\}$. Hence, by LaSalle invariance principle, every trajectory in $\Omega_r$ will approach (0, 0, 0, 0) asymptotically. Note that when $\varepsilon \to 0^+$, $\Re_\varepsilon \to (-\pi/2, \pi/2)\times\Re$. As a result, we can enlarge the region of attraction by letting $\varepsilon \to 0^+$. Since $\Omega_r$ is chosen to be compact, we also have exponential stability over $\Omega_r$ by Lemma 1 in the Appendix.

## 5. Example 3: Ball and Beam

Consider the ball and beam system as shown in Fig. 3. Given that the length of the beam is $\ell$, the configuration space is $[-\ell,\ell]\times[-\pi/2,\pi/2]$ after nondimensionalization of the time and torque [11]. The Lagrangian of this system is given by:

$$L(q,\dot{q}) = \frac{1}{2}\{(\dot{q}^1)^2 + (\ell^2 + (q^1)^2)(\dot{q}^2)^2\} - gq^1\sin q^2,$$

where $g$ is the gravitational constant. The linearization at the equilibrium point (0, 0) is controllable, so we can apply the energy shaping method. The two matching conditions

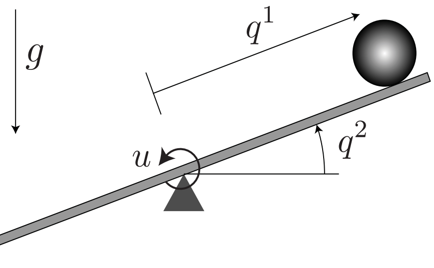



**Fig. 3.** The ball and beam system

are

$$\left(\ell^2 + (q^1)^2\right)\hat{T}_{11}\frac{\partial \hat{T}_{11}}{\partial q^1} + \hat{T}_{12}\left(\frac{\partial \hat{T}_{11}}{\partial q^2} - \frac{2q^1\hat{T}_{12}}{\ell^2 + (q^1)^2}\right) = 0,$$

$$\left(\ell^2 + (q^1)^2\right)\hat{T}_{11}\frac{\partial \hat{V}}{\partial q^1} + \hat{T}_{12}\frac{\partial \hat{V}}{\partial q^2} - g\left(\ell^2 + (q^1)^2\right)\sin q^2 = 0.$$

We may try an ansatz for $\hat{T}_{12}$ first and then solve for $\hat{T}_{11}$, assuming $\hat{T}_{11} = \hat{T}_{11}(q^1)$. We thus have the following general solutions [11] for the kinetic matching PDE:

$$\hat{T}_{11} = A_0\sqrt{\left(\ell^2 + (q^1)^2\right)}, \hat{T}_{12} = \frac{A_0}{\sqrt{2}}\left(\ell^2 + (q^1)^2\right).$$

For simplicity, we now take $A_0 = \sqrt{2}$ implying $\hat{T}_{11} = \sqrt{2(\ell^2 + (q^1)^2)}$ and $\hat{T}_{12} = \ell^2 + (q^1)^2$. The resulting potential energy, by solving the second matching condition, takes the form

$$\hat{V} = g(1-\cos q^2) + f\left(q^2 - \frac{1}{\sqrt{2}}\ln\frac{q^1 + \sqrt{\ell^2 + (q^1)^2}}{\ell}\right).$$

Again, we take $f(x) = x^2$ to ensure that $\hat{V}$ has a minimum at $q$ = 0. The positive definiteness requirement for $\hat{T}$ is met by taking $\hat{T}_{22} = (\ell^2 + (q^1)^2)^{\frac{3}{2}}$. Notice that the resulting $\hat{T}$ is positive definite everywhere. Furthermore, compared to the method in [11], we have freedom over the choice of $\hat{T}_{22}$. Now, the corresponding mass matrix is

$$\hat{m} = \frac{1}{\sqrt{2}-1}\begin{bmatrix} \frac{1}{\sqrt{\ell^2 + (q^1)^2}} & -1 \\ -1 & \sqrt{2\{\ell^2 + (q^1)^2\}} \end{bmatrix}.$$

With all these at hand, we can calculate the gyroscopic terms. By definition, we have

$$\hat{S}_{111} = 0,$$

$$\hat{S}_{ij1} = \frac{3\sqrt{2}-4}{2(\sqrt{2}-1)}\sqrt{\ell^2+(q^1)^2}\,q^1,$$

for all $i \neq j$. The $\hat{A}_{ijk}$ terms can be computed as follows:

$$\hat{A}_{111} = \hat{A}_{222} = 0,$$
$$\hat{A}_{112} = -\frac{3\sqrt{2}-4}{\sqrt{2}-1}\sqrt{\ell^2+(q^1)^2}\,q^1,$$
$$\hat{A}_{221} = -\frac{3\sqrt{2}-4}{\sqrt{2}-1}\{\ell^2+(q^1)^2\}q^1,$$
$$\hat{A}_{121} = \hat{A}_{211} = \hat{A}_{221} = -\frac{3\sqrt{2}-4}{2(\sqrt{2}-1)}\sqrt{\ell^2+(q^1)^2}\,q^1,$$
$$\hat{A}_{122} = \hat{A}_{212} = -\frac{3\sqrt{2}-4}{4(\sqrt{2}-1)}\{\ell^2+(q^1)^2\}q^1.$$

We thus obtain the gyroscopic force terms as follows:

$$\hat{C}_{111} = \hat{C}_{222} = 0,$$
$$\hat{C}_{112} = -\frac{(2+\sqrt{2})q^1}{2\{\ell^2+(q^1)^2\}},$$
$$\hat{C}_{121} = \hat{C}_{211} = \frac{(2+\sqrt{2})q^1}{4\{\ell^2+(q^1)^2\}},$$
$$\hat{C}_{122} = \hat{C}_{212} = \frac{(3\sqrt{2}-4)^2 q^1}{4\sqrt{\ell^2+(q^1)^2}\,(\sqrt{2}-1)^4},$$
$$\hat{C}_{221} = -\frac{(3\sqrt{2}-4)^2 q^1}{2\sqrt{\ell^2+(q^1)^2}\,(\sqrt{2}-1)^4}.$$

Combining these gyroscopic force terms together, we can now obtain the expression for the gyroscopic force $\hat{F}$:

$$\hat{F}_1 = -Gyro(q,\dot{q})\cdot\dot{q}^2,\ \hat{F}_2 = Gyro(q,\dot{q})\cdot\dot{q}^1$$

Where

$$Gyro = \frac{(3\sqrt{2}-4)q^1\left(\dfrac{1-\sqrt{2}}{\sqrt{\ell^2+(q^1)^2}}\dot{q}^1 + \left(3\sqrt{2}-4\right)\dot{q}^2\right)}{2(\sqrt{2}-1)^4\sqrt{\ell^2+(q^1)^2}}.$$

Now, for the control force, we first compute the control bundle $\hat{W}$:

$$\hat{W} = span\left\{\begin{bmatrix} -\dfrac{1}{\sqrt{2\{\ell^2+(q^1)^2\}}} & 1\end{bmatrix}^T\right\}.$$

Then, we choose the dissipative control force $\hat{u}$ by

$$\hat{u} = -\begin{bmatrix} \dfrac{1}{2\{\ell^2+(q^1)^2\}} & -\dfrac{1}{\sqrt{2\{\ell^2+(q^1)^2\}}} \\ -\dfrac{1}{\sqrt{2\{\ell^2+(q^1)^2\}}} & 1 \end{bmatrix}\begin{bmatrix}\dot{q}^1 \\ \dot{q}^2\end{bmatrix},$$

from which one can compute the corresponding control law *u*. Local exponential stability is guaranteed by Theorem 2 and one can find a compact region of attraction by applying LaSalle invariance principle. By Lemma 1 in the Appendix, it becomes a region of exponential convergence as well.

## 6. Example 4: Furuta Pendulum

We now come to study the energy shaping problem for the Furuta pendulum. The configuration space of the Furuta pendulum is $Q = (-\pi,\pi]\times(-\pi,\pi]$. Following the notation in [12] (with some minor changes), the Lagrangian for the Furuta pendulum is given by

$$L(q,\dot{q}) = \frac{1}{2}\alpha(\dot{q}^1)^2 + \beta\dot{q}^1\dot{q}^2\cos q^1 + \frac{1}{2}(\gamma+\alpha\sin^2 q^1)(\dot{q}^2)^2 - D\cos q^1,$$

where $\alpha = m\ell^2, \beta = m\ell R, \gamma = (M+m)R^2, D = mg\ell$ and the parameters are defined in Fig. 4. In [12], the Furuta pendulum is shaped by observing that it can be transformed via feedback to a system equivalent to an inverted pendulum on a cart with some gyroscopic force terms. Here,we will solve the energy shaping problem, using the standardized method of solving matching PDEs. Since the linearization of the system at $(q,\dot{q}) = (0,0)$ is a controllable system, the energy shaping method applies by Theorem 2. For the sake of simplicity in later computations, we can divide the equations of motion by the parameter $\alpha$ so as to obtain the following mass matrix and potential energy:

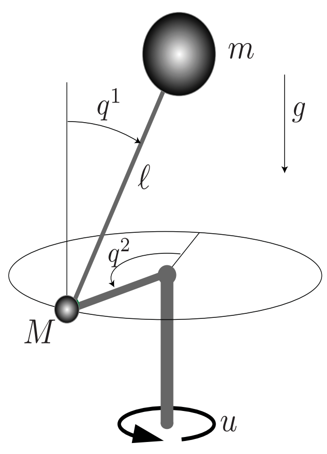



**Fig. 4.** The Furuta pendulum

$$m=\begin{bmatrix} 1 & A\cos q^1 \\ A\cos q^1 & B+\sin^2 q^1 \end{bmatrix}, V(q)=C\cos q^1,$$

for some $A,B,C > 0$. The resulting matching PDEs are

$$\begin{aligned}&\left(\hat{T}_{11}(B+\sin^2 q^1)-A\hat{T}_{12}\cos q^1\right)\\ &\left(\frac{\partial \hat{T}_{11}}{\partial q^1}+\frac{2A\sin q^1(\hat{T}_{11}A\cos q^1-\hat{T}_{12})}{-(B+1)+(A^2+1)\cos^2 q^1}\right)\\ &+\left(-\hat{T}_{11}A\cos q^1+\hat{T}_{12}\right)\\ &\left(\frac{\partial \hat{T}_{11}}{\partial q^2}+\frac{2\cos q^1\sin q^1(-\hat{T}_{11}A\cos q^1+\hat{T}_{12})}{-(B+1)+(A^2+1)\cos^2 q^1}\right)=0,\end{aligned} \tag{21}$$

$$\left(\hat{T}_{11}(B+\sin^2 q^1)-A\hat{T}_{12}\cos q^1\right)\frac{\partial \hat{V}}{\partial q^1}+\left(-\hat{T}_{11}A\cos q^1+\hat{T}_{12}\right)\frac{\partial \hat{V}}{\partial q^2}$$
$$-\sin q^1\{-(B+1)+(A^2+1)\cos^2 q^1\}=0\,. \tag{22}$$

We first solve (21). Notice that solutions of the form $\hat{T}_{11}=A_1+A_2\cos^2 q^1$ and $\hat{T}_{12}=B_1\cos q^1$ cannot give a potential energy $\hat{V}$ in (22) with a minimum at $q = 0$. We guess that a possible candidate can be the following

$$\begin{aligned}\hat{T}_{11}&=\frac{X_1-X_2\cos^2 q^1}{Z_1-Z_2\cos^2 q^1},\\ \hat{T}_{12}&=\frac{X_3\cos q^1}{Z_1-Z_2\cos^2 q^1}.\end{aligned} \tag{23}$$

Substitute this pair into (21), and we have the following relation on the coefficients:

$$X_1=\frac{A(B+1)X_2+(A^2+1)X_3}{A(A^2+1)}, \tag{24}$$

$$Z_2=\frac{AX_2Z_1}{A(B+1)X_2+(A^2+1)X_3}, \tag{25}$$

With $X_1$ and $Z_2$ defined in (24) and (25), the $\hat{T}_{\alpha i}$ entries become

$$\hat{T}_{11}=\frac{A(B+1)X_2+(A^2+1)X_3}{A(A^2+1)Z_1}\cdot\frac{A(B+1)+(A^2+1)\frac{X_3}{X_2}-A(A^2+1)\cos^2 q^1}{A(B+1)+(A^2+1)\frac{X_3}{X_2}-A\cos^2 q^1},$$

$$\hat{T}_{12}=\frac{X_3}{Z_1}\cdot\frac{\{A(B+1)X_2+(A^2+1)X_3\}\cos q^1}{A(B+1)X_2+(A^2+1)X_3-AX_2\cos^2 q^1}$$

and then one can solve for the potential energy:

$$\hat{V}=\frac{CZ_1A(A^2+1)\cos q^1}{A(B+1)X_2+(A^2+1)X_3}$$
$$+f\left(\frac{A^2X_2\tan^{-1}\left(\frac{AX_2\sin q^1}{\sqrt{AX_2\{(A^2+1)X_3+ABX_2\}}}\right)}{\sqrt{AX_2\{(A^2+1)X_3+ABX_2\}}}+q^2\right),$$

where $f = f(x)$ is any smooth function which attains its minimum value at $q = 0$. We can take $f(x)=x^2$ for the sake of simplicity. Recall that $\hat{T}$ should be positive definite around q = 0, and $\hat{V}$ has a minimum at q = 0.

$$\frac{(A^2+1)X_3-A(A^2-B)X_2}{ABX_2+(A^2+1)X_3}<0, \tag{26}$$

by evaluating the expression of $\hat{T}_{11}$ at (0, 0). The requirement on $\hat{V}$ implies the following constraints:

$$\frac{Z_1}{A(B+1)X_2+(A^2+1)X_3}<0, \tag{27}$$

$$\frac{CZ_1(A^2+1)((A^2+1)X_3+ABX_2)^2}{A(B+1)X_2+(A^2+1)X_3}<2A^3X_2^2. \tag{28}$$

Notice that (28) is always true due to (27) and the fact that $A,\ C > 0$. Since $-\frac{AB}{A^2+1}<\frac{A(A^2-B)}{A^2+1}$, inequality (26) implies that

$$-\frac{AB}{A^2+1}<\frac{X_3}{X_2}<\frac{A(A^2-B)}{A^2+1}.$$

As a result, the expression $A(B+1)+\frac{X_3}{X_2}(A^2+1)$ that appears in $\hat{T}_{11}$ has the following bounds:

$$A<A(B+1)+\frac{X_3}{X_2}(A^2+1)<A(A^2+1),$$

implying that the denominator of $\hat{T}_{11}$ is never zero.

Hence, using (27), $\hat{T}_{11}>0$ if and only if

$$A(B+1)+\frac{X_3}{X_2}(A^2+1)-A(A^2+1)\cos^2 q^1<0$$
$$\Leftrightarrow \cos^2 q^1>\frac{B+1}{A^2+1}+\frac{1}{A}\frac{X_3}{X_2}.$$

Furthermore, since $X_3/X_2>-AB/(A^2+1)$, $\cos^2 q^1$ is always bounded below by $1/(A^2+1)$. This implies that to enlarge the region of attraction as much as possible, one may choose $X_2$ and $X_3$ so that $X_3/X_2$ is close to $-AB/(A^2+1)$. For instance, one may choose $X_2 = A^2+1$ and $X_3 = -rAB$ where $r < 1$ is close to 1. Then by (27) $Z_1$ must be negative as the denominator in (27) is positive. After setting $\hat{T}$, one can solve the energy shaping problem as in the previous examples, which is left to the readers.

## 7. Conclusions and Future Work

In this paper we introduced a standardized procedure for shaping a controlled Lagrangian system, and illustrated this procedure using four examples. Recently we discovered some criteria for shaping a mechanical system with two degrees of underactuation and with more than three degrees of freedom [9], so a similar energy shaping procedure can be proposed in this case. Nevertheless, energy shaping for higher degrees of underactuation still remains largely unsolved. We plan to investigate this issue in the future.

## Acknowledgments

This work was supported by Hallym University Research Fund 2013 (HRF-201303-016).

## Appendix

**Lemma 1.** Consider a differential equation on $R^n$

$$\dot{x} = f(x), \qquad f(0) = 0.$$

Suppose that the origin is exponentially stable. If a compact set is a region of asymptotic convergence, then it is a region of exponential convergence.

**Proof.** By the assumption of exponential stability, there are $r > 0, M_0 > 0$ and $\lambda > 0$ such that

$$|x(t,x_0)| \le M_0 e^{-\lambda t}|x_0|$$

for each $t > 0$ and each initial point $x_0 \in B_r$, where $B_r = \{x \mid |x| < r\}$. By the assumption on $\Omega$ and continuity of $x(\cdot;\cdot)$, for each $y \in \Omega \setminus B_r$ there is an open neighborhood $U_y$ of $y$ and time $T_y > 0$ such that $x(T_y; x_0) \in B_r$ for all $x_0 \in U_y \cap (\Omega \setminus B_r)$. Hence, for each $x_0 \in U_y \cap (\Omega \setminus B_r)$

$$\begin{aligned} |x(t,x_0)| &= |x(t-T_y; x(T_y;x_0))| \\ &\le M_0 e^{-\lambda(t-T_y)}|x(T_y;x_0)| \\ &\le M_0 e^{-\lambda(t-T_y)} r \\ &\le M_0 e^{\lambda T_y} e^{-\lambda t}|x_0| \end{aligned}$$

for all $t \ge 0$, since $r \le |x_0|$. Since $\{B_r, U_y \mid y \in \Omega \setminus B_r\}$ is an open cover of the compact set $\Omega$, there is a finite subcover $\{B_r, U_{y_1}, \cdots, U_{y_k}\}$ of $\Omega$.

Let $M = M_0 \max\{e^{\lambda T_{y_i}} \mid i = 1, \cdots, k\}$ It is then easy to see that

$$|x(t,x_0)| \le M_0 e^{-\lambda t}|x_0|$$

for each $t \ge 0$, and each initial point $x_0 \in \Omega$, which proves that $\Omega$ is a region of exponential convergence.

## References

[1] A. Bloch, M. Leok, J.E. Marsden, and D. Zenkov, “Controlled Lagrangians and stabilization of the discrete cart-pendulum system,” 2005, pp. 6579-6584.

[2] A. Bloch, M. Leok, J.E. Marsden, and D. Zenkov, “Matching and stabilization of discrete mechanical systems,” Proc. Appl. Math. Mech., vol. 7, pp. 1 030 603-1 030 604, 2007.

[3] A. Bloch, M. Leok, J.E. Marsden, and D. Zenkov, “Controlled Lagrangians and stabilization of discrete mechanical systems,” Discrete and Continuous Dynamical Systems - Series S, vol.3, no. 1, pp.19-36, 2010.

[4] A. Bloch, N. Leonard, and J.E. Marsden, “Stabilization of mechanical systems using controlled Lagrangians,” in Proc. 36th IEEE Conference on Decision and Control, 1997.

[5] A. Bloch, N. Leonard, and J.E. Marsden, “Controlled Lagrangians and the stabilization of mechanical systemsn I: The first matching theorem,” IEEE Trans. Automat. Control, vol. 45, pp. 2253-2270, 2000.

[6] D. E. Chang, “Some results on stabilizability of controlled Lagrangian systems by energy shaping,” in Proc. 17th IFAC World Congress, Seoul Korea, 2008.

[7] D. E. Chang, “The method of controlled Lagrangian systems: Energy plus force shaping,” SIAM J. Control and Optimization, vol. 48, no. 8, pp. 4821-4845, 2010.

[8] D.E. Chang, “Stabilizability of controlled Lagrangian systems of two degrees of freedom and one degree of underactuation,” IEEE Trans. Automatic Control, vol. 55, no. 3, pp. 664-673, 2010.

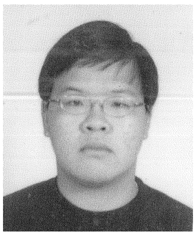
[9] W. M. Ng, D. E. Chang and G. Labahn, “Energy shaping for systems with two degrees of underactuation and more than three degrees of freedom” SIAM J. Optimization and Control, 51(2), pp. 881-905, 20 13.

[10] M. W. Spong, P. Corke, and R. Lozano, “Nonlinear control of the inertia wheel pendulum,” Automatica, vol. 37, pp. 1845-1851, 1999.

[11] R. Ortega, M. W. Spong, F. Gomez-Estern, and G. Blankenstein, “Stabilization of a class of underactuated

mechanical systems via interconnection and damping assignment," IEEE Trans. Automatic Control, vol. 47, pp. 1218-1233, 2002.

[12] S. Nair and N. Leonard, "A normal form for energy shaping: application to the Furuta pendulum," in Proceedings of 41st IEEE Conf. on Decision and Control, Las Vegas, USA, 2002.10



**Wai Man Ng** He received his B.S. and M.Phil. degrees in mathematics from the Chinese University of Hong Kong in 2004 and 2006 respectively, and his Ph.D. degree in applied mathematics from the University of Waterloo, Canada, in 2012. His research interests include partial differential equations and control theory.

**Dong Eui Chang** He received the B.S degree in control and instrumentation engineering and the M.S. degree from electrical engineering, both, from Seoul National University and the Ph.D. in control & dynamical systems from the California Institute of Technology. He is currently associate professor in applied mathematics at the University of Waterloo, Canada. His research interests lie in control, mechanics and various engineering applications.

**Seong-Ho Song** He received the B.S, M.S, and Ph. D degree in measurement and control engineering from Seoul National University. His research interests are nonlinear control, mechatronics, and vision systems.