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H_2 Design of Decoupled Control Systems Based on Directional Interpolations

Kiheon Park† **and Jin-Geol Kim***

Abstract – H_2 design of decoupled control systems is treated in the generalized plant model. The existence condition of a decoupling controller is stated and a parameterized form of all achievable decoupled closed loop transfer matrices is presented by using the directional interpolation approaches under the assumption of simple transmission zeros. The class of all decoupling controllers that yield finite cost function is obtained as a parameterized form and an illustrative example to find the optimal controller is provided.

Keywords: Linear multivariable control, Decoupling design, Interpolation condition, H_2 design

1. Introduction

One important characteristic of the multivariable systems is coupling interactions between input and output variables. Efforts to eliminate these interactions lead to finding controllers that make the transfer matrix from the inputs to the outputs diagonal. Once a closed loop system is decoupled, engineers can exploit the well-established design methods of single-input-single-output control system for each channel. The existence condition of a decoupling controller is now well known. For two-degree-of-freedom (2DOF) configuration, Doseor and Gündes [1] and Lee and Bongiorno [2, 3] show that a decoupling controller always exists if the plant is internally stabilizable. On the other hand, a decoupling controller of the 1DOF control system does not always exist. For 1DOF configuration, necessary and sufficient conditions for the existence of decoupling controllers are presented by [4, 5, 6]. While the existence condition of the decoupling controllers has been sufficiently studied, not many papers treat performance issues of decoupled systems. The robust stability problem of decoupling controllers is first addressed by Safonov and Chen [7]. They obtain the stabilizing controllers maximizing stability margin in the H_∞ norm context under decoupling and output regulation constraints. Brinsmead and Goodwin [8] investigate the inherent limits of a decoupled system via H_2 cost of tracking error. Optimal H_2 design considering both of the tracking error and the plant saturation in decoupling problems is treated by Lee and Bongiorno [2, 3] for 2DOF and 3DOF systems and by Youla and Bongiorno [5] and Bongiorno and Youla [9] for 1DOF configuration with non-unity feedback.

Decoupling design in the generalized plant model is very compact and effective in that the derived formulas are applied to the most various models including 1DOF, 2DOF, and 3DOF configurations with non-unity feedback cases. The existence condition of a decoupling controller for the generalized plant is treated by [10, 11]. As for H_2 decoupling design, Park [12] extends the work of Youla and Bongiorno [5] to the generalized plant model. However, the freedom of controller configuration in [12] is limited to one and hence the potential of the generalized plant model for including all possible feedback configurations is not fully appreciated. In [13], optimal H_2 block decoupling problem is treated with the general setting of 2DOF controller configuration. The class of all decoupling controllers is parameterized by a free parameter and this parameter is used to obtain the optimal controller which minimizes the H_2 norm of the transfer matrix from the reference input to the error. In formulating the cost functional, however, the control variable is not considered.

In this paper, H_2 design of decoupled system for the generalized plant model is treated based on directional interpolation approaches [14]. The assumption of one-degree-of-freedom configuration in [12] is eliminated and the only assumption needed on the plant is the condition of simple transmission zeros. The approach using the vector operation in [10, 11] is not taken here and hence dimension inflation problem is also avoided to describe the existence condition of a decoupling controller. In this paper, the class of all decoupled transfer matrices is parameterized and the optimal H_2 controller is obtained together with the ones that yield finite H_2 cost. It is shown that the optimal controller is strictly proper under the reasonable order assumptions on the generalized plant.

Notations; Throughout the paper, we consider only real rational matrices whose elements are from the set of all real rational functions, which are not necessarily proper.

† Corresponding Author: School of Electronics and Electrical Engineering, SungKyunKwan University, Korea. (khpark@skku.edu)
* School of Electrical Engineering, Inha University, Korea. (john@inha.ac.kr)
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Since this set is the quotient field associated with the ring of real polynomials, we will adopt fractional representations of a real rational matrix using real polynomial matrices. For any real rational matrix $G(s)$, the notation $G_*(s)$ stands for $G'(-s)$ ($G'(s)$ denotes the transpose of $G(s)$). In the partial fraction expression of $G(s)$, the contribution made by all its finite poles in $\mathrm{Re}\,s \le 0$ and $\mathrm{Re}\,s > 0$ are denoted by $\{G\}_+$ and $\{G\}_-$, respectively. The notation $G(s) \le 0(s^\nu)$ means that no entry in $G(s)$ grows faster than s^ν as $s \to \infty$. A rational matrix $G(s)$ is said to be stable if it is analytic in $\mathrm{Re}\,s \ge 0$. The Kronecker product of two matrices is denoted as $G \otimes R$. The *Schur* product $G \circ R$ of two equi-size matrices $G = [g_{ij}]$ and $R = [r_{ij}]$ is the matrix whose *i*-row, *j*-column entry is $g_{ij}\, r_{ij}$. The Khatri-Rao product of two matrices is denoted as $G \odot R$ and is the matrix whose i-column is given by $g_i \otimes r_i$ where g_i and r_i the i-column of G and i-column of R, respectively [15]. The notation $vec\,G$ implies the vector formed by stacking all the columns of the matrix G. For a diagonal matrix, $vecd\,G$ denotes the vector formed by stacking all the diagonal elements of the matrix G. $T_{ba}(s)$ represents the transfer matrix from a to b. The notations C, C_+ and $\bar{C}_+$ denotes the complex number plane, the open right half plane of C and the closure of C_+, respectively. The notation ξ^* denotes the conjugate transpose of a vector ξ.

2. Decoupling Problem and its Solution

The model under consideration is shown in Fig. 1. The vectors $r(s)$ and $w(s)$ are the exogenous inputs. The vector $v(s)$ is the output variable in regard of decoupling design. The vector $z(s)$ is the regulated variable. The vectors $u(s)$ and $y(s)$ are the control input and the measured variable, respectively. The variables $r(s)$ and $v(s)$ are the ones such that the transfer matrix $T_{vr}(s)$ is to be decoupled. In most cases, $r(s)$ is the reference input and $v(s)$ is the plant output.

The transfer matrix of the generalized plant is given by

$$\begin{bmatrix} v \\ z \\ y \end{bmatrix} = P \begin{bmatrix} r \\ w \\ u \end{bmatrix}, P = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} \tag{1}$$

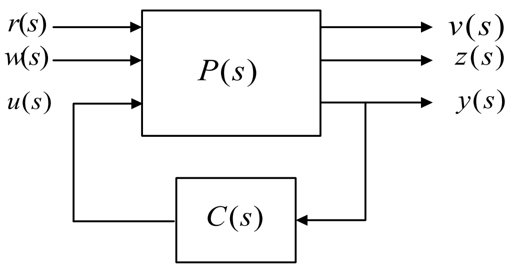

Fig. 1. The generalized plant model.

The variables v and r have the same dimension $n \times 1$. The variables w and u have the dimensions $q_1 \times 1$ and $q_2 \times 1$, respectively. The variables z and y have the dimensions $l_1 \times 1$ and $l_2 \times 1$, respectively. In this paper, we consider the multi-objective design of decoupling and H_2 cost minimization. The following assumption is necessary and sufficient for the existence of a stabilizing controller [16]. In the below, the notation Ψ_P denotes the characteristic denominator [17] of the rational matrix $P(s)$ and $\Psi_P{}^+$ is the polynomial which absorbs all the zeros of Ψ_P in $\bar{C}_+$.

Assumption 1: The general plant block $P(s)$ is free of hidden modes in $\bar{C}_+$ and $\Psi_P{}^+ = \Psi_{P_{22}}{}^+$.

Let

$$P_{22} = A^{-1}(s)B(s) = B_1(s)A_1^{-1}(s) \tag{2}$$

denote polynomial coprime fractional expressions. There always exist polynomial matrices $X(s), Y(s), X_1(s)$ and $Y_1(s)$ such that

$$\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix}\begin{bmatrix} A_1 & -Y \\ B_1 & X \end{bmatrix} = \begin{bmatrix} A_1 & -Y \\ B_1 & X \end{bmatrix}\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3}$$

with $\det X(s) \cdot \det X_1(s) \not\equiv 0$. It is well known [18] that the condition $\Psi_P^+ = \Psi_{P_{22}}^+$ in Assumption 1 is equivalent to the one that the three matrices

$$\begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} - \begin{bmatrix} P_{02} \\ P_{12} \end{bmatrix} A_1 Y_1 \begin{bmatrix} P_{20} & P_{21} \end{bmatrix}, \begin{bmatrix} P_{02} \\ P_{12} \end{bmatrix} A_1 \quad \text{and} \quad A \begin{bmatrix} P_{20} & P_{21} \end{bmatrix} \tag{4}$$

are stable. As explained, the transfer matrix $T_{vr}(s)$ is the one to be decoupled (i.e., to be diagonalized) and it is given by

$$T_{vr}(s) = P_{00} + P_{02}(I - CP_{22})^{-1} CP_{20} \ . \tag{5}$$

In the generalized plant model in Fig. 1, decoupling design is to find stabilizing controllers $C(s)$ that make the transfer matrix $T_{vr}(s)$ diagonal and invertible. The approach taken here for solving the decoupling problem is to characterize the diagonal matrices $T_{vr}(s)$ that admit stabilizing controllers $C(s)$ in (5) and hence we define the realizability of a diagonal matrix $T(s)$ as follows:

Definition 1: A diagonal stable rational matrix $T(s)$ is said to be realizable for the given plant $P(s)$ if there exists a stabilizing controller $C(s)$ that realizes the transfer matrix $T_{vr}(s)$ of the system as the matrix $T(s)$.

In decoupling design, we ask $T_{vr}(s)$ to be diagonal and invertible so that the normal rank of $T_{vr}(s)$ should be n.

In almost all cases, the matrix P_{00} is a null matrix and in this case $T_{vr}(s) = P_{02}(I - CP_{22})^{-1}CP_{20}$. In view of this, it can be concluded that we need assumptions on the rank of the matrices P_{02} and P_{20}. In this paper, we assume the following.

Assumption 2: $n = q_2 = l_2$ and P_{02} and P_{20} are invertible.

Now we seek to find the realizability condition for diagonal matrices $T(s)$. Consider the class of all stabilizing controllers characterized by the formula

$$C(s) = -(X_1 - KB)^{-1}(Y_1 + KA) \tag{6}$$
$$= -(Y + A_1K)(X - B_1K)^{-1}, \tag{7}$$

where $K(s)$ arbitrary real rational stable matrices such that $\det(X_1 - KB) \not\equiv 0$ and $\det(X - B_1K) \not\equiv 0$. In this case we have

$$T_{vr} = T_{00} - T_{02}KT_{20}, \tag{8}$$

where

$$T_{00} = P_{00} - P_{02}A_1Y_1P_{20}, \tag{9}$$
$$T_{02} = P_{02}A_1 \text{ and } T_{20} = AP_{20}. \tag{10}$$

Notice that T_{00}, T_{02} and T_{20}, and hence $T_{vr}(s)$, are stable by the properties in (4). Since $C(s)$ in (6) characterizes the class of all stabilizing controllers, the formula for $T_{vr}(s)$ in (8) describes the structure of realizable $T(s)$ and this is the basis to start for characterizing realizable $T(s)$. Before we proceeding further, we assume the following.

Assumption 3: The matrices T_{02} and T_{20} have the distinct simple transmission zeros $z_i \in \overline{C}_+$, $i = 1 \to m_1$, and $\hat{z}_i \in \overline{C}_+$, $i = 1 \to m_2$, respectively , and $z_i \neq \hat{z}_j$, for any i and j.

Since T_{02} and T_{20} are invertible and z_i and $\hat{z}_i$ are zeros of T_{02} and T_{20}, respectively, we can find nonzero vectors ξ_i and μ_i [19] such that

$$\xi_i^* T_{02}(z_i) = 0 \text{ and } T_{20}(\hat{z}_i)\mu_i = 0. \tag{11}$$

Hence, for a diagonal stable matrix $T(s)$ to be realizable, it is necessary from (8) and (11) that

$$\xi_i^* T(z_i) = \varepsilon_i^* \text{ where } \varepsilon_i^* = \xi_i^* T_{00}(z_i),\ i = 1 \to m_1, \tag{12.a}$$
$$T(\hat{z}_i)\mu_i = \delta_i \text{ where } \delta_i = T_{00}(\hat{z}_i)\mu_i,\ i = 1 \to m_2. \tag{12.b}$$

Since a realizable $T(s)$ is a diagonal matrix, say $T(s) = diag\{t_k(s)\}$, $k = 1 \to n$, the above directional interpolation conditions reduce to the interpolation constraints to the scalar function $t_k(s), k = 1 \to n$. Let's denote the $k-th$ element of a (row or column) vector x_i as x_{ik}. The conditions in (12) are changed to

$$\overline{\xi}_{ik}\, t_k(z_i) = \overline{\varepsilon}_{ik},\ i = 1 \to m_1, \tag{13.a}$$
$$\mu_{ik}\, t_k(\hat{z}_i) = \delta_{ik},\ i = 1 \to m_2, \tag{13.b}$$

for $k = 1 \to n$. Obviously, a rational function $t_k(s)$ to satisfy the interpolation conditions in (13) does not exist if

$$\xi_{ik} = 0 \text{ and } \varepsilon_{ik} \neq 0 \text{, or, } \mu_{ik} = 0 \text{ and } \delta_{ik} \neq 0 \tag{14}$$

for some i. Since the interpolation condition for $t_k(s)$ in (13) is a necessary condition for the existence of a decoupling controller, if the interpolation problem in (13) does not have a solution $t_k(s)$ for some k, a decoupling controller does not exist. On the other hand, if the data sets $\{\xi_{ik}, \varepsilon_{ik}\}$ and $\{\mu_{jk}, \delta_{jk}\}$ are free from the non-existence condition in (14), a solution $t_k(s)$ exists. That is, a decoupling controller exists in the following cases:

$$\xi_{ik} \neq 0 \text{ and } \mu_{jk} \neq 0 \text{ for any } i \text{ and } j, \tag{15.a}$$

or,

$$\text{if } \xi_{ik} = 0 \text{ for some } i\text{, then } \varepsilon_{ik} = 0\text{, and,} \tag{15.b}$$
$$\text{if } \mu_{jk} = 0 \text{ for some } j\text{, then } \delta_{jk} = 0, \tag{15.c}$$

for $1 \le k \le n$. This existence condition is conveniently described by the following rank description. That is, a decoupling controller exists if and only if

$$rank\, \xi_{ik} = rank[\xi_{ik}\ \ \varepsilon_{ik}] \text{ and } rank\, \mu_{jk} = rank[\mu_{jk}\ \ \delta_{jk}] \tag{15.d}$$

for any i, j and k. From now on we will assume that the data sets satisfy the conditions in (15.d).

In the next, we seek to characterize all rational $t_k(s)$ satisfying the interpolation conditions in (13). When both ξ_{ik} and ε_{ik} are zeros as in (15.b), this imposes no interpolation constraint at z_i in (13.a) on $t_k(s)$ and the same is true for μ_{jk} and δ_{jk} in (15.c). So let's define the polynomials $d_{kl}(s)$ and $d_{kr}(s)$ as follows;

$$d_{kl}(s) = \prod_{i=1}^{m_1}(s - z_i)^{\alpha_i},\ \alpha_i = \begin{cases} 1 & if\ \xi_{ik} \neq 0 \\ 0 & if\ \xi_{ik} = 0 \end{cases} \tag{16.a}$$

$$d_{kr}(s) = \prod_{j=1}^{m_2}(s - \hat{z}_j)^{\beta_j},\ \beta_j = \begin{cases} 1 & if\ \mu_{jk} \neq 0 \\ 0 & if\ \mu_{jk} = 0 \end{cases} \tag{16.b}$$

We can now describe all solutions of $t_k(s), k = 1 \to n$, satisfying the interpolation conditions of

$$t_k(z_i) = \overline{\varepsilon}_{ik} / \overline{\xi}_{ik},\ for\ i\ such\ that\ \xi_{ik} \neq 0, \tag{17.a}$$

and

$$t_k(\hat{z}_j) = \delta_{jk} / \mu_{jk}, \text{ for } j \text{ such that } \mu_{jk} \neq 0, \quad (17.b)$$

as

$$t_k(s) = t_{k0}(s) + d_{kl}(s) t_{ka}(s) d_{kr}(s), \quad (18)$$

where $t_{k0}(s)$ is any stable rational function satisfying the interpolation conditions in (17) (when there are no interpolation conditions in (17), we set $t_{k0}(s)=0$) and $t_{ka}(s)$ any arbitrary stable rational function. Since we don't exclude improper $T_{vr}(s)$ from our consideration, $t_{k0}(s)$ can be chosen as a polynomial and an easy choice in this case is the one obtained from Lagrange interpolation formula [20]. Hence we can express $T(s)$ as

$$T(s) = \Delta_0(s) + \Delta_l(s)\Delta(s)\Delta_r(s) \quad (19)$$

where

$$\Delta_0(s) = diag\{t_{10}(s), t_{20}(s), \cdots, t_{n0}(s)\}, \quad (20)$$

$$\Delta_l(s) = diag\{d_{1l}(s), d_{2l}(s), \cdots, d_{nl}(s)\}, \quad (21)$$

$$\Delta_r(s) = diag\{d_{1r}(s), d_{2r}(s), \cdots, d_{nr}(s)\}, \quad (22)$$

and $\Delta(s)$ is an arbitrary $n \times n$ diagonal stable rational matrix. In (19), when $T_{02}(s)$ ($T_{20}(s)$) does not have a zero in $\bar{C}_+$, $\Delta_l = I$ ($\Delta_r = I$).

The matrix $T(s)$ formulated by the parameterized form in (19) is sufficient to be realizable as $T_{vr}(s)$. In fact, consider the matrix $K(s)$ obtained from (8) with $T_{vr}(s)$ in (19). It can be shown in the below that the matrix $K(s)$ formed by

$$K(s) = T_{02}^{-1}(T_{00} - \Delta_0 - \Delta_l \Delta \Delta_r)T_{20}^{-1} \quad (23)$$

$$= K_0(s) - K_a(s)\Delta(s)K_b(s), \quad (24)$$

is stable where

$$K_0 = T_{02}^{-1}(T_{00} - \Delta_0)T_{20}^{-1}, \quad (25)$$

$$K_a = T_{02}^{-1}\Delta_l \text{ and } K_b = \Delta_r T_{20}^{-1}. \quad (26)$$

Since z_i and $\hat{z}_i$ are simple zeros of T_{02} and T_{20}, respectively, they are simple poles of T_{02}^{-1} and T_{20}^{-1}, respectively. Consider their partial fractional expressions

$$T_{02}^{-1} = \sum_{i=1}^{m_1} \frac{M_i}{s - z_i} + F_1(s), \quad T_{20}^{-1} = \sum_{j=1}^{m_2} \frac{N_j}{s - \hat{z}_j} + F_2(s), \quad (27)$$

where F_1 and F_2 are stable. Using the results in Lemma 1 (Appendix), We can easily show that K_a and K_b are stable since $M_i \Delta_l(z_i) = 0$ (notice that $\xi_{ik} d_{kl}(z_i) = 0$ for any i and k) and $\Delta_r(\hat{z}_j)N_j = 0$. To show that K_0 is stable, we insert the equalities in (27) into (25) so that

$$\begin{aligned} K_0 = &(\sum_{i=1}^{m_1} \frac{M_i}{s - z_i})(T_{00} - \Delta_0)F_2(s) + F_1(s)(T_{00} - \Delta_0)(\sum_{j=1}^{m_2} \frac{N_j}{s - \hat{z}_j}) \\ &+ (\sum_{i=1}^{m_1} \frac{M_i}{s - z_i})(T_{00} - \Delta_0)(\sum_{j=1}^{m_2} \frac{N_j}{s - \hat{z}_j}) \\ &+ F_1(s)(T_{00} - \Delta_0)F_2(s). \end{aligned} \quad (28)$$

It now follows from (64) of Lemma 1 (Appendix) that $M_i(T_{00}(z_i) - \Delta_0(z_i)) = k_i \xi_i^*(T_{00}(z_i) - \Delta_0(z_i)) = k_i(\varepsilon_i^* - \varepsilon_i^*) = 0$ and hence the matrix $M_i(T_{00}(s) - \Delta_0(s))$ has the factor $(s - z_i)$. Similarly, we can show that the matrix $(T_{00}(s) - \Delta_0(s))N_j$ has the factor $(s - \hat{z}_j)$. By these observations, we can also conclude that the matrix $M_i(T_{00}(s) - \Delta_0(s))N_j$ has the factor $(s - z_i)(s - \hat{z}_j)$. Therefore, the first, the second, and the third terms of (28) are stable. The fourth term is obviously stable and this completes the proof and now we can state the following theorem without further proof.

Theorem 1: Under Assumptions 1~3, a decoupling controller for the plant (1) exists if and only if the data sets $\{\xi_{ik}, \varepsilon_{ik}\}$ and $\{\mu_{jk}, \delta_{jk}\}$ satisfy the conditions in (15.d). When decoupling controllers exist, the class of all decoupled transfer matrices $T_{vr}(s)$ is characterized by the formula

$$T_{vr}(s) = T(s) = \Delta_0(s) + \Delta_l(s)\Delta(s)\Delta_r(s) \quad (29)$$

as in (19).

From the previous developments, we see that not all non-minimum phase zeros of $T_{02}(s)$ and $T_{20}(s)$ appear as the zeros of realizable $T_{vr}(s)$. In view of (18), the non-minimum phase zero z_i ($\hat{z}_j$) appears as a zero at the k-th channel of $T_{vr}(s)$, $t_k(s)$, only when

$$\xi_{ik} \neq 0 \text{ and } \varepsilon_{ik} = 0 \ (\mu_{ik} \neq 0 \text{ and } \delta_{ik} = 0), \quad (30)$$

which is the condition that both $t_{k0}(s)$ and $d_{kl}(s)$ ($d_{kr}(s)$) have the factor $(s - z_i)$ ($(s - \hat{z}_j)$).

3. H_2 Design of Decoupled Control Systems

In this section, we formulate an H_2 design problem for the decoupled systems and present its solution. The H_2 design problem for the system in Fig. 1 is to find the decoupling controller which minimizes a given quadratic cost associated with the regulated variable z when the system is stimulated by the exogenous input $[r'(s)\ w'(s)]'$. To allow the exogenous input to include shape-deterministic components, we assume that $q(t) = [r'(t)\ w'(t)]'$ is the output of the square block $P_q(s)$

driven by a stochastic vector $q_o(t)$ so that

$$q(s) = \left[r'(s) \ w'(s) \right]' = P_q(s) \ q_o(s) . \tag{31}$$

Although allowing the block $P_q(s)$ to possess the poles in $\overline{C}_+$ makes the H_2 problem more general [21], we assume here for simplicity that $P_q(s)$ is stable and $q_o(t)$ is a white noise vector with spectral density of unity. It now follows that

$$\begin{bmatrix} z \\ y \end{bmatrix} = \tilde{P} \begin{bmatrix} q_o \\ u \end{bmatrix}, \quad \tilde{P} = \begin{bmatrix} \tilde{P}_{11} & P_{12} \\ \tilde{P}_{21} & P_{22} \end{bmatrix}, \tag{32}$$

where

$$\tilde{P}_{11} = \left[P_{10} \ P_{11} \right] P_q \ \text{ and } \ \tilde{P}_{21} = \left[P_{20} \ P_{21} \right] P_q . \tag{33}$$

Let's denote the transfer matrix from q_o to z as $T_{zo}(s)$. With the setup in (31) and the assumption on $q_o(t)$, a meaningful quadratic cost that considers the transient performance and the steady state performance is given by

$$E = \left\| T_{zo} \right\|^2 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} Tr(T_{zo*}(s) T_{zo}(s)) ds , \tag{34}$$

where

$$T_{z0}(s) = \tilde{P}_{11} + P_{12}(I - CP_{22})^{-1} C \tilde{P}_{21} . \tag{35}$$

The H_2 problem here is to find the decoupled loop transfer matrix $T(s)$ that minimizes the cost index in (34). Since P_{02} and P_{20} are invertible, it follows from (5) that

$$(I - CP_{22})^{-1} C = P_{02}^{-1} (T_{vr} - P_{00}) P_{20}^{-1} \tag{36}$$

and inserting (36) with the parameterized formula for $T_{vr}(s)$ in (29) to (35) yields

$$T_{z0}(s) = T_{11}(s) + T_a(s) \varDelta(s) T_b(s) , \tag{37}$$

where

$$T_{11} = \tilde{P}_{11} - P_{12} P_{02}^{-1} (P_{00} - \varDelta_0) P_{20}^{-1} \tilde{P}_{21} , \tag{38}$$

$$T_a = P_{12} P_{02}^{-1} \varDelta_l \ , \ T_b = \varDelta_r P_{20}^{-1} \tilde{P}_{21} , \tag{39}$$

And $\varDelta(s)$ an arbitrary diagonal stable rational matrix. Knowing that $K_0(s)$, $K_a(s)$ and $K_b(s)$ in (25) and (26) are stable, we can easily show that T_{11}, T_a and T_b are stable (see Lemma 2 in Appendix). Notice that the expression for $T_{zo}(s)$ in (37) is the standard form to develop the optimal H_2 solution and it can be obtained under the following standard assumptions [12].

Assumption 4: $\tilde{P}_{11}$ is strictly proper.

Assumption 5: $T_a(j\omega)$ has full column rank for finite ω.

Assumption 6: $T_b(j\omega)$ has full row rank for finite ω.

Assumption 7: $P_{12}(s)$ and $\tilde{P}_{21}(s)$ behave as $P_{12}(s) \to M_{12} s^k$ and $\tilde{P}_{21}(s) \to M_{21} s^l$ as $s \to \infty$ with the condition that M_{12} and M_{21} have full column rank and full row rank, respectively, and $k + l \geq 0$. The transfer matrices $P_{00}(s)$, $P_{02}(s)$, $P_{20}(s)$ and $P_{22}(s)$ are proper.

Let $\Omega(s)$ be the Wiener-Hopf spectral factor of the equation

$$(T_b T_{b*})' \circ (T_{a*} T_a) = \Omega_*(s) \Omega(s) \tag{40}$$

and define

$$U(s) = (T_b' \odot T_a) \Omega^{-1} . \tag{41}$$

It should be noticed that $U_* U = I$ and hence U is inner. Now we present the H_2 solution formula whose proof is omitted since it proceeds as similarly as that of [12].

Theorem 2: Suppose that Assumptions 1~7 are satisfied and the plant $P(s)$ admits the existence condition in (15.d). The class of all decoupling loop transfer matrices that yield finite cost is given by

$$vecd \ T(s) = \varDelta_l \varDelta_r \Omega^{-1} (\gamma_0 + \{\gamma\}_- - \{\sigma\}_+ + f) \tag{42}$$

where

$$\sigma = \Omega_*^{-1} \varDelta_{r*} \varDelta_{l*} \ \{ (\tilde{P}_{11} \tilde{P}_{21*} P_{20*}^{-1})' \circ (P_{02*}^{-1} P_{12*}) \} vecd \, I_{l_1} \tag{43}$$

$$\gamma = \Omega \varDelta_l^{-1} \varDelta_r^{-1} vecd \varDelta_0 - \gamma_0 , \tag{44}$$

$$\gamma_0 = U_* vec(T_a \varDelta_l^{-1} P_{00} \varDelta_r^{-1} T_b) , \tag{45}$$

and $f(s)$ an arbitrary strictly proper stable vector. The optimal loop transfer matrix is the one with $f = 0$ and the cost E for the transfer matrix in (42) is given by $E = \tilde{E} + \|f\|^2 \geq \tilde{E}$, where $\tilde{E}$ denotes the cost for the optimal one. The loop transfer matrix $T(s)$ in (42) is proper. The corresponding controller $C(s)$ is obtained form the equation

$$C = P_{02}^{-1} (T - P_{00}) P_{20}^{-1} (I + P_{22} P_{02}^{-1} (T - P_{00}) P_{20}^{-1})^{-1} \tag{46}$$

and it is strictly proper.

We remark that when P_{00} is diagonal, γ_0 in (45) simplifies to $\gamma_0 = \Omega \, vecd \, (\varDelta_l^{-1} \varDelta_r^{-1} P_{00})$. In most cases, $P_{00} = 0$ and in this case $T(s)$ in (42) is strictly proper.

4. An Illustrative Example

As a brief example to show the procedures to obtain the optimal transfer matrix $T(s)$ in (42), we consider the case that the generalized plant is given by

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|c|c} 0 & 0 & \hat{P} \\ \hline I & 0 & -\hat{P} \\ 0 & 0 & I \\ \hline I & -I & -\hat{P} \end{array}\right]. \quad (47)$$

This case corresponds to the one-degree-of-freedom controller system in which a measurement noise exists and the regulated variables are set as the tracking error and the plant input. Decoupling and minimization problems are as explained in the previous sections. Suppose that $P_q(s) = diag\{P_r(s), I\}$ and $\hat{P}$ and P_r are given as

$$\hat{P} = \begin{bmatrix} \frac{s-2}{s+2} & \frac{-1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+1} \end{bmatrix}, \quad P_r = \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}. \quad (48)$$

Since $\Psi_P{}^{+} = \Psi_{P_{22}}{}^{+} = 1$ in (47), Assumption 1 is satisfied and Assumption 2 is obviously satisfied. We can find coprime fractions for $-\hat{P}(s)$ as

$$A = \begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix}, \quad B = \begin{bmatrix} 2-s & 0 \\ 0 & -1 \end{bmatrix}, \quad (49)$$

$$A_1 = \begin{bmatrix} s+2 & -0.25 \\ 0 & s+1 \end{bmatrix} \text{ and } B_1 = \begin{bmatrix} 2-s & 0.25 \\ 0 & -1 \end{bmatrix}. \quad (50)$$

We see that $T_{02} = B_1$ has a simple zero at $z_1 = 2$ and $T_{20} = A$ has no non-minimum phase zero (Assumption 3 is satisfied). The output zero vector ξ_1^* and the value ε_1^* are obtained as $\xi_1^* = \begin{bmatrix} 1 & 0.25 \end{bmatrix}$ and

$$\varepsilon_1^* = \xi_1^* T_{00}(2) = \xi_1^* B_1(2) Y_1(2) = \begin{bmatrix} 0 & 0 \end{bmatrix}. \quad (51)$$

The interpolation conditions for $t_1(s)$ and $t_2(s)$ are given by $t_1(2) = 0$ and $t_2(2) = 0$. Therefore $t_1(s) = t_2(s) = 0$ and the decoupled transfer matrix in (29) is obtained as $T(s) = \Delta_0 + \Delta_l \Delta \Delta_r$ with $\Delta_0 = 0$, $\Delta_l = (s-2)I_2$ and $\Delta_r = I_2$. It can be confirmed that this result is identical with that of [5] in which all realizable $T(s)$ is described by the formula $T = \Delta_\theta \Delta_\alpha + \Delta_\theta \Delta \Delta_\psi$. In fact, we can easily obtain that $\Delta_\alpha = 0$, $\Delta_\theta = (s-2) I_2$, and $\Delta_\psi = I_2$ by following the definitions of them in [5].

It now follows from (39) that

$$T_a{}' = \begin{bmatrix} 2-s & 0 & s+2 & 0 \\ 0 & 2-s & 1 & (s+1)(s-2) \end{bmatrix} \quad (52)$$

and

$$T_b = \begin{bmatrix} \frac{1}{s+2} & 0 & -1 & 0 \\ 0 & \frac{1}{s+1} & 0 & -1 \end{bmatrix}. \quad (53)$$

It is easy to verify that Assumptions 4~7 are satisfied and the Wiener-Hopf factor Ω and other values in Theorem 2 are obtained as

$$\Omega = diag\{\sqrt{2}(s+\sqrt{5}), \frac{(s+2)(s+\sqrt{3})^2}{s+1}\}, \quad (54)$$

$$\{\sigma\}_+ = \begin{bmatrix} \frac{k_1}{s+2} & \frac{k_2}{s+1} \end{bmatrix}', \quad \gamma = 0 \text{ and } \gamma_0 = 0 \quad (55)$$

With $k_1 = \frac{-1}{\sqrt{2}(2+\sqrt{5})}$ and $k_2 = \frac{-1}{(1+\sqrt{3})^2}$. The optimal decoupled transfer matrix in (42) is given by

$$T = diag\{\frac{k_1(2-s)}{\sqrt{2}(s+2)(s+\sqrt{5})}, \frac{k_2(2-s)}{(s+2)(s+\sqrt{3})^2}\} \quad (56)$$

and the corresponding controller matrix $C(s)$ can be calculated from (46).

5. Conclusion and Discussion

Decoupling design of lineal multivariable control systems is treated for the generalized plant model within the H_2 framework. A necessary and sufficient condition for the existence of decoupling controllers is obtained based on interpolation approaches. It is shown that directional interpolation problems associated with the decoupling design are changed to simple interpolation constrains of scalar functions whose solutions can be easily obtained. The class of all decoupled closed loop transfer matrices is parameterized and the optimal transfer matrix is obtained using this parameterized formula. The existence condition of a decoupling controller for the generalized plant is treated also in [10, 11] by using vector operations. A major disadvantage of those works is that the vector operation causes dimension inflation and hence the method suffers a difficulty when applied to large-size plants. The existence condition formula developed in this paper is free of dimension inflation.

In this paper, the matrices T_{02} and T_{20} are assumed to be square but extension to the rectangular case can be readily done if the transformation of T_{02} and T_{20} using unimodular matrices in [3, 9, 11] is adopted. The constraint that $z_i \neq \hat{z}_j$ for any i, j in Assumption 3 can be loosened but it requires more complex descriptions in Theorem 1 and it will be presented in future publications. In Assumption 3, non-minimum phase zeros of T_{02} and T_{20} are assumed to be simple and one of future research topics would be generalization of the results in this paper to the multiple zero case. It is expected that the methods of the multiple directional interpolation approach [22] and the generalized characteristic vectors in [23, 24] will play a role for the generalization.

Appendix

Lemma 1: Suppose that $G(s)$ is an $n \times n$ stable rational matrix with full rank and it has distinct simple zeros $z_i \in \bar{C}_+$, $i = 1 \to m$. Let ξ_i^* be an output zero vector of z_i such that $\xi_i^* G(z_i) = 0$ and $G^{-1}(s)$ be denoted by the partial fractional expression

$$G^{-1}(s) = \sum_{i=1}^{m} \frac{M_i}{s - z_i} + F(s), \tag{57}$$

where M_i is the residue matrix at z_i and $F(s)$ is a stable matrix. The j-th row of M_i is of the form $k_{ij}\xi_i^*$, $k_{ij} \in C$ so that

$$M_i = k_i \xi_i^*, \;\; k_i = \left[k_{i1} \; k_{i2} \cdots k_{in}\right]' \neq 0 \tag{58}$$

Proof: Notice that the rank of $G(z_i)$ is $n-1$, which can be easily concluded by considering the Smith-McMillan form of $G(s)$. Next, since $G^{-1}G = I$, it follows that the matrix

$$\sum_{i=1}^{m} \frac{M_i}{s - z_i} G(s) = I - F(s)G(s) \tag{59}$$

must be stable and it is required that $M_i\, G(s)/(s - z_i)$ be stable for each i. This requires that $M_i\, G(z_i) = 0$ and the fact that the nullity of $G(z_i)$ is 1 yields that each row of M_i is either 0 or proportional to ξ_i^*. Q.E.D.

The similar result for input zero vector can be stated. Let μ_i be an column vector such that $G(z_i)\mu_i = 0$ and N_i be the residue matrix of $G^{-1}(s)$ at z_i. Then

$$N_i = \mu_i \hat{k}_i, \;\; \hat{k}_i = [\hat{k}_{i1} \; \hat{k}_{i2} \cdots \hat{k}_{in}] \neq 0 \;\;, \;\; \hat{k}_{ij} \in C. \tag{60}$$

Lemma 2: The matrices T_{11}, T_a and T_b in (38) and (39) are stable.

Proof: It is previously shown that the matrix K_0 in (25) is stable. That is, the matrix

$$K_0 = T_{02}^{-1}(T_{00} - \Delta_0)T_{20}^{-1} = A_1^{-1}P_{02}^{-1}(P_{00} - \Delta_0)P_{20}^{-1}A^{-1} - Y_1 A^{-1} \tag{61}$$

is stable. It follows from (38) that

$$T_{11} = [P_{10} \; P_{11}]P_q - P_{12}P_{02}^{-1}(P_{00} - \Delta_0)P_{20}^{-1}[P_{20} \; P_{21}]P_q \tag{62}$$

$$= [P_{10} \; P_{11}]P_q - P_{12}A_1A_1^{-1}P_{02}^{-1}(P_{00} - \Delta_0)P_{20}^{-1}A^{-1}A[P_{20} \; P_{21}]P_q \tag{63}$$

$$= [P_{10} \; P_{11}]P_q - P_{12}A_1(Y_1A^{-1} + K_0)A[P_{20} \; P_{21}]P_q \tag{64}$$

$$= \{[P_{10} \; P_{11}] - P_{12}A_1Y_1[P_{20} \; P_{21}] - P_{12}A_1K_0A[P_{20} \; P_{21}]\}P_q \tag{65}$$

Clearly, this matrix is stable by the stable properties in (4) and the fact that K_0 and P_q are stable. Stabilities of T_a and T_b are obvious since $T_a = P_{12}P_{02}^{-1}\Delta_l = P_{12}A_1K_a$ and $T_b = \Delta_r P_{20}^{-1}\tilde{P}_{21} = K_b A\tilde{P}_{21}$, where K_a, K_b, $P_{12}A_1$, and $A\tilde{P}_{21}$ are stable. Q.E.D.

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Kiheon Park He received his B.S. and M.S. degrees in Electrical Engineering from Seoul National University, Korea, in 1978 and 1980, respectively, and his Ph.D. degree in System Engineering from Polytechnic University, NY, in 1987. From 1988 to 1990, he worked for the Electronic and Telecommunication Research Institute (ETRI), Dae-jeon, Korea. Since 1990, he has been with the School of Electronics and Electrical Engineering at SungKyunKwan University, Suwon, Korea. His research interests include linear multivariable control theory, optimal Wiener-Hopf control and decoupling controller design.

Jin-Geol Kim He received the B.S. degree in Electrical Engineering from Seoul National University, Seoul, Korea in 1978. He received the M.S. and Ph.D. degrees in Electrical and Computer Engineering from the University of Iowa, Iowa City, IA, U.S.A. in 1984 and 1988, respectively. He also received the M.S. degree in Mathematics from the University of Iowa, Iowa City, IA, U.S.A. in 1985. Since 1988 he has been with the School of Electrical Engineering, Inha University, Inchon, Korea. His main research interests include nonlinear control algorithms, mobile/biped robots, and intelligent control systems.