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WEAK SOLUTIONS OF GRADIENT FLOW OF LANDAU-DE GENNES ENERGY

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Abstract. Taking into account the flexoelectric effects, we consider a gradient flow of Landau-de Gennes energy which generalizes the Oseen-Frank energy. In this article, we discuss existence of weak solutions of the gradient flow in an appropriate function space.

1. Introduction

Molecules in Nematic Liquid Crystals are described by a traceless symmetric second order tensor

$$\mathbf{Q} = \int_{\mathbb{S}^2} \ell \otimes \ell f(\ell)\, d\ell - \frac{1}{3} I, \tag{1.1}$$

where f is a probability distribution function satisfying $f(\ell) = f(-\ell)$ for all $\ell \in \mathbb{S}^2$. Shapes of molecules are characterized by three eigenvalues of $\mathbf{Q}$ and the direction of a molecule is defined by the unit eigenvector whose corresponding eigenvalue has the largest magnitude. The order tensor $\mathbf{Q}$ is a measure of the local degree of orientational order in liquid crystals. The liquid crystal is said to be uniaxial if two eigenvalues of $\mathbf{Q}$ are equal, and it is biaxial when $\mathbf{Q}$ has three distinct eigenvalues. The tensor $\mathbf{Q}$ is zero in the isotropic phase. Since $\mathbf{Q}$ is a symmetric matrix, all eigenvalues of $\mathbf{Q}$ are real and expressed in term of $\mathbf{Q}$ as [4]

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$$\begin{cases} \lambda_1 = \frac{2\sqrt{\mathrm{tr}\,\mathbf{Q}^2}}{\sqrt{6}}\cos\alpha, \\ \lambda_2 = \frac{2\sqrt{\mathrm{tr}\,\mathbf{Q}^2}}{\sqrt{6}}\left(-\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha\right), \\ \lambda_3 = \frac{2\sqrt{\mathrm{tr}\,\mathbf{Q}^2}}{\sqrt{6}}\left(-\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha\right), \end{cases}$$

where

$$\cos(3\alpha) = -\frac{\sqrt{6}\mathrm{tr}\,\mathbf{Q}^3}{\mathrm{tr}\,\mathbf{Q}^2\sqrt{\mathrm{tr}\mathbf{Q}^2}}, \quad \sin(3\alpha) = \sqrt{1 - \frac{6(\mathrm{tr}\,\mathbf{Q}^3)^2}{(\mathrm{tr}\,\mathbf{Q}^2)^3}}, \;\; \alpha \in \left[0, \frac{\pi}{3}\right].$$

It follows from $\mathrm{tr}\mathbf{Q} = 0$ and $\mathbf{Q} = \mathbf{Q}^T$ that $6(\mathrm{tr}\,\mathbf{Q}^3)^2 \leq (\mathrm{tr}\,\mathbf{Q}^2)^3$. Moreover, $\mathbf{Q}$ has two distinct eigenvalues if and only if $6(\mathrm{tr}\,\mathbf{Q}^3)^2 = (\mathrm{tr}\,\mathbf{Q}^2)^3$. From (1.1), it can be easily seen that $-\frac{1}{3} \leq \lambda_i \leq \frac{2}{3}$ for $i = 1, 2, 3$. It then follows that $\mathrm{tr}\,\mathbf{Q}^2 \leq \frac{1}{6}$.

If $\mathbf{Q}$ is expressed by

$$\mathbf{Q} = S_1\left(\mathbf{m}\otimes\mathbf{m} - \frac{1}{3}\mathbf{I}\right) + S_2\left(\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}\mathbf{I}\right),$$

where $\{\mathbf{m}, \mathbf{n}, \mathbf{m}\times\mathbf{n}\}$ is an orthonormal basis for $\mathbf{R}^3$ consisting of unit eigenvectors of $\mathbf{Q}$, then the eigenvalues are

$$\frac{1}{3}(2S_1 - S_2), \quad -\frac{1}{3}(S_1 + S_2), \quad \frac{1}{3}(2S_2 - S_1).$$

In the Landau-de Gennes theory, neglecting the higher derivatives and powers of $\mathbf{Q}$ the free energy density $\mathcal{F}$ for nematic liquid crystals is given by

$$\begin{aligned} &\mathcal{F}(\mathbf{Q}, \nabla\mathbf{Q}) \\ &= \frac{1}{2}\left(L_1 Q_{\alpha\beta,\gamma}Q_{\alpha\beta,\gamma} + L_2 Q_{\alpha\beta,\beta}Q_{\alpha\gamma,\gamma} + L_3 Q_{\alpha\beta,\gamma}Q_{\alpha\gamma,\beta}\right) + f_{bulk}(\mathbf{Q}), \end{aligned}$$

where

$$f_{bulk}(\mathbf{Q}) = \frac{A}{2}\mathrm{tr}\,\mathbf{Q}^2 - \frac{B}{3}\mathrm{tr}\,\mathbf{Q}^3 + \frac{C}{4}\left(\mathrm{tr}\,\mathbf{Q}^2\right)^2.$$

The bulk energy f_{bulk} is a potential function for uniaxial nematic liquid crystals, meaning that f_{bulk} favors molecules to be uniaxial. In order to study biaxial liquid crystals, we need to add higher powers of $\mathbf{Q}$ to f_{bulk}. In liquid crystals, there exists a polarization induced by a splay and bending distortion [2, 1]. Such a polarization is called flexoelectric polarization which is analogous to piezoelectric polarization in solids. The flexoelectric polarization can be written in terms of $\mathbf{Q}$ as

$$\mathbf{P}^f = (P_1, P_2, P_3),$$
$$P_i = \epsilon_3 Q_{ij,j} + \epsilon_4 Q_{jk} Q_{ij,k} + \epsilon_5 Q_{ij} Q_{jk,k} + \text{ higher order.}$$

Due to the appearance of the flexoelectric polarization, the following electrostatic equations (Maxwell's equations) will be taken into account in the system

$$\nabla \cdot (\boldsymbol{\epsilon}(\mathbf{Q})\mathbf{E}) = -\nabla \cdot \mathbf{P}^f, \quad \nabla \times \mathbf{E} = 0, \tag{1.2}$$

where $\epsilon(\mathbf{Q})$ is the dielectric permittivity tensor given by

$$\epsilon(\mathbf{Q}) = \epsilon_0 \mathbf{I} + \epsilon_1 \mathbf{Q} + \epsilon_2 \mathbf{Q}^2.$$

Hence the electrostatic energy is

$$f_{elec} = -\frac{1}{2}(\epsilon(\mathbf{Q})\mathbf{E}) \cdot \mathbf{E} - \mathbf{P}^f \cdot \mathbf{E}.$$

If we let

$$\mathbf{Q} = \frac{3}{2} S(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}),$$

then

$$\epsilon(\mathbf{Q})\mathbf{E} \cdot \mathbf{E} = \left(\epsilon_0 - \frac{\epsilon_1}{2} S + \frac{\epsilon_2}{4} S^2\right) |\mathbf{E}|^2 + \frac{3}{2} S \left(\epsilon_1 + \frac{\epsilon_2}{2} S\right) (\mathbf{n} \cdot \mathbf{E})^2,$$
$$\mathbf{P}^f = e_{11}(\nabla \cdot \mathbf{n})\mathbf{n} + e_{33}\mathbf{n} \times \nabla \times \mathbf{n},$$
$$e_{11} = \frac{3}{2}\epsilon_3 S + \frac{3}{4}(2\epsilon_5 - \epsilon_4) S^2, \quad e_{33} = \frac{3}{2}\epsilon_3 S + \frac{3}{4}(2\epsilon_4 - \epsilon_5) S^2.$$

Then the permittivity $\epsilon_\perp$ and dielectric anisotropic constant ϵ_a are defined by

$$\epsilon_\perp = \epsilon_0 - \frac{\epsilon_1}{2} S + \frac{\epsilon_2}{4} S^2, \quad \epsilon_a = \frac{3}{2} S \left(\epsilon_1 + \frac{\epsilon_2}{2} S\right).$$

Now, since eigenvalues of $\mathbf{Q}$ are in between $-\frac{1}{3}$ and $\frac{2}{3}$, we impose the following condition for strong ellipticity of (1.2)

$$3\epsilon_0 > \epsilon_1 \quad \text{if } \epsilon_1 > 0, \qquad \text{and} \qquad 3\epsilon_0 > -2\epsilon_1 \quad \text{if } \epsilon_1 \le 0.$$

Since some material can have $\epsilon_1 > 0$ and $S > 0$, we have to include ϵ_2-term in order to satisfy solvability condition $\epsilon_\perp > |\epsilon_a|$. For a sake of simplicity, we take $\epsilon_4 = \epsilon_5 = 0$ so that equations (1.2) become

$$\nabla \cdot \left[\left(\epsilon_0 \mathbf{I} + \epsilon_1 \mathbf{Q} + \epsilon_2 \mathbf{Q}^2\right) \nabla\varphi\right] = -\epsilon_3 \nabla \cdot (\nabla \cdot \mathbf{Q}), \tag{1.3}$$

where $\nabla \cdot \mathbf{Q} = Q_{1j,j}\mathbf{e}_x + Q_{2j,j}\mathbf{e}_y + Q_{3j,j}\mathbf{e}_z$, $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is a set of unit vectors in x, y, z axes respectively, and φ is an electric potential function, i.e. $\mathbf{E} = \nabla\varphi$.

By Maxwell's equation, the electrostatic energy functional can be written as

$$\int_\Omega f_{elec}\, dx = -\frac{1}{2}\int_\Omega (\nabla\cdot\mathbf{Q})\cdot\nabla\varphi\, dx$$

The total energy functional $\mathcal{E}$ is

$$\mathcal{E}(\mathbf{Q},\varphi) = \int_\Omega \left\{\frac{1}{2}L|\nabla\mathbf{Q}|^2 + \frac{A}{2}\mathrm{tr}\,\mathbf{Q}^2 - \frac{B}{3}\mathrm{tr}\,\mathbf{Q}^3 + \frac{C}{4}\left(\mathrm{tr}\,\mathbf{Q}^2\right)^2 - \frac{1}{2}\epsilon_3(\nabla\cdot\mathbf{Q})\cdot\nabla\varphi\right\} dx.$$

In the absence of a flow, equations for dynamic problems are

$$\frac{\partial\mathbf{Q}}{\partial t} = L\Delta\mathbf{Q} - A\mathbf{Q} + B\left(\mathbf{Q}^2 - \frac{\mathrm{tr}\,\mathbf{Q}^2}{3}\mathbf{I}\right) - C(\mathrm{tr}\,\mathbf{Q}^2)\mathbf{Q} - \frac{1}{2}\epsilon_3\left(\nabla^2\varphi - \frac{1}{3}\Delta\varphi\mathbf{I}\right) \text{ in } \Omega. \tag{1.4}$$

subject to

$$\nabla\cdot(\epsilon(\mathbf{Q})\nabla\varphi) = -\epsilon_3\nabla\cdot(\nabla\cdot\mathbf{Q}) \quad \text{in } \Omega, \tag{1.5}$$

and boundary conditions

$$\begin{cases} \frac{\partial\mathbf{Q}(x,t)}{\partial\nu} = \mathbf{0} & \text{on } \Gamma, \quad \mathbf{Q}(x,t) = \mathbf{Q}_1(x) \quad \text{on } \partial\Omega\setminus\Gamma, \\ \varphi(x,t) = \varphi_0(x) & \text{on } \Gamma, \quad \frac{\partial\varphi(x,t)}{\partial\boldsymbol{\nu}} = 0 \quad \text{on } \partial\Omega\setminus\Gamma, \end{cases} \tag{1.6}$$

where $\mathbf{Q}_1$ is fixed.

From now on, we study existence of weak solutions of the system (1.4)-(1.5) with the boundary conditions (1.6).

2. A priori estimates

In this section, we study a priori estimates for solutions which will be used in the next section. Let us introduce

$$W^{1,2}(\Omega;\mathcal{S}_0) = \left\{\mathbf{Q} : ||\mathbf{Q}||_{L^2(\Omega)} + ||\nabla\mathbf{Q}||_{L^2(\Omega)} < \infty, \mathbf{Q}:\Omega\to\mathcal{S}_0\right\},$$

$$H^1_\Gamma(\Omega) = \left\{\psi\in H^1(\Omega) : \psi = 0 \text{ on } \Gamma,\ \frac{\partial\psi}{\partial\boldsymbol{\nu}} = 0 \text{ on } \partial\Omega\setminus\Gamma\right\}.$$

For any $p > 0$, and $t > 0$, we denote by $L^p(0,t;\mathcal{V})$ the space of all functions $\mathbf{Q} : (0,t)\to\mathcal{V}$ such that

$$\int_0^t ||\mathbf{Q}||_{\mathcal{V}}\, dt < \infty,$$

where $\mathcal{V}$ is a function space equipped with its norm $||\cdot||_{\mathcal{V}}$. We look for a weak solution of the system (1.4),(1.5), and (1.6). In other words, the problem is to find $\mathbf{Q} \in L^2\left(0,T;W^{1,2}(\Omega;\mathcal{S}_0)\right)$ and $\varphi \in L^2(0,T;H^1(\Omega))$ satisfying

(2.1)

$$\begin{cases} \int_\Omega \left\{ \left(\frac{\partial \mathbf{Q}}{\partial t} + L\nabla\mathbf{Q} + A\mathbf{Q} - B\mathbf{Q}^2 + C(\operatorname{tr}\mathbf{Q}^2)\mathbf{Q} \right) \cdot \mathbf{T} \right\} dx \\ \qquad\qquad = \frac{1}{2}\epsilon_3 \int_\Omega \nabla\varphi \cdot (\nabla\cdot\mathbf{T})\, dx, \\ \int_\Omega \left(\epsilon(\mathbf{Q})\nabla\varphi\right)\cdot\nabla\psi\, dx = \int_\Gamma \left(\epsilon(\mathbf{Q}_1)\nabla\varphi_0\cdot\boldsymbol{\nu}\right)\psi\, dA - \int_\Omega (\nabla\cdot\mathbf{Q})\cdot\nabla\psi\, dx, \\ \qquad \frac{\partial \mathbf{Q}(x,t)}{\partial \nu} = \mathbf{0} \quad \text{on } \Gamma, \quad \mathbf{Q}(x,t) = \mathbf{Q}_1(x) \quad \text{on } \partial\Omega\setminus\Gamma, \\ \qquad \varphi(x,t) = \varphi_0(x) \quad \text{on } \Gamma, \quad \frac{\partial\varphi(x,t)}{\partial\boldsymbol{\nu}} = 0 \quad \text{on } \partial\Omega\setminus\Gamma, \end{cases}$$

for all $\mathbf{T} \in W^{1,2}(\Omega;\mathcal{S}_0)$ and $\psi \in H^1_\Gamma(\Omega)$.

Lemma 2.1. *Let $(\mathbf{Q},\varphi)$ be a solution pair of functions to (1.4),(1.5), and (1.6). Then*

$$\mathbf{Q} \in L^2(0,T;W^{1,2}(\Omega;\mathcal{S}_0)) \cap L^4(0,T;L^4(\Omega;\mathcal{S}_0)), \qquad \varphi \in L^2(0,T;H^1(\Omega)).$$

Proof. Let $(\mathbf{Q},\varphi)$ be a solution pair of functions to (1.4),(1.5), and (1.6). Multiplying each equation in (1.4) by Q_{ij} and integrating by parts followed by summing up, we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega |\mathbf{Q}|^2\, dx + &\int_\Omega \left(L|\nabla\mathbf{Q}|^2 + A\operatorname{tr}\mathbf{Q}^2 - B\operatorname{tr}\mathbf{Q}^3 + C(\operatorname{tr}\mathbf{Q}^2)^2\right) dx \\ &= \frac{1}{2}\epsilon_1 \int_\Omega \nabla\varphi\cdot(\nabla\cdot\mathbf{Q})\, dx. \end{aligned} \tag{2.2}$$

Similarly, multiplying (1.5) by φ and integrating by parts yield

$$\int_\Omega (\epsilon(\mathbf{Q})\nabla\varphi)\cdot\nabla\varphi\, dx = -\epsilon_3\int_\Omega (\nabla\cdot\mathbf{Q})\cdot\nabla\varphi\, dx. \tag{2.3}$$

Combining (2.2) with (2.3) we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega |\mathbf{Q}|^2\, dx + &\int_\Omega \left(L|\nabla\mathbf{Q}|^2 + C(\operatorname{tr}\mathbf{Q}^2)^2 + \frac{1}{2}\left(\epsilon(\mathbf{Q})\nabla\varphi\right)\cdot\nabla\varphi\right) dx \\ &= \int_\Omega \left(-A\operatorname{tr}\mathbf{Q}^2 + B\operatorname{tr}\mathbf{Q}^3\right) dx. \end{aligned} \tag{2.4}$$

By Hölder inequality, choose $\eta > 0$ such that $C - B\eta^2 > 0$ and

$$\int_\Omega \operatorname{tr}\mathbf{Q}^3\, dx \le \int_\Omega \left\{\frac{1}{\eta^2}\operatorname{tr}\mathbf{Q}^2 + \eta^2(\operatorname{tr}\mathbf{Q}^2)^2\right\} dx. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\frac{d}{dt}\int_\Omega |\mathbf{Q}|^2\,dx + \int_\Omega \left(L|\nabla \mathbf{Q}|^2 + \tilde{C}(\operatorname{tr}\mathbf{Q}^2)^2 + \frac{1}{2}\left(\epsilon(\mathbf{Q})\nabla\varphi\right)\cdot\nabla\varphi \right) dx \le \mathcal{M}||\mathbf{Q}||_{L^2} + \mathcal{D}, \tag{2.6}$$

where $\tilde{C} = C - \frac{1}{\eta^2}$, and $\mathcal{M} = -A + \frac{B}{\eta^2}$. Hence we get

$$\frac{d}{dt}||\mathbf{Q}||^2_{L^2} \le \mathcal{M}||\mathbf{Q}||^2_{L^2} + \mathcal{D},$$

and Grownwall's inequality leads us to have

$$||\mathbf{Q}(t)||^2_{L^2} \le ||\mathbf{Q}(0)||^2_{L^2} e^{\mathcal{M}t} + \frac{\mathcal{D}}{\mathcal{M}}\left(e^{\mathcal{M}t} - 1\right). \tag{2.7}$$

This implies that

$$\sup_{0\le t\le T} ||\mathbf{Q}(t)||^2_{L^2} \le ||\mathbf{Q}(0)||^2_{L^2} e^{\mathcal{M}T} + \frac{\mathcal{D}}{\mathcal{M}}\left(e^{\mathcal{M}T} - 1\right),$$

and integrating (2.6) with respect to t yields

$$\int_0^T \int_\Omega \left(L|\nabla \mathbf{Q}|^2 + \tilde{C}(\operatorname{tr}\mathbf{Q}^2)^2 + \frac{1}{2}\left(\epsilon(\mathbf{Q})\nabla\varphi\right)\cdot\nabla\varphi \right) dx\,dt < \infty.$$

Since $(\epsilon(\mathbf{Q})\nabla\varphi)\cdot\nabla\varphi \ge \lambda||\nabla\varphi||^2$ for some $\lambda > 0$, by Poincare inequality we have

$$\mathbf{Q} \in L^2(0,T; W^{1,2}(\Omega;\mathcal{S}_0)) \cap L^4(0,T; L^4(\Omega;\mathcal{S}_0)), \varphi \in L^2(0,T; H^1(\Omega)).$$

□

3. Existence of weak solution

THEOREM 3.1. *For any given $T > 0$, $\mathbf{Q}_0 \in L^2(\Omega;\mathcal{S}_0)$, there exists a solution pair $(\mathbf{Q},\varphi)$ to (2.1) such that $\mathbf{Q} \in L^(0,T; W^{1,2}(\Omega;\mathcal{S}_0)$ and $\varphi \in L^2(0,T; H^1(\Omega))$. Moreover, if $\mathbf{Q}_0 \in W^{1,2}(\Omega;\mathcal{S}_0)$, then*

$$\mathbf{Q} \in \mathcal{C}(0,T; W^{1,2}(\Omega;\mathcal{S}_0)) \cap L^4(0,T; L^4(\Omega;\mathcal{S}_0)), \frac{\partial \mathbf{Q}}{\partial t} \in L^2(0,T; L^2(\Omega;\mathcal{S}_0)).$$

Proof. We use the Galerkin Method [6] to obtain a weak solution $(\mathbf{Q},\varphi)$ to (2.1). We first approximate $W^{1,2}(\Omega,\mathcal{S}_0)$ and $H^1(\Omega)$ by increasing sequences of finite dimensional subspaces $\mathcal{X}^m \subset W^{1,2}(\Omega,\mathcal{S}_0)$, and $\mathcal{Y}^m \subset H^1(\Omega)$ such that

$$\cup_{m=1}^\infty \mathcal{X}^m = W^{1,2}(\Omega,\mathcal{S}_0), \quad \cup_{m=1}^\infty \mathcal{Y}^m = H^1(\Omega).$$

For each $m \in \mathbb{N}$, let $\{\mathbf{x}_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^m$ be orthonormal bases for $\mathcal{X}^m$ and $\mathcal{Y}^m$, respectively. We first seek a solution pair $(\mathbf{Q}^m, \varphi^m)$ in $\mathcal{X}^m \times \mathcal{Y}^m$ in the form

$$\mathbf{Q}^m(x,t) = \sum_{i=1}^m p_i(t)\mathbf{x}_i(x), \quad \varphi^m(x,t) = \sum_{i=1}^m q_i(t) y_i(x).$$

Substituting $(\mathbf{Q}^m, \varphi^m)$ for $(\mathbf{Q}, \varphi)$ in (2.1), and taking $\mathbf{T} = \mathbf{x}_j, \psi = y_k$, we obtain a system of nonlinear ordinary differential equations for $\{p_i(t), q_i(t)\}_{i=1}^m$. It follows from the standard theory of ODEs that the new system has a unique solution on some interval $[0, t_m] \subset [0, T]$. By lemma 2.1, we know that

$$\sup_{0 \le t \le T} \{||\mathbf{Q}^m(t)||_{L^2}, ||\varphi^m(t)||_{L^2}\} < \infty.$$

We extend $\mathbf{Q}^m, \varphi^m$ to the interval $[0,T]$ by the standard continuation method [3, 6]. Apply Lemma 2.1 again to show that $\{\mathbf{Q}^m\}_{m\in\mathbb{N}}$ is bounded in $L^2(0,T; W^{1,2}(\Omega; \mathcal{S}_0)) \cap L^4(0,T; L^4(\Omega; \mathcal{S}_0))$, and $\{\varphi^m\}_{m\in\mathbb{N}}$ is bounded in $L^2(0,T; H^1(\Omega))$. Note that $\{(\mathrm{tr}\,(\mathbf{Q}^m)^2)\mathbf{Q}^m\}_{m\in\mathbb{N}}$ is bounded in $L^{\frac{4}{3}}((0,T)\times\Omega))$.

We can extract a subsequence (not relabeled) $\{(\mathbf{Q}^m, \varphi^m)\}_{m\in\mathbb{N}}$ such that

$$\begin{cases} \mathbf{Q}^m \rightharpoonup \bar{\mathbf{Q}} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; \mathcal{S}_0)), \\ \mathbf{Q}^m \rightharpoonup \bar{\mathbf{Q}} \text{ weakly in } L^4(0,T; L^4(\Omega; \mathcal{S}_0)), \\ (\mathrm{tr}\,(\mathbf{Q}^m)^2)\mathbf{Q}^m \rightharpoonup \mathbf{P} \text{ weakly in } L^{\frac{4}{3}}((0,T)\times\Omega)), \\ \varphi^m \rightharpoonup \bar{\varphi} \text{ weakly in } L^2(0,T; H^1(\Omega)). \end{cases} \tag{3.1}$$

Using the Sobolev imbedding $W^{1,2} \subset L^4$[5], we obtain imbeddings

$$L^4(0,T; W^{1,2}(\Omega; \mathcal{S}_0)) \hookrightarrow L^4((0,T)\times\Omega; \mathcal{S}_0),$$
$$L^{\frac{4}{3}}((0,T)\times\Omega; \mathcal{S}_0) \hookrightarrow L^{\frac{4}{3}}(0,T; \left[W^{1,2}(\Omega; \mathcal{S}_0)\right]').$$

It follows that $\left\{\frac{\partial \mathbf{Q}^m}{\partial t}\right\}_{m\in\mathbb{N}}$ is bounded in $L^{\frac{4}{3}}(0,T; \left[W^{1,2}(\Omega; \mathcal{S}_0)\right]')$. Since $\{\mathbf{Q}^m\}_{m\in\mathbb{N}}$ is bounded in $L^2(0,T; W^{1,2}(\Omega; \mathcal{S}_0))$, Aubin's compactness shows that

$$\mathbf{Q}^m \to \bar{\mathbf{Q}} \text{ strongly in } L^2(0,T; L^2(\Omega; \mathcal{S}_0)).$$

This concludes that $(\mathrm{tr}\,\bar{\mathbf{Q}}^2)\bar{\mathbf{Q}} = \mathbf{P}$, and therefore $(\bar{\mathbf{Q}}, \bar{\varphi})$ is a weak solution pair. □

COROLLARY 1. *There exists a weak solution pair* $(\mathbf{Q}, \varphi)$ *which belongs to* $L^2(0,\infty; W^{1,2}(\Omega; \mathcal{S}_0) \times L^2(0,\infty; H^1(\Omega))$ *to (1.4),(1.5), and (1.6).*

Proof. As in the proof of lemma 2.1, multiplying (1.4),(1.5) by $\mathbf{Q}$ and φ followed by integration by parts we obtain

(3.2)
$$\int_\Omega \left(L|\nabla \mathbf{Q}|^2 + A\,\mathrm{tr}\,\mathbf{Q}^2 - B\,\mathrm{tr}\,\mathbf{Q}^3 + C(\mathrm{tr}\,\mathbf{Q}^2)^2 + \frac{1}{2}\left(\epsilon(\mathbf{Q})\nabla\varphi\right)\cdot\nabla\varphi \right) dx + \frac{d}{dt}\int_\Omega |\mathbf{Q}|^2\,dx = 0$$

Since $C > 0$, there exists $\mathcal{D}$ such that $A\,\mathrm{tr}\,\mathbf{Q}^2 - B\,\mathrm{tr}\,\mathbf{Q}^3 + C(\mathrm{tr}\,\mathbf{Q}^2)^2 \geq -\mathcal{D}$. It follows from (3.2) and the Poincare inequality that

$$\frac{d}{dt}\int_\Omega |\mathbf{Q}|^2\,dx + \mathcal{M}\int_\Omega |\mathbf{Q}|^2\,dx \leq \mathcal{D}|\Omega|,$$

where $\mathcal{M} = \frac{L}{\mathcal{K}} > 0$ with the Poincare constant $\mathcal{K}$. By Grownwall's inequality we have

$$||\mathbf{Q}(t)||_{L^2} \leq ||\mathbf{Q}(0)||_{L^2} e^{-\mathcal{M}t} + \frac{\mathcal{D}|\Omega|}{\mathcal{M}}\left(1 - e^{-\mathcal{M}t}\right).$$

Therefore $\sup_{0\leq t<\infty} ||\mathbf{Q}(t)||_{L^2} \leq \frac{\mathcal{D}|\Omega|}{\mathcal{M}}$ and the proof is complete. □

Next, we prove that such a weak solution is unique and it converges to an equilibrium solution of the energy functional $\mathcal{E}$.

THEOREM 3.2. *If $\epsilon_1 = \epsilon_2 = 0$ in (1.5), then there exists a unique weak solution to (1.4),(1.5), and (1.6).*

Proof. Let $(\mathbf{Q}_1, \varphi_1)$ and $(\mathbf{Q}_2, \varphi_2)$ be two weak solutions. Then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{Q}|^2\,dx + \int_\Omega \left[L|\nabla\mathbf{Q}|^2 + \left(f'_{bulk}(\mathbf{Q}_1) - f'_{bulk}(\mathbf{Q}_2)\right)\cdot\mathbf{Q}\right] = \frac{1}{2}\epsilon_3\int_\Omega \nabla\varphi\cdot(\nabla\cdot\mathbf{Q})\,dx,$$

$$\int_\Omega (\epsilon_0\nabla\varphi)\cdot\nabla\varphi\,dx = -\epsilon_3\int_\Omega (\nabla\cdot\mathbf{Q})\cdot\nabla\varphi\,dx,$$

where $\mathbf{Q} = \mathbf{Q}_1 - \mathbf{Q}_2$, $\varphi = \varphi_1 - \varphi_2$. Plugging the second equation into the first one, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{Q}|^2\,dx + \int_\Omega \left(L|\nabla\mathbf{Q}|^2 + \frac{1}{2}\epsilon_0|\nabla\varphi|^2\right) dx \\
&= \int_\Omega \left[\left(f'_{bulk}(\mathbf{Q}_1) - f'_{bulk}(\mathbf{Q}_2)\right)\cdot\mathbf{Q}\right] dx \\
&\leq M\int_\Omega |\mathbf{Q}|^2\,dx \text{ for some } M > 0.
\end{aligned}$$

Hence $||\mathbf{Q}||_{L^2} \leq ||Q(0)||_{L^2} e^t = 0$ so that $\mathbf{Q}_1 = \mathbf{Q}_2$ and $\varphi_1 = \varphi_2$. □

THEOREM 3.3. *If $\mathbf{Q}_0 \in W^{1,2}(\Omega, \mathcal{S}_0)$, then there is a subsequence of solutions to (1.4) which converges to a solution of the steady state problem as $t \to \infty$.*

Proof. Multiplying individual equation by $\frac{\partial Q_{ij}}{\partial t}$ and integrating by parts followed by summing up, we obtain

$$\begin{aligned}\int_\Omega |\mathbf{Q}_t|^2 &= -\frac{d}{dt}\int_\Omega \left[\frac{L}{2}|\nabla\mathbf{Q}|^2 + A\mathrm{tr}\,\mathbf{Q}^2 - B\,\mathrm{tr}\,\mathbf{Q}^3 + C\left(\mathrm{tr}\,\mathbf{Q}^2\right)^2\right] dx \\ &\quad + \epsilon_3 \int_\Omega \nabla\cdot\mathbf{Q}_t\cdot\nabla\varphi\,dx - \frac{d}{dt}\int_\Omega |\nabla\varphi|^2\,dx \\ &= \epsilon_3\int_\Omega \nabla\cdot\mathbf{Q}_t\cdot\nabla\varphi\,dx.\end{aligned}$$

Then

$$\begin{aligned}&\int_\Omega |\mathbf{Q}_t|^2 \\ &= -\frac{d}{dt}\int_\Omega \left[\frac{L}{2}|\nabla\mathbf{Q}|^2 + A\mathrm{tr}\,\mathbf{Q}^2 - B\,\mathrm{tr}\,\mathbf{Q}^3 + C\left(\mathrm{tr}\,\mathbf{Q}^2\right)^2 + |\nabla\varphi|^2\right] dx\end{aligned}$$

and

$$\begin{aligned}&\int_0^T\int_\Omega |\mathbf{Q}_t|^2\,dx\,dt \\ &= -\int_\Omega \left[\frac{L}{2}|\nabla\mathbf{Q}|^2 + A\mathrm{tr}\,\mathbf{Q}^2 - B\,\mathrm{tr}\,\mathbf{Q}^3 + C\left(\mathrm{tr}\,\mathbf{Q}^2\right)^2 + |\nabla\varphi|^2\right]_{t=T} dx \\ &\quad + \int_\Omega \left[\frac{L}{2}|\nabla\mathbf{Q}|^2 + A\mathrm{tr}\,\mathbf{Q}^2 - B\,\mathrm{tr}\,\mathbf{Q}^3 + C\left(\mathrm{tr}\,\mathbf{Q}^2\right)^2 + |\nabla\varphi|^2\right]_{t=0} dx \\ &\leq \int_\Omega \left[\frac{L}{2}|\nabla\mathbf{Q}|^2 + A\mathrm{tr}\,\mathbf{Q}^2 - B\,\mathrm{tr}\,\mathbf{Q}^3 + C\left(\mathrm{tr}\,\mathbf{Q}^2\right)^2 + |\nabla\varphi|^2\right]_{t=0} dx \\ &\quad + M|\Omega|,\end{aligned}$$

where M is the minimum value of $A\mathrm{tr}\,\mathbf{Q}^2 - B\,\mathrm{tr}\,\mathbf{Q}^3 + C\left(\mathrm{tr}\,\mathbf{Q}^2\right)^2$.

Hence we obtain $\mathbf{Q}_t \in L^2(0,\infty; L^2(\Omega;\mathcal{S}_0))$. This shows that

$$\int_\Omega |\mathbf{Q}_t(x,t_i)|^2\,dx \to 0 \text{ as } i\to\infty,$$

for almost all sequence $\{t_i\}_{i\in\mathbb{N}}$ satisfying $t_i \to \infty$ as $i \to \infty$. Furthermore, we also get

$$(\nabla\mathbf{Q}, \nabla\varphi) \in L^\infty(0,\infty; L^2(\Omega;\mathcal{S}_0)) \times L^\infty(0,\infty; L^2(\Omega)).$$

By Poincare inequality, we have

$$(\mathbf{Q},\varphi) \in L^\infty(0,\infty;W^{1,2}(\Omega;\mathcal{S}_0)) \times L^\infty(0,\infty;W^{1,2}(\Omega))$$

and there is a sequence $\{t_i\}_{i\in\mathbb{N}}$ with $t_i \to \infty$ as $i \to \infty$ such that

$$(\mathbf{Q}(x,t_i),\varphi(x,t_i)) \rightharpoonup (\bar{\mathbf{Q}},\bar{\varphi}) \text{ weakly in } W^{1,2} \text{ as } t_i \to \infty.$$

Since $(\mathbf{Q},\varphi)$ is a weak solution pair,

$$\begin{cases} \left\langle \frac{\partial \mathbf{Q}}{\partial t}, \tilde{\mathbf{Q}} \right\rangle + \left\langle L\nabla\mathbf{Q} + A\mathbf{Q} - B\mathrm{tr}\mathbf{Q}^2 + C(\mathrm{tr}\mathbf{Q}^2)\mathbf{Q}, \nabla\tilde{\mathbf{Q}} \right\rangle \\ \quad +\epsilon \left\langle \nabla\varphi, \nabla\cdot\tilde{\mathbf{Q}} \right\rangle = 0, \\ \int_\Omega \left(\nabla\varphi\cdot\nabla\psi + \nabla\cdot\mathbf{Q}\cdot\nabla\psi\right)\,dx = 0, \end{cases}$$

for all $\tilde{\mathbf{Q}} \in W^{1,2}(\Omega,\mathcal{S}_0), \psi \in W^{1,2}(\Omega)$. Passing to the limit as $t_i \to \infty$, we obtain

$$\begin{cases} \int_\Omega \left(L\nabla\bar{\mathbf{Q}} + A\bar{\mathbf{Q}} - B\mathrm{tr}\,\bar{\mathbf{Q}}^2 + C(\mathrm{tr}\,\bar{\mathbf{Q}}^2)\bar{\mathbf{Q}}\right)\cdot\nabla\tilde{\mathbf{Q}} \\ \quad +\epsilon\int_\Omega \nabla\bar{\varphi}\cdot(\nabla\cdot\tilde{\mathbf{Q}})\,dx = 0, \\ \int_\Omega \left(\nabla\bar{\varphi}\cdot\nabla\psi + \nabla\cdot\bar{\mathbf{Q}}\cdot\nabla\psi\right)\,dx = 0. \end{cases}$$

This completes the proof. □

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