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ON A GENERAL CLASS OF OPTIMAL FOURTH-ORDER MULTIPLE-ROOT FINDERS

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Abstract. A general class of two-point optimal fourth-order methods is proposed for locating multiple roots of a nonlinear equation. We investigate convergence analysis and computational properties for the family. Special and simple cases are considered for real-life applications. Numerical experiments strongly verify the convergence behavior and the developed theory.

1. Introduction

In many scientific problems, it is often worth to develop a generic family of iterative methods for a given nonlinear equation $f(x) = 0$. Such root-finding methods can be found in works done by Dong.[2], Li et al.[4-5], Neta et al.[6-8], Sharma[9], Victory et al.[10] and Zhou et al.[13]. Special attention is paid to the work of Zhou et al. who have recently carried out an analysis on developing a class of fourth-order optimal multiple root-finders shown below by (1.1):

$$\begin{cases} y_n = x_n - \gamma \cdot \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - Q(\frac{f'(y_n)}{f'(x_n)}) \cdot \frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{1.1}$$

where $\gamma \in \mathbb{R}$ and $Q(\cdot) \in \mathbf{C}^2(\mathbb{R})$ with $\mathbf{C}^2(\mathbb{R})$ as a class of functions whose derivatives exist up to the second order and are continuous on $\mathbb{R}$.

Definition 1.1. (Error equation, asymptotic error constant, order of convergence)
Let $x_0, x_1, \cdots, x_n, \cdots$ be a sequence of numbers converging to α. Let $e_n = x_n - \alpha$ for $n = 0, 1, 2, \cdots$. If constants $p \geq 1,\ c \neq 0$ exist in such a way that $e_{n+1} = c\ {e_n}^p + O(e_n)^{p+1}$ called the *error equation*, then p and

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$\eta = |c|$ are said to be the *order of convergence* and the *asymptotic error constant*, respectively. It is easy to find $c = \lim_{n\to\infty} \frac{e_{n+1}}{{e_n}^p}$. Some authors call c the asymptotic error constant.

It has been shown that (1.1) defines an optimal fourth-order methods with $\gamma = \frac{2m}{m+2}$ and requirements

$$\begin{cases} Q(u) = m, \\ Q'(u) = -\frac{1}{4}m^{3-m}(m+2)^m, \\ Q''(u) = \frac{1}{4}m^4\left(\frac{m}{m+2}\right)^{-2m}, \end{cases} \tag{1.2}$$

where $u = \left(\frac{m}{m+2}\right)^{m-1}$. The treatment, unfortunately, has not investigated the corresponding error equation for (1.1), whose information is usually very valuable for the complete analysis of convergence order.

In this paper, we concern ourselves with the extension of (1.1) by developing a more general family of fourth-order methods which are of optimal order in the sense of Kung-Traub[11]. Assuming that the multiplicity m of a root α is known for a nonlinear equation $f(x) = 0$, we propose a general class of two-point optimal fourth-order methods for locating multiple roots in the following form:

$$\begin{cases} y_n = x_n - \gamma \cdot \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - K_f(h_n, v_n),\ h_n = \frac{f(x_n)}{f'(x_n)},\ v_n = \frac{f'(y_n)}{f'(x_n)}, \end{cases} \tag{1.3}$$

where $\gamma \in \mathbb{R}$ and $K_f : \mathbb{C}^2 \to \mathbb{C}$: is analytic in a region containing $(0, \rho)$ with $\rho \in \mathbb{C}$ being a constant satisfying $\frac{f'(y_n)}{f'(x_n)} = \rho + O(e_n),\ e_n = x_n - \alpha$. When $K_f(h, v) = h \cdot T_f(v)$ is chosen, observe that method (1.3) indeed reduces to (1.1) by taking $Q(v) = \gamma + T_f(v)$. By noting $\frac{f(x_n)}{f'(x_n)} = O(e_n),\ \frac{f'(y_n)}{f'(x_n)} = \rho + O(e_n)$, it is useful to develop $K_f(h, v)$ about $(0, \rho)$ up to fourth-order terms in order to design fourth-order methods. In the course of development, we will get as much information about the structure of K_f as possible. We will also derive explicitly the corresponding error equations for (1.3) in general terms.

Observe that proposed scheme (1.3) requires three new function evaluations for $f(x_n), f'(x_n), f'(y_n)$ at two points x_n, y_n per iteration. Consequently, it has an optimal convergence of order four. In the following section, further analysis with this observation will lead to the successful development of a new optimal family of fourth-order methods.

2. Method development and convergence analysis

The following Theorem 2.1 best describes the method development and convergence analysis regarding proposed scheme (1.3).

THEOREM 2.1. *Assume that* $f : \mathbb{C} \to \mathbb{C}$ *has a multiple root* α *of multiplicity* m *for a given* $m \in \mathbb{N}$ *and is analytic in a region containing* α. *Let* $\kappa = \left(\frac{m}{m+2}\right)^m$ *and* $\Delta = f^{(m)}(\alpha)$ *and* $\theta_j = \frac{f^{(m+j)}(\alpha)}{f^m(\alpha)}$ for $j = 1, 2, 3, \cdots$. *Let* $\theta_1\theta_2\theta_3 \neq 0$ *and* x_0 *be an initial guess chosen in a sufficiently small neighborhood of* α. *Let* $K_f : \mathbb{C}^2 \to \mathbb{C}$ *be analytic in a region containing* $(0, \rho)$ *with* $\rho = \left(\frac{m}{m+2}\right)^{m-1}$. *Let* $K_{ij} = \frac{1}{i!j!}\frac{\partial^{i+j}K_f(h,v)}{\partial h^i \partial v^j}\Big|_{(h=0,v=\rho)}$ *for* $0 \leq i, j \leq 4$. *If* $K_{00} = K_{01} = K_{02} = K_{03} = K_{20} = K_{21} = K_{30} = 0, K_{10} = \frac{m^2}{2+m}, K_{11} = -\frac{m^3}{4\kappa}, K_{12} = \frac{m^4}{8\kappa^2}$, *then iterative scheme (1.3) defines a family of fourth-order multipoint optimal methods satisfying the error equations below: for* $n = 0, 1, 2, \cdots$

$$e_{n+1} = -\left(\frac{K_{40}}{m^4} + \frac{\theta_1\theta_2}{m(m+1)^2(m+2)} - \frac{m\theta_3}{(m+1)(m+2)^3(m+3)} - \frac{4K_{31}\theta_1\kappa}{m^6(m+1)} + \Omega\right) e_n^4 + O(e_n^5), \tag{2.1}$$

where $\Omega = \frac{16K_{22}\theta_1^2\kappa^2}{m^8(m+1)^2} - \frac{\theta_1^3(12m^5-2m^6+2m^7+2m^8+m^9+192K_{13}\kappa^3)}{3m^{10}(m+1)^3} + \frac{256K_{04}\theta_1^4\kappa^4}{m^{12}(m+1)^4}$.

Proof. Taylor series expansion of $f(x_n)$ about α up to $(m+4)^{th}$-order terms yields with $f(\alpha) = 0$:

$$f(x_n) = \frac{\Delta e_n^m}{m!}\{1 + A_1e_n + A_2e_n^2 + A_3e_n^3 + A_4e_n^4 + O(e_n^5)\}, \tag{2.2}$$

where $A_j = \frac{m!\theta_j}{(m+j)!}$ for $j = 1, 2, \cdots$. With $B_j = \frac{m+j}{m}A_j$ for $j = 1, 2, \cdots$, we also find:

$$f'(x_n) = \frac{\Delta e_n^{m-1}}{(m-1)!}\{1 + B_1e_n + B_2e_n^2 + B_3e_n^3 + B_4e_n^4 + O(e_n^5)\}. \tag{2.3}$$

For simplicity, e_n will be denoted by e throughout the proof. With the aid of symbolic computation of Mathematica, we have with $h_n = \frac{f(x_n)}{f'(x_n)}$:

$$y_n = x_n - \gamma \cdot h_n = \alpha + te + K_1(1-t)e^2 + K_2(1-t)e^3 + K_3(1-t)e^4 + K_4(1-t)e^5 + O(e^6), \tag{2.4}$$

where $t = 1 - \frac{\gamma}{m}$, $K_1 = B_1 - A_1$, $K_2 = B_2 - A_2 - B_1K_1$, $K_3 = B_3 - A_3 - B_2K_1 - B_1K_2$, $K_4 = B_4 - A_4 - B_3K_1 - B_2K_2 - B_1K_3$.

In view of the fact that $f(y_n) = f(x_n)|_{e_n \to (y_n - \alpha)}$, we get:

$$\begin{aligned}
f(y_n) = \frac{\Delta e^m}{m!}[t^m + t^{m-1}(A_1 t^2 + K_1 m(1-t))e + \frac{1}{2}t^{m-2}[K_1^2 m(m-1)(-1+t)^2 \\
-2A_1K_1(m+1)(-1+t)t^2 + 2t(A_2t^3 + K_2m(1-t))]e^2 \\
+\frac{1}{6}t^{m-3}[-K_1^3m(m-1)(m-2)(-1+t)^3 + 3A_1K_1^2m(m+1)(-1+t)^2t^2 \\
-6K_1(-1+t)t(A_2(m+2)t^3 + K_2m(-1+m+t-mt)) \\
+6t^2(K_3m(1-t) + t(-A_1K_2(m+1)(-1+t) + A_3t^3))]e^3 \\
+\frac{1}{24}t^m[12A_2(m+2)(-1+t)(K_1^2(m+1)(-1+t) - 2K_2t) \\
+\frac{1}{t^4}(K_1^4m(m-1)(m-2)(m-3)(-1+t)^4 \\
-12K_1^2K_2m(m-1)(m-2)(-1+t)^3t - 4A_1K_1^3m(m^2-1)(-1+t)^3t^2 \\
-24K_1(-1+t)t^2(K_3m(-1+m+t-mt) + t(-A_1K_2m(m+1)(-1+t) \\
+A_3(m+3)t^3)) + 12t^2(K_2^2m(m-1)(-1+t)^2 \\
+2t(K_4m(1-t) + t(-A_1K_3(m+1)(-1+t) + A_4t^4))))]e^4 + O(e^5)]. \quad (2.5)
\end{aligned}$$

Similarly, we can obtain $f'(y_n) = f'(x_n)|_{e_n \to (y_n - \alpha)}$ in terms of Δ, m, t, A_i, K_i. It follows that

$$\begin{aligned}
v_n &= \frac{f'(y_n)}{f'(x_n)} \\
&= t^{m-1} + (-1+t)t^{m-2}(K_1(1-m) + B_1t)e \\
&\quad +\frac{1}{2}(-1+t)t^{m-3}[K_1^2(m-1)(m-2)(-1+t) \\
&\quad +2B_1K_1t(-1+m-mt) + 2t(K_2(1-m) \\
&\quad +t(-B_1^2 + B_2(1+t)))]e^2 + \Psi_3e^3 + \Psi_4e^4 + O(e^5), \quad (2.6)
\end{aligned}$$

where $\Psi_3 = \frac{1}{6}(-1+t)t^{m-4}[-K_1^3(m-1)(m-2)(m-3)(-1+t)^2 + 3B_1K_1^2(m-1)(2+m(-1+t))(-1+t)t + K_3(1-m) + B_1^3t + B_3t(1+t+t^2) - B_1(K_2(1+m(-1+t)) + B_2t(2+t))) + 6K_1t(K_2(m-1)(m-2)(-1+t) + t(B_1^2(1+m(-1+t)) + B_2(m-1-(m+1)t^2)))]$,
$\Psi_4 = \frac{1}{24}t^{m-5}[K_1^4(m-1)(m-2)(m-3)(m-4)(-1+t)^3 - 4B_1K_1^3(m-1)(m-2)(3+m(-1+t))(-1+t)^2t + 24K_1t^2(K_3(m-1)(m-2)(-1+t) + B_1^3t(-1+m-mt) + B_3t(m-1-(m+2)t^3) + B_1(K_2(m-1)(2+m(-1+t))(-1+t) + B_2t(2+t^2+m(-2+t+t^2)))) + 12t^2(K_2^2(m-1)(m-2)(-1+t) + 2K_2t(B_1^2(1+m(-1+t)) + B_2(m-1-(m+1)t^2)) + 2t(K_4(1-m) - B_1^4t + B_1^2B_2t(3+t) + t(1+t)(-B_2^2 + B_4(1+t^2)) - B_1(K_3(1+m(-1+t)) + B_3t(2+t+t^2)))) + 12K_1^2(-1+t)t(-K_2(m-1)(m-2)(m-3)(-1+t) + t(-B_1^2(m-1)(2+m(-1+t)) + B_2(-2+m(3-m+(1+m)t^2))))]$.

By use of the fact that $O(t^{m-1}) = O(h) = O(e)$, Taylor expansion of $K_f(h,v)$ about $(0, t^{m-1})$ up to the fourth-order terms in both variables yields: after removing terms of degree higher than 4 in e by setting $K_{\ell j} = 0$ for all ℓ, j satisfying $\ell + j \geq 5$

$$\begin{aligned} K_f(h,v) = {} & K_{00} + K_{01}(v - t^{m-1}) + K_{02}(v - t^{m-1})^2 + K_{03}(v - t^{m-1})^3 \\ & + K_{04}(v - t^{m-1})^4 + [K_{10} + K_{11}(v - t^{m-1}) + K_{12}(v - t^{m-1})^2 \\ & + K_{13}(v - t^{m-1})^3]h + [K_{20} + K_{21}(v - t^{m-1}) + K_{22}(v - t^{m-1})^2]h^2 \\ & + [K_{30} + K_{31}(v - t^{m-1})]h^3 + K_{40}h^4 + O(e^5). \end{aligned}$$

By use of (2.2), (2.3) and (2.6) with $K_f(h_n, v_n)$, we obtain x_{n+1} in (1.3) as follows:

$$\begin{aligned} x_{n+1} = {} & \alpha - K_{00} + \big[-\frac{K_{10}}{m} + t - K_{01}(-1+t)t^{m-2}(K_1(1-m) + B_1 t)\big] e \\ & + \phi_2 e^2 + \phi_3 e^3 + \phi_4 e^4 + O(e^5), \qquad (2.7) \end{aligned}$$

where ϕ_i is a multivariate function in $t, m, A_j, B_j, K_j, K_{jk}$ denoted by $\phi_i(t, m, A_j, B_j, K_j, K_{jk})$ with $2 \leq j, k \leq 4$ for $2 \leq i \leq 4$. Identifying $e_{n+1} = x_{n+1} - \alpha$ in (2.7), we first set $K_{00} = 0$ and $-\frac{K_{10}}{m} + t - K_{01}(-1+t)t^{m-2}(K_1(1-m) + B_1 t) = 0$, together with $\phi_2 = \phi_3 = 0$ to obtain fourth-order convergence. As a result, we find that

$$K_{10} = mt - \frac{K_{01}(-1+t)t^{m-2}(m(-1+t)+1+t)}{m+1}\theta_1. \qquad (2.8)$$

Substituting $A_1, A_2, B_1, B_2, K_1, K_2$ and (2.8) with $K_{00} = 0$ into $\phi_2 = 0$ yields after simplifications:

$$\begin{aligned} & -\frac{K_{20}}{m^2} + \frac{m + K_{11}(1-t)t^{m-2}(m(-1+t)+1+t)}{m^2(m+1)}\theta_1 \\ & + \frac{K_{01}t^{m-2}(2 - 2m + 3mt - (2+m)t^3)}{m(m+1)(m+2)(m(t-1)+1+t)}\theta_2 + \psi \cdot (t-1)t^{m-3} \cdot \theta_1^2 = 0, \quad (2.9) \end{aligned}$$

with $\psi = \frac{K_{01}(2+m(m-3-5(m-1)t+4(1+m)t^2)) - 2K_{02}(-1+t)t^{m-1}(m(-1+t)+1+t)^2}{2m^2(m+1)^2}$.

For Eqn.(2.9) to hold independently of θ_1, θ_2, we must have:

$$\begin{aligned} K_{11} &= \frac{mt^{2-m}}{(t-1)(1+m(t-1)+t)}, \\ K_{20} &= K_{01} = K_{02} = 0, K_{10} = mt. \qquad (2.10) \end{aligned}$$

Similarly by substituting $A_1, A_2, B_1, B_2, K_1, K_2$, (2.8) and (2.10) with $K_{00} = 0$ into $\phi_3 = 0$, we find after simplifications:

$$-\frac{K_{30}}{m^3} - \frac{K_{21}(-1+t)t^{m-2}(m(-1+t)+1+t)}{m^3(m+1)}\theta_1$$

$$+\frac{t(m-(2+m)t)}{m(m+1)(m+2)}\theta_2+\psi_1\cdot\theta_1^2+\psi_2\cdot\theta_1^3=0, \qquad (2.11)$$

with $\psi_1=\frac{2K_{12}(t-1)^2t^{2m-3}(m(t-1)+1+t)^3+m(3m-2-m^2+(m-1)(3m+2)t-2(m+1)^2t^2)}{2tm^3(m+1)^2(m(t-1)+t+1)}$,
$\psi_2=\frac{K_{03}(t-1)^3t^{3m-7}(m(t-1)+t+1)^3}{m^3(m+1)^3}$.

For Eqn.(2.11) to be independently of θ_1,θ_2, we get with $\kappa=(\frac{m}{m+2})^m$:

$$t=\frac{m}{m+2},K_{03}=0,K_{21}=0,K_{30}=0,K_{12}=\frac{m^4}{8\kappa^2}. \qquad (2.12)$$

Finally by substituting $A_1,A_2,A_3,B_1,B_2,B_3,K_1,K_2,K_3$, (2.8), (2.10) and(2.12) with $K_{00}=0$ into ϕ_4, we find after simplifications:

$$\phi_4=-\frac{K_{40}}{m^4}-\frac{\theta_1\theta_2}{m(1+m)^2(2+m)}+\frac{m\theta_3}{(m+1)(m+2)^3(m+3)}+\frac{4K_{31}\theta_1\kappa}{m^6(m+1)}$$
$$-\frac{16K_{22}\theta_1^2\kappa^2}{m^8(m+1)^2}-\frac{256K_{04}\theta_1^4\kappa^4}{m^{12}(m+1)^4}$$
$$+\frac{\theta_1^3(12m^5-2m^6+2m^7+2m^8+m^9+192K_{13}\kappa^3)}{3m^{10}(m+1)^3}. \qquad (2.13)$$

Consequently, (2.7) now implies $e_{n+1}=\phi_4e_n^4+O(e_n^5)$, yielding the desired equation (2.1). Since the proposed methods require one-function and two-derivative evaluation per iteration, they are optimal in the sense of Kung-Traub. This completes the proof. □

3. Special cases of fourth-order methods

In this section, some special cases of fourth-order methods are presented from $K_f(h,v)$ of proposed methods (1.3). They include the existing optimal methods that have been developed by many researchers. As requirements for the fourth-order convergence of $K_f(h,v)$ described in Theorem 2.1, the relations

$$K_{00}=K_{01}=K_{02}=K_{03}=K_{20}=K_{21}=K_{30}=0,$$
$$K_{10}=\frac{m^2}{m+2},K_{11}=-\frac{m^3}{4\kappa},K_{12}=\frac{m^4}{8\kappa^2} \qquad (3.1)$$

easily determine the exact form of each special case with $\gamma=\frac{2m}{m+2},\kappa=\left(\frac{m}{m+2}\right)^m$.

Case 1:

$$K_f(h,v)=T_f(v)\cdot h \text{ with } T_f(v)=\frac{a_1v^2+a_2v+a_3}{b_1v^2+b_2v+b_3}. \qquad (3.2)$$

Parameters:

$$\begin{cases} a_3 = \kappa\frac{-a_2m(2+m)^2(4+2m+m^2)-(-16b_1m^2+a_1(2+m)(48+40m+20m^2+6m^3+m^4)\kappa}{m^3(2+m)^2}, \\ b_2 = \frac{-2a_2m(2+m)-2(2a_1(2+m)(3+m)+b_1m^2(-2+2m+m^2)\kappa}{m^4}, \\ b_3 = \kappa\frac{2a_2m(2+m)^2+(b_1m^2(-8+8m+6m^2+m^3)+4a_1(16+18m+7m^2+m^3))\kappa}{m^4(2+m)}. \end{cases}$$

Three coefficients can be determined in terms of remaining three coefficients.

Asymptotic error Constants:

$$\eta = \phi_4 = \lambda\theta_1^3 - \frac{1}{m(m+1)^2(m+2)}\theta_1\theta_2 + \frac{m}{(m+1)(m+2)^3(m+3)}\theta_3,$$

where $\lambda = \frac{a_2m(2+m)^2(-2+2m+2m^2+m^3)+2(-2b_1m^2(-2+5m+2m^2+m^3)+a_1\nu)\kappa}{3m^4(1+m)^3(a_2m(2+m)^2+2(-2b_1m^2+a_1(12+14m+6m^2+m^3))\kappa)}$ with $\nu = (-48 - 16m + 40m^2 + 50m^3 + 28m^4 + 8m^5 + m^6)$.

We also find the weighting function $Q(v) = \gamma + T_f(v)$ with $\gamma = \frac{2m}{m+2}$ in the second step of (1.1). Among many subcases of Case 1, we consider some useful subcases with each subcase number abbreviated by S.C. In Table 1, a summary of eight subcases of Case 1 is well displayed with specific forms of $T_f(v) = \frac{a_1v^2+a_2v+a_3}{b_1v^2+b_2v+b_3}$ and $Q(v) = \gamma + T_f(v)$ as well as asymptotic error constants $\eta = \lambda_1\theta_1^3 + \lambda_2\theta_1\theta_2 + \lambda_3\theta_3$. The results of the first five subcases are in agreement with those of five cases studied by Zhou et al.[13]. Especially Subcase 3 implies the optimal fourth-order iterative methods developed by Sharma et al.[9]. Both Subcase 4 and Subcase 6 imply also the optimal fourth-order iterative methods developed by Li et al.[4-5].

Case 2:

$$K_f(h, v) = \left(\frac{a_2 + a_3v^3}{a_1 + v^2}\right)h \tag{3.3}$$

Parameters:

$$a_1 = -\frac{(m^3 + 4m^2 + 4m - 8)\kappa^2}{m^2(m+4)}, a_2 = \frac{(m^4 + 6m^3 + 22m^2 + 48m + 64)\kappa^2}{3(m+2)(m+4)},$$

$$a_3 = -\frac{m^3(m^2 + 2m - 2)}{3(m+2)^2(m+4)\kappa}.$$

Asymptotic error Constants:

$$\eta = \lambda_1\theta_1^3 - \frac{1}{m(m+1)^2(m+2)}\theta_1\theta_2 + \frac{m}{(m+1)(m+2)^3(m+3)}\theta_3,$$

with $\lambda_1 = \frac{m^8+10m^7+44m^6+106m^5+140m^4+64m^3-72m^2-32m+128}{3m^5(m+1)^3(m+2)^2(m^2+4m+6)}$.

4. Algorithm, numerical results, and discussions

By use of *Mathematica*[12] program, we have performed numerical experiments with 500 precision digits, being large enough to minimize round-off errors. For accurate computation of asymptotic error constants and asymptotic order of convergence, the zero α, however, was given with 550 significant digits, whenever its exact value is not known; the error bound $\epsilon = \frac{1}{2} \times 10^{-200}$ was used. All numerical experiments have been carried out on a personal computer equipped with an AMD 3.1 Ghz dual-core processor and Windows 32-bit XP operating system.

Iterative methods associated with current numerical experiments are identified as follows:

Method **NET**: $x_{n+1} = x_n - \frac{f(x_n)}{a_1 f'(x_n)+a_2 f'(y_n)+a_3 f'(\eta_n)}$,
$y_n = x_n - a\frac{f(x_n)}{f'(x_n)}, \eta_n = x_n - b\frac{f(x_n)}{f'(x_n)} - c\frac{f(x_n)}{f'(y_n)}$,
with a, b, c, a_1, a_2, a_3 as constant real parameters.

Method **SHA**: $x_{n+1} = y_n - T_f(v_n)h_n$, with $y_n = x_n - \gamma h_n$ and $T_f(v)$ chosen as Subcase 3 of Case 1.

Method **ZCS**: $x_{n+1} = y_n - T_f(v_n)h_n$, with $y_n = x_n - \gamma h_n$ and $T_f(v)$ chosen as Subcase 4 of Case 1.

Method **YK1**: $x_{n+1} = y_n - T_f(v_n)h_n$, with $y_n = x_n - \gamma h_n$ and $T_f(v)$ chosen as Subcase 8 of Case 1.

Method **YK2**: $x_{n+1} = y_n - K_f(h_n, v_n)$, with $y_n = x_n - \gamma h_n$ and $K_f(h, v)$ chosen as Case 2.

Note that Method **NET** is not optimal requiring one-function and three-derivative evaluation per iteration. It employs the following parameters in the current experiments:

$$a = \frac{m}{2}, \ b = \begin{cases} -2^{m-1}c + 2m \text{ for } m \neq 3; \\ \frac{12}{5} - 4c \text{ for } m = 3. \end{cases} \tag{4.1}$$

TABLE 1. Forms of $T_f(v),\ Q(v)$ of each subcase for Case 1

S.C.	$T_f(v) = \frac{a_1 v^2 + a_2 v + a_3}{b_1 v^2 + b_2 v + b_3}$		$Q(v) = \gamma + T_f(v)$
1	$a_1 v^2 + a_2 v + a_3$		$Av^2 + Bv + C$
	$a_1 = \frac{m^4}{8\kappa^2}$	$b_1 = 0$	$A = a_1$
	$a_2 = -\frac{m^3(m+3)}{4\kappa}$	$b_2 = 0$	$B = a_2$
	$a_3 = \frac{m^2(m^3+8m^2+20m+24)}{8(m+2)}$	$b_3 = 1$	$C = \frac{m(m^3+6m^2+8m+8)}{8}$
2	$a_1 v + a_2 + \frac{a_3}{v}$		$A + \frac{B}{v} + C$
	$a_1 = \frac{m^4}{8\kappa}$	$b_1 = 0$	$A = a_1$
	$a_2 = -\frac{m^3(m^2+5m+8)}{4(m+2)}$	$b_2 = 1$	$B = a_3$
	$a_3 = \frac{1}{8} m(m+2)^3 \kappa$	$b_3 = 0$	$C = -\frac{(m^3+3m^2+2m-4)}{4}$
3	$a_1 + \frac{a_2}{v} + \frac{a_3}{v^2}$		$A + \frac{B}{v} + \frac{C}{v^2}$
	$a_1 = \frac{m^3(m^2+2m-4)}{8(m+2)}$	$b_1 = 1$	$A = \frac{m(m^3-4m+8)}{8}$
	$a_2 = -\frac{(m+2)^2 m(m-1)\kappa}{4}$	$b_2 = 0$	$B = a_2$
	$a_3 = \frac{1}{8} m(m+2)^3 \kappa^2$	$b_3 = 0$	$C = a_3$
4	$\frac{a_1 v^2 + a_2 v + a_3}{b1 v^2 + b2 v}$		$\frac{A}{v} + \frac{1}{B+Cv}$
	$a_1 = -\frac{2m^2\beta}{(m+2)\kappa}$	$b_1 = \frac{m^2\beta}{\kappa}$	$A = \frac{(m-2)m(m+2)^3\kappa}{2(m^3-4m+8)}$
	$a_2 = -\frac{m^2(m+4)(m^2-8)\beta}{2(m+2)}$	$b_2 = -\frac{(m^3-4m+8)\beta}{m}$	$B = b_2$
	$a_3 = \frac{(m-2)(m+2)^3\beta\kappa}{2}$	$b_3 = 0$	$C = b_1$
5	$\frac{a_2 v + a_3}{b_2 v + 1}$		$\frac{B+Cv}{1+Av}$
	$a_1 = 0$	$b_1 = 0$	$A = -\frac{1}{\kappa}$
	$a_2 = -\frac{m^3}{2(m+2)\kappa}$	$b_2 = -\frac{1}{\kappa}$	$B = -\frac{m^2}{2}$
	$a_3 = -\frac{m(m^2+2m+4)}{2(m+2)}$	$b_3 = 1$	$C = \frac{m(m-2)}{2\kappa}$
6	$\frac{v+a_3}{b_2 v + b_3} = \frac{1}{b_2}\left(1 + \frac{a_3 - b_3/b_2}{v + b_3/b_2}\right)$		$A + \frac{1}{B+Cv}$
	$a_1 = 0$	$b_1 = 0$	$A = -\frac{m(m-2}{2}$
	$a_2 = 1$	$b_2 = -\frac{2(m+2)}{m^3}$	$B = -\frac{1}{m}$
	$a_3 = -\frac{(m^2+2m+4)\kappa}{m^2}$	$b_3 = \frac{2(m+2)\kappa}{m^3}$	$C = \frac{1}{m\kappa}$
7	$\frac{a_2 v + a_3}{v^2 + b_3}$		$\frac{2m}{m+2} + \frac{Bv+C}{v^2+A}$
	$a_1 = 0$	$b_1 = 1$	$A = b_3$
	$a_2 = -\frac{m(m^2+2m-4)\kappa}{m+2}$	$b_2 = 0$	$B = a_2$
	$a_3 = \frac{m(m^3+6m^2+14m+16)\kappa^2}{(m+2)^2}$	$b_3 = -\frac{(m^2+2m-4)\kappa^2}{m(m+2)}$	$C = a_3$
8	$\frac{a_1 v^2 + a_3}{v^2 + b_3} = a1 + \frac{a_3 - a_1 b_3}{v^2 + b_3}$		$A + \frac{C}{v^2+B}$
	$a_1 = -\frac{m^2(m^2+2m-2)}{2(m+2)(m+3}$	$b_1 = 1$	$A = -\frac{m(m^2-6)}{2(m+3)}$
	$a_2 = 0$	$b_2 = 0$	$B = b_3$
	$a_3 = \frac{(m+2)(m^2+2m+6)\kappa^2}{2(m+3)}$	$b_3 = -\frac{(m-1)(m+2)^2\kappa^2}{m^2(m+3)}$	$C = \frac{2(m+2)^3\kappa^2}{(m+3)^2}$

$$\begin{cases} a_1 = \frac{-2^m c(m-1)(m-2)+6m(4m^2-3m-2)}{24m^2(m^2-2m-2)},\ a_2 = \frac{2^{m-4}[2^m c(m-1)(m-2)-32m(m+1)]}{3m^2(m^2-2m-2)}, \\ a_3 = -\frac{(-1)^{-m}(m-2)}{12m(m^2-2m-2)} \text{ for } m \neq 3 \\ a_1 = -\frac{1}{24} - \frac{25c}{27},\ a_2 = \frac{4(12+25c)}{27},\ a_3 = -\frac{125}{72} \text{ for } m = 3. \end{cases} \tag{4.2}$$

$$\begin{cases} c = -\frac{2^{1-m}m(9m^2-12m+7\pm\sqrt{33m^4-168m^3+318m^2+72m-239})}{(m-1)(m-2)(m-3)} & \text{for } m > 3, \\ c = \text{free real constant} & \text{for } m = 3, \\ c = \frac{264}{19} & \text{for } m = 2, \\ c = 18 & \text{for } m = 1. \end{cases} \tag{4.3}$$

REMARK 4.1. Observe that the values of c, a_1, a_2, a_3 can be more generally determined in terms of m by selecting a special form of b and exhibit better selections than those values suggested by Neta[6].

DEFINITION 4.2. (Asymptotic Convergence Order)
Assume that the asymptotic error constant $\eta = \lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^p}$ is known as described in Definition 1.1. Then we can define the *asymptotic convergence order* $p_a = \lim_{n\to\infty} \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$, being abbreviated by A.C.O.

In order to verify the fourth-order convergence of (1.3) to be seen in Table 2, four test functions $F_1(x) - F_2(x)$ are presented below:

$$\begin{cases} F_1(x) = [\cos\frac{\pi x}{2} + 2x - \pi]^5; & \alpha \approx 2.06795083703446, m = 5 \\ F_2(x) = [\cos x^2 - x\log(1 + x^2 - \pi) + 1]^2(x^2 - \pi); & \alpha = \sqrt{\pi}, m = 3 \end{cases} \tag{4.4}$$

where $\log z$ $(z \in \mathbb{C})$ represents a principal analytic branch such that $-\pi \le Im(\log z) < \pi$. Tables 2 lists iteration indexes n, approximate zeros x_n, residual errors $|f(x_n)|$, errors $|e_n| = |x_n - \alpha|$ and computational asymptotic error constants $\eta_n = |\frac{e_n}{e_{n-1}{}^4}|$ as well as the theoretical asymptotic error constant η and computational asymptotic convergence order $p_n = \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$. Initial guesses x_0 were selected close to α not only to guarantee the convergence of (1.3) and but also to clearly observe the convergence of the computed asymptotic error constants requiring small-number divisions. Computational asymptotic error constants agree up to 10 significant digits with theoretical ones.

Additional test functions are given below:

$$\begin{cases} f_1(x) = (\sin^2 x - x^2 + 1)^2; & \alpha \approx 1.40449164821534, m = 2 \\ f_2(x) = (x - \pi + \sin x \log x^2 + 1)^2; & \alpha = \pi, m = 2 \\ f_3(x) = (2x + e^{-x} + \sin x^2 - 3)^6; & \alpha \approx 0.924463112118051, m = 6 \\ f_4(x) = (x^{10} - \sqrt{3}x^3 \cos\frac{\pi x}{6} + \frac{1}{x^2+1})(x-1)^4; & \alpha = 1, m = 5 \\ f_5(x) = \cos(x^2 - 2x + \frac{52}{49}) - \log(x^2 - 2x + \frac{101}{49}) - 1; \alpha = 1 + i\frac{\sqrt{3}}{7}, i = \sqrt{-1}, m = 1, \end{cases}$$

where $\log z$ $(z \in \mathbb{C})$ represents a principal analytic branch such that $-\pi \le Im(\log z) < \pi$.

The values of $|x_n - \alpha|$ for additional test functions are listed in Table 3 for fourth-order methods **NET**, **SHA**, **ZCS** and **YK1, YK2**. As Table 3 suggests, proposed methods show favorable performance as compared with **NET**. Under the same order of convergence, one should note that the speed of local convergence of $|x_n - \alpha|$ is dependent on c_j,

TABLE 2. Convergence for sample test functions $F_1(x) - F_2(x)$ with methods **YK1 − YK2**

$\binom{\mathbf{MT}}{F_i}$	n	x_n	$\lvert f(x_n)\rvert$	$\lvert e_n\rvert$	$\lvert \frac{e_n}{e_{n-1}{}^4}\rvert$	η	p_n
$\binom{\mathbf{YK1}}{F_1}$	0	1.98	0.000194797	0.0879508			
	1	2.06799668257943	9.68×10^{-21}	4.58×10^{-5}	0.7661913267	0.5782727709	3.88425
	2	2.06795083703446	5.19×10^{-87}	2.55×10^{-18}	0.5781901293		4.00001
	3	2.06795083703446	4.32×10^{-352}	2.46×10^{-71}	0.5782727709		4.00000
	4	2.06795083703446	2.05×10^{-1412}	2.12×10^{-283}			
$\binom{\mathbf{YK2}}{F_2}$	0	1.8	0.00265039	0.0275461			
	1	1.77245143577374	1.97×10^{-15}	2.41×10^{-6}	4.194664758	3.532062747	3.95213
	2	1.77245385090552	2.42×10^{-64}	1.20×10^{-22}	3.532011206		4.00000
	3	1.77245385090552	5.59×10^{-260}	7.36×10^{-88}	3.532062747		4.00000
	4	1.77245385090552	1.57×10^{-1042}	1.03×10^{-348}			

MT:Method; $p_n = \frac{\log\lvert e_n/\eta\rvert}{\log\lvert e_{n-1}\rvert}$, $e_n = x_n - \alpha$.

TABLE 3. Comparison of $\lvert x_n - \alpha\rvert$ for $f_1(x) - f_6(x)$ among fourth-order methods

f	x_0	$\lvert x_n-\alpha\rvert$	**NET**	**SHA**	**ZCS**	**YK1**	**YK2**
f_1	1.45	$\lvert x_1-\alpha\rvert$	2.99e-6*	3.43e-6	2.99e-6	3.10e-6	2.94e-6
		$\lvert x_2-\alpha\rvert$	6.49e-23	1.29e-22	6.49e-23	7.80e-23	5.99e-23
		$\lvert x_3-\alpha\rvert$	1.44e-89	2.63e-88	1.44e-89	3.11e-89	*1.02e-89*
		$\lvert x_4-\alpha\rvert$	3.53e-356	4.33e-351	3.53e-356	7.93e-355	8.84e-357
f_2	3.00	$\lvert x_1-\alpha\rvert$	3.02e-4	3.19e-4	3.02e-4	3.06e-4	3.00e-4
		$\lvert x_2-\alpha\rvert$	3.15e-15	4.15e-15	3.15e-15	3.38e-15	3.05e-15
		$\lvert x_3-\alpha\rvert$	3.75e-59	1.18e-58	3.75e-59	5.00e-59	*3.27e-59*
		$\lvert x_4-\alpha\rvert$	7.53e-235	7.62e-233	7.53e-235	2.40e-234	4.33e-235
f_3	0.875	$\lvert x_1-\alpha\rvert$	5.11e-6	2.34e-6	2.34e-6	2.34e-6	2.34e-6
		$\lvert x_2-\alpha\rvert$	2.13e-21	1.83e-23	1.83e-23	1.83e-23	1.83e-23
		$\lvert x_3-\alpha\rvert$	6.48e-83	6.91e-92	6.89e-92	6.89e-92	*6.88e-9*2
		$\lvert x_4-\alpha\rvert$	5.50e-329	1.39e-365	1.38e-365	1.37e-365	1.37e-365
f_4	1.08	$\lvert x_1-\alpha\rvert$	2.76e-3	2.59e-4	2.55e-4	2.53e-4	2.52e-4
		$\lvert x_2-\alpha\rvert$	1.98e-8	7.07e-14	6.48e-14	6.30e-14	6.15e-14
		$\lvert x_3-\alpha\rvert$	5.01e-29	3.90e-52	2.71e-52	2.40e-52	2.18e-52
		$\lvert x_4-\alpha\rvert$	2.05e-111	3.61e-205	8.32e-206	5.14-206	*3.45e-206*
		$\lvert x_5-\alpha\rvert$	5.80e-441				
f_5	0.97+	$\lvert x_1-\alpha\rvert$	2.91e-5	4.10e-5	1.20e-4	4.10e-5	3.68e-5
	$0.22i$	$\lvert x_2-\alpha\rvert$	6.32e-18	3.36e-17	6.28e-15	3.36e-17	1.97e-17
		$\lvert x_3-\alpha\rvert$	1.38e-68	1.51e-65	4.70e-56	1.51e-65	*1.63e-66*
		$\lvert x_4-\alpha\rvert$	3.17e-271	6.27e-259	1.48e-220	6.27e-259	7.66e-263

* 2.99e-6 denotes 2.99 $\times10^{-6}$

namely $f(x)$ and α. Tables 3 and 4 well exhibit fourth-order convergence of proposed scheme (1.3). As can be seen in Table 5, the CPU times(measured in seconds) of **NET** are mostly increased by an approximate factor of between 3 and 6, as compared with proposed methods

Table 4. Comparison of computational asymptotic convergence order $p_n = \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$

f	x_0	p_n	**NET**	**SHA**	**ZCS**	**YK1**	**YK2**
f_1	1.45	p_1	4.04837	4.04850	4.04837	4.04831	4.04823
		p_2	4.00000	4.00000	4.00000	4.00000	4.00000
		p_3	4.00000	4.00000	4.00000	4.00000	4.00000
f_2	3.00	p_1	3.64910	3.64397	3.64910	3.64791	3.64989
		p_2	4.00017	4.00018	4.00017	4.00017	4.00017
		p_3	4.00000	4.00000	4.00000	4.00000	4.00000
f_3	0.875	p_1	4.43038	4.14855	4.14848	4.14846	4.14844
		p_2	3.99999	4.00000	4.00000	4.00000	4.00000
		p_3	4.00000	4.00000	4.00000	4.00000	4.00000
f_4	1.08	p_1	4.62467	4.35547	4.35699	4.35755	4.35801
		p_2	3.99308	4.00043	4.00042	4.00042	4.00042
		p_3	4.00000	4.00000	4.00000	4.00000	4.00000
		p_4	4.00000				
f_5	$0.97+$ $0.22i$	p_1	3.93551	3.92454	3.88212	3.92454	3.92753
		p_2	3.99998	3.99997	3.99985	3.99997	3.99997
		p_3	4.00000	4.00000	4.00000	4.00000	4.00000

Table 5. Comparison of CPU times among derivative-free eighth-order methods

f	x_0	**NET**	**SHA**	**ZCS**	**YK1**	**YK2**
f_1	1.45	0.032	**0.031**	0.062	0.031	0.031
f_2	3.00	**0.047**	0.094	0.141	0.094	0.078
f_3	0.875	0.203	0.078	0.109	**0.062**	0.062
f_4	1.00	0.984	0.125	0.234	0.156	**0.110**
f_5	$0.97+0.22i$	0.172	0.047	0.125	**0.078**	0.078

YK1-YK2. The least errors or CPU times are highlighted in boldface or italicized numbers in Tables 3 and 5.

Although being limited to the current test functions, **YK2** has shown best accuracy. Since computational accuracy generally depends on the iterative methods, the sought zeros and the test functions as well as close initial approximations, one should be aware that no iterative method always shows best accuracy for all the test functions. The efficiency indices for the proposed family of methods (1.3) are found to be $4^{1/3}$, being optimal in the sense of Kung-Traub and better than $4^{1/4}$ for non-optimal scheme **NET**. The explicit form of error equation (2.1) ensures the convergence order of method (1.3). Efficient computing time ensures a better implementation of (1.3), from a practical point of view, as compared to existing methods **NET, SHA, ZCS**. The current analysis can be extended to a development of higher-order multiple-root finders for nonlinear equations.

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