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AN AFFINE MODEL OF $X_0(pq)+q$

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Abstract. In this article, we show that the modular equation $\Phi_p^{T_q}(X,Y)$ of Thompson series T_q corresponding to $\Gamma_0(q)+q$ gives an affine model of a modular curve $X_0(pq)+q$.

1. Introduction

Let $T_{(n)}$ be the fundamental Thompson series corresponding to $\Gamma_0(n)$. Chen and Yui [1] constructed the modular polynomial $\Phi_{mn}(X,Y) \in \mathbb{Z}[X,Y]$ for $T_{(n)}$ and investigated their properties, where m is any positive integer coprime to n. The author and Koo [2] showed that the modular polynomial $\Phi_{nm}(X,Y)$ for $T_{(n)}$ gives an affine model of the modular curve $X_0(mn)$ defined over $\mathbb{Q}$.

Let $\Gamma_0(q)+q$ be the group generated by a Hecke group $\Gamma_0(q)$ and a Frike involution $\left(\begin{smallmatrix} 0 & -1 \\ q & 0 \end{smallmatrix}\right)$ and let $\Gamma_0(pq)+q$ be the group generated by a Hecke group $\Gamma_0(pq)$ and an Atkin-Lehner involution $\left(\begin{smallmatrix} q & 1 \\ xpq & yq \end{smallmatrix}\right)$. Let T_q be the fundamental Thompson series corresponding to $\Gamma_0(q)+q$. Chen and Yui [1] also constructed the modular polynomial $\Phi_p^{T_q}(X,Y) \in \mathbb{Z}[X,Y]$, where p,q are distinct primes. Let $X_0(pq)+q$ be a modular curve $\Gamma_0(pq)+q\backslash\mathfrak{H}^*$.

In this article, we show that the modular equation $\Phi_p^{T_q}(X,Y)$ gives an affine model of modular curve $X_0(pq)+q$ over $\mathbb{Q}$ (Theorem 2.4).

Throughout this article, we adopt the following notations:

- $q \in \{3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 41,\ 47,\ 59,\ 71\}$
- p : a prime number coprime to q
- $x, y \in \mathbb{Z}$ such that $yq - xp = 1$

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- $\mathfrak{H}$: the upper half complex plane
- $\mathfrak{H}^*$: the extended upper half complex plane
- $\Gamma_0(q)+q$: the group generated by a Hecke group $\Gamma_0(q)$ and a Frike involution $\left(\begin{smallmatrix} 0 & -1 \\ q & 0 \end{smallmatrix}\right)$
- $\Gamma_0(pq)+q$: the group generated by a Hecke group $\Gamma_0(pq)$ and an Atkin-Lehner involution $\left(\begin{smallmatrix} q & 1 \\ xpq & yq \end{smallmatrix}\right)$
- $X_0(pq)+q$: a modular curve $\Gamma_0(pq)+q\backslash\mathfrak{H}^*$
- $K(X_0(pq)+q)$: the function field of $X_0(pq)+q$
- T_q : the fundamental Thompson series corresponding to $\Gamma_0(q)+q$
- $q_h = e^{2\pi iz/h}$ $(z \in \mathfrak{H})$

2. An affine model of $X_0(pq)+q$

At first, we show that $T_q(z)$ and $T_q(pz)$ generate the function field of $X_0(pq)+q$. We need the following two Lemmas.

LEMMA 2.1. *The coset* $\Gamma_0(pq)\left(\begin{smallmatrix} q & 1 \\ xpq & yq \end{smallmatrix}\right)$ *has no parabolic element.*

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ cpq & d \end{smallmatrix}\right)$ be any element of $\Gamma_0(pq)$. If $\gamma\left(\begin{smallmatrix} q & 1 \\ xpq & yq \end{smallmatrix}\right)$ is parabolic element, then $(qa+bxpq+pqc+dyq)^2 = 4q$, from where we have $q=2$. This is a contradiction. Thus we have the result. □

LEMMA 2.2. $\{\infty, 1/q\}$ *is the set of representatives of all* $\Gamma_0(pq)+q$*-inequivalent cusps.*

Proof. By Lemma 2.1, the cardinality of the set of representatives of all $\Gamma_0(pq)+q$-inequivalent cusps is half of that of the set of representatives of all $\Gamma_0(pq)$-inequivalent cusps (see [5, Proposition 1.37]). Thus the cardinality of the set of representatives of all $\Gamma_0(pq)+q$-inequivalent cusps is 2. Now it suffices to show that ∞ and $1/q$ are $\Gamma_0(pq)+q$-inequivalent. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be any element of $\Gamma_0(pq)$. If $\gamma\infty = 1/q$, then $c = \pm q$ (which is a contradiction). If $\gamma\left(\begin{smallmatrix} q & 1 \\ xpq & yq \end{smallmatrix}\right)\infty = 1/q$, then p divides a (which is also a contradiction). Thus ∞ and $1/q$ are $\Gamma_0(pq)+q$-inequivalent. □

For a cusp x, take $\sigma \in SL_2(\mathbb{Z})$ such that $\sigma x = \infty$. Then there exists $h > 0$ such that

$$\sigma\Gamma_0(pq)_x\sigma^{-1}\cdot\{\pm 1\} = \{\pm\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)^m : m \in \mathbb{Z}\}.$$

We call h the *width* of a cusp x in $\Gamma_0(pq)$. By Lemma 2.1, the width of a cusp x in $\Gamma_0(pq)+q$ is equal to that in $\Gamma_0(pq)$ (see [4, page 41]). Thus, we have the following table.

TABLE 1

a cusp	∞	$1/q$
the width	1	p

PROPOSITION 2.3. $K(X_0(pq)+q)=\mathbb{C}(T_q(z),T_q(pz))$.

Proof. The following equations show that $T_q(z)$ and $T_q(pz)$ are invariant under the action of $\Gamma_0(pq)+q$.

$$\begin{aligned} T_q(z)|_{\left(\begin{smallmatrix} q & 1\\ xpq & yq\end{smallmatrix}\right)} &= T_q(z)|_{\left(\begin{smallmatrix} 0 & -1\\ q & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} xp & y\\ -q & -1\end{smallmatrix}\right)} = T_q(z)|_{\left(\begin{smallmatrix} xp & y\\ -q & -1\end{smallmatrix}\right)} = T_q(z),\\ T_q(pz)|_{\left(\begin{smallmatrix} q & 1\\ xpq & yq\end{smallmatrix}\right)} &= T_q(z)|_{\left(\begin{smallmatrix} 0 & -1\\ q & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} xp & y\\ -pq & -p\end{smallmatrix}\right)} = T_q(z)|_{\left(\begin{smallmatrix} x & y\\ -q & -p\end{smallmatrix}\right)\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)}\\ &= T_q(pz),\\ T_q(pz)|_{\left(\begin{smallmatrix} a & b\\ pqc & d\end{smallmatrix}\right)} &= T_q\Big(\frac{a(pz)+bp}{qc(pz)+d}\Big) = T_q(pz) \quad \text{if } ad-bpqc=1. \end{aligned}$$

Now, we observe the behavior of $T_q(z)$ and $T_q(pz)$ at cusps ∞ and $1/q$.

- ∞ : Write $T_q(z)=\frac{1}{q_1}+\sum_{n\geq 1} b_n q_1^n (b_n\in\mathbb{Z})$, then

$$T_q(pz)=\frac{1}{q_1^p}+\sum_{n\geq 1} b_n q_1^{pn}.$$

- $1/q$: $\left(\begin{smallmatrix} 1 & 0\\ q & 1\end{smallmatrix}\right)$ sends ∞ onto $1/q$. Then we have $T_q(z)|_{\left(\begin{smallmatrix} 1 & 0\\ q & 1\end{smallmatrix}\right)}=T_q(z)$ and

$$T_q(pz)|_{\left(\begin{smallmatrix} 1 & 0\\ q & 1\end{smallmatrix}\right)} = T_q(z)|_{\left(\begin{smallmatrix} p & -y\\ q & -x\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y\\ 0 & p\end{smallmatrix}\right)} = T_q\Big(\frac{z+y}{p}\Big) = \frac{b}{q_p}+\sum_{n\geq 1} b^n c_n q_p^n,$$

where b is a nonzero constant. Therefore, $T_q(z)$ and $T_q(pz)$ are modular functions of $\Gamma_0(pq)+q$ and the orders of $T_q(z)$ and $T_q(pz)$ at cusps ∞ and $1/q$ are given as

TABLE 2

a cusp	∞	$1/q$
the order of $T_q(z)$ at a cusp	-1	$-p$
the order of $T_q(pz)$ at a cusp	$-p$	-1

For any $f\in K(X_0(pq)+q)$, let $d(f)$ be the total degree of poles of f. Then $[K(X_0(pq)+q):\mathbb{C}(f)]=d(f)$ (see [5, Proposition 2.11]). Since

$T_q(z)$ and $T_q(pz)$ have no poles on the complex upper half plane, from TABLE 2 we obtain that

$$d(T_q(z)^2 + T_q(pz)) = 3p, \quad d(T_q(z) + T_q(pz)) = 2p, \quad d(T_q(z)) = p + 1.$$

Thus we have $\gcd(d(T_q(z)^2 + T_q(pz)), d(T_q(z) + T_q(pz)), d(T_q(z))) = 1$, which implies that $K(X_0(pq)+q) = \mathbb{C}(T_q(z), T_q(pz))$. Because $[K(X_0(pq) +q) : \mathbb{C}(T_q(z), T_q(pz))]$ divides $[K(X_0(pq) + q) : \mathbb{C}(T_q^2(z) + T_q(pz))]$, $[K(X_0(pq) + q) : \mathbb{C}(T_q(z) + T_q(pz))]$ and $[K(X_0(pq) + q) : \mathbb{C}(T_q(z))]$. □

In [1], Chen-Yui construct the modular equation $\Phi_p^{T_q}(X, Y) \in \mathbb{Z}[X, Y]$ of $T_q(z)$ which is an irreducible symmetric polynomial in two variables X, Y such that

$$\Phi_p^{T_q}(X, T_q) = \prod_{w \in \Omega(p)} (X - T_q \circ w),$$

where $\Omega(p) = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : ad = p, a > 0, 0 \le b < d, \gcd(a, b, d) = 1\}$. The fact that $\Phi_p^{T_q}(T_q(pz), T_q(z)) = 0$ and Proposition 2.3 imply the following.

THEOREM 2.4. *The modular equation $\Phi_p^{T_q}(X, Y)$ of T_q gives an affine model of a modular curve $X_0(pq) + q$.*

Finally we give several examples of the modular equation $\Phi_p^{T_q}(X, Y)$ using Mahler's explicit description of modular equations for "basic" S_p-series ([3, p.90-93]).

EXAMPLE 2.5.

- $p = 2,\ q = 3.$ $\Phi_2^{T_3}(X, Y) = X^3 + (-Y^2 + 1566)X^2 + (17343Y + 741474)X + (Y^3 + 1566Y^2 + 741474Y + 28166076)$ gives an affine model of $X_0(6) + 3$.
- $p = 2,\ q = 7.$ $\Phi_2^{T_7}(X, Y) = X^3 + (-Y^2 + 102)X^2 + (407Y + 3810)X + (Y^3 + 102Y^2 + 3810Y + 27100)$ gives an affine model of $X_0(14) + 7$.
- $p = 3,\ q = 7.$ $\Phi_3^{T_7}(X, Y) = X^4 + (-Y^3 + 153Y + 612)X^3 + (2043Y^2 + 37080Y + 168507)X^2 + (153Y^3 + 37080Y^2 + 652391Y + 2982276)X + (Y^4 + 612Y^3 + 168507Y^2 + 2982276Y + 13600628)$ gives an affine model of $X_0(21) + 7$.

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