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THE RELATION BETWEEN HENSTOCK INTEGRAL AND HENSTOCK DELTA INTEGRAL ON TIME SCALES

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Abstract. In this paper, we define an extension $f^* : [a,b] \to \mathbb{R}$ of a function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ for a time scale $\mathbb{T}$ and show that f is Henstock delta integrable on $[a,b]_{\mathbb{T}}$ if and only if f^* is Henstock integrable on $[a,b]$.

1. Introduction and preliminaries

The Henstock delta integral on time scales was introduced by Allan Peterson and Bevan Thompson [6].

In this paper, we investigate the relation between the Henstock integral and Henstock delta integral on time scales.

First, we introduce some concepts related to the notion of time scales. A time scale $\mathbb{T}$ is any closed nonempty subset of $\mathbb{R}$, with the topology inherited from the standard topology on the real numbers $\mathbb{R}$. For each $t \in \mathbb{T}$, we define the forward jump operator $\sigma(t)$ by

$$\sigma(t) = \inf\{z > t : z \in \mathbb{T}\}$$

and the backward jump operator $\rho(t)$ by

$$\rho(t) = \sup\{z < t : z \in \mathbb{T}\}$$

where $\inf \phi = \sup \mathbb{T}$ and $\sup \phi = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say the t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense,while if $\rho(t) = t$, we say that t is left-dense. The forward graininess function

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$\mu(t)$ is defined by $\mu(t) = \sigma(t) - t$, and the backward graininess function $\nu(t)$ is defined by $\nu(t) = t - \rho(t)$.

For $a, b \in \mathbb{T}$, we define the time scale interval in $\mathbb{T}$ by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

2. The Henstock and Henstock delta integrals

DEFINITION 2.1. ([6]) $\delta = (\delta_L, \delta_R)$ is a $\triangle$-gauge on $[a, b]_{\mathbb{T}}$ by $\delta_L(t) > 0$ on $(a, b]_{\mathbb{T}}$, $\delta_R(t) > 0$ on $[a, b)_{\mathbb{T}}$, $\delta_L(a) \geq 0, \delta_R(b) \geq 0$, and $\delta_R(t) \geq \mu(t)$ for each $t \in [a, b)_{\mathbb{T}}$.

DEFINITION 2.2. ([6]) A collection $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^{n}$ of tagged intervals is a Henstock partition of $[a, b]_{\mathbb{T}}$ if $\bigcup_{i=1}^{n}[t_{i-1}, t_i]_{\mathbb{T}} = [a, b]_{\mathbb{T}}$, $[t_{i-1}, t_i]_{\mathbb{T}} \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)]$ and $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}}$ for each $i = 1, 2, \cdots, n$.

For Henstock partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^{n}$, we write

$$S(f, \mathcal{P}) = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

whenever $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$.

DEFINITION 2.3. ([6]). A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is Henstock delta integrable (or $H_{\triangle}$-integrable) on $[a, b]_{\mathbb{T}}$ if there exists a number A such that for each $\epsilon > 0$ there exists a $\triangle$-gauge δ on $[a, b]_{\mathbb{T}}$ such that

$$\Big| S(f, \mathcal{D}) - A \Big| < \epsilon$$

for every δ-fine Henstock partition $\mathcal{D}$ of $[a, b]_{\mathbb{T}}$. The number A is called the $H_{\triangle}$-integral of f on $[a, b]_{\mathbb{T}}$, and we write $A = (H_{\triangle}) \int_a^b f \triangle t$.

Recall that $f : [a, b] \to \mathbb{R}$ is Henstock integrable (or H-integrable) on $[a, b]$ if there exists a number A such that for each $\epsilon > 0$ there exists a gauge $\delta : [a, b] \to \mathbb{R}^+$ on $[a, b]$ such that

$$\Big| S(f, \mathcal{P}) - A \Big| < \epsilon$$

for every δ-fine Henstock partition $\mathcal{P}$ of $[a, b]$.

The proof of the following theorem can be found in [5].

THEOREM 2.4. [5] *A function $f : [a, b] \to \mathbb{R}$ is H-integrable on $[a, b]$ if and only if f is $H_{\triangle}$-integrable on $[a, b]$.*

Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a function on $[a,b]_{\mathbb{T}}$, and let $\{(a_k, b_k)\}_{k=1}^{\infty}$ be the sequence of intervals contiguous to $[a,b]_{\mathbb{T}}$ in $[a,b]$.

Define a function $f^* : [a,b] \to \mathbb{R}$ on $[a,b]$ by

$$f^*(t) = \begin{cases} f(a_k) & \text{if} \quad t \in (a_k, b_k) \quad \text{for some} \quad k \\ f(t) & \text{if} \quad t \in [a,b]_{\mathbb{T}} \end{cases}$$

Then we have the following theorem.

THEOREM 2.5. [5] *If $f^* : [a,b] \to \mathbb{R}$ is H-integrable on $[a,b]$, then $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is $H_\triangle$-integrable on $[a,b]_{\mathbb{T}}$ and $(H_\triangle) \int_a^b f \triangle t = (H) \int_a^b f^*$.*

Now we can prove the following theorem.

THEOREM 2.6. *If $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is H_Δ-integrable, then $f^* : [a,b] \to \mathbb{R}$ is H-integrable and $(H) \int_a^b f^* = (H_\Delta) \int_a^b f \Delta t$.*

Proof. Let $\epsilon > 0$. Since f is H_Δ-integrable, there exists a Δ-gauge $\delta = (\delta_L, \delta_R)$ on $[a,b]_{\mathbb{T}}$ with $\delta_R(t) = \sigma(t) - t$ if t is a right-scattered point of $[a,b]_{\mathbb{T}}$ such that

$$\left| f(Q) - (H_\Delta) \int_a^b f \Delta t \right| < \epsilon$$

for each $\delta-$fine partition of $[a,b]_{\mathbb{T}}$.

Let $\{(a_k, b_k)\}_{k=1}^{\infty}$ be the sequence of intervals contiguous to $[a,b]_{\mathbb{T}}$ in $[a,b]$. Define a gauge $\delta^* = (\delta_L^*, \delta_R^*)$ on $[a,b]$ as follows;

For each left-dense point t of $[a,b]_{\mathbb{T}}$, choose $s_1 \in [a,b]_{\mathbb{T}}$ such that $t - \delta_L(t) \le s_1 < t$ and let $\delta_L^*(t) = t - s_1$.

For each right-dense point t of $[a,b]_{\mathbb{T}}$, choose $s_2 \in [a,b]_{\mathbb{T}}$ such that $t < s_2 \le t + \delta_R(t)$ and let $\delta_R^*(t) = s_2 - t$.

For each $k \in \mathbb{N}$, let $\delta_R^*(a_k) = \frac{b_k - a_k}{3}$ and

$$\delta_L^*(b_k) = \min\left\{ \frac{\epsilon}{2^k(|f(a_k)| + |f(b_k)| + 1)}, \frac{b_k - a_k}{3}, \delta_L(b_k) \right\}.$$

For each $t \in (a_k, b_k)$, let $\delta_L^*(t) = \frac{1}{2}(t - a_k)$ and $\delta_R^*(t) = \frac{1}{2}(b_k - t), k = 1, 2, 3 \cdots$.

Assume that $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ is a δ^*-fine partition of $[a,b]$. By separating the tagged interval $(\xi_i, [t_{i-1}, t_i])$ into $(\xi_i, [t_{i-1}, \xi_i])$ and $(\xi_i, [\xi_i, t_i])$ if necessary, we may assume that $\xi_i = t_{i-1}$ or $\xi_i = t_i$ for each $i = 1, 2, \cdot, n$.

Let $K = \{k \in \mathbb{N} |\ (a_k, b_k) \cap \{t_1, t_2, \cdots, t_{n-1}\} \ne \emptyset\}$ and let K be ordered as $K = \{k_1, k_2, k_3, \cdots, k_v\}$ such that $a_{k_1} < a_{k_2} < \cdots < a_{k_v}$. For each $j (1 \le j \le v)$, denote

$$p_j = \min\{i|\ t_i \in (a_{k_j}, b_{k_j}), 1 \le j \le n\} \quad \text{and}$$
$$q_j = \max\{i|\ t_i \in (a_{k_j}, b_{k_j}), 1 \le j \le n\}.$$

Obviously, we have $\xi_i \in [a,b]_{\mathbb{T}}$ if $t_{i-1} \in [a,b]_{\mathbb{T}}$ and $t_i \in [a,b]_{\mathbb{T}}$.

Let $Q_0 = \{(\xi_i, [t_{i-1}, t_i])|\ t_{i-1} \in [a,b]_{\mathbb{T}} \text{ and } t_i \in [a,b]_{\mathbb{T}}\}$. Then Q_0 is a $\delta-$fine partial partition of $[a,b]_{\mathbb{T}}$. Note that $\xi_{p_j} = t_{p_j-1} \in [a,b]_{\mathbb{T}}$ and $\xi_{q_j+1} = t_{q_j+1} \in [a,b]_{\mathbb{T}}$ for each $j(1 \le j \le v)$.

Let $\mathcal{P}_j = \{(\xi_i, [t_{i-1}, t_i])|p_j \le i \le q_{j+1}\}$ for each $j = 1, 2, \cdots, v$. Then $f^*(\mathcal{P}) = f(Q_0) + \sum_{j=1}^{v} f^*(\mathcal{P}_j)$.

First, assume that $f^*(\mathcal{P}) \le (H_\Delta)\int_a^b f\Delta t$. Define $Q_1, Q_2, \cdots, Q_v$ as follows;

Case 1. $t_{p_j-1} = a_{k_j}$ and $t_{q_j+1} = b_{k_j}$. Let $Q_j = \{(\xi_{p_j}, [a_{k_j}, b_{k_j}])\}$. Then Q_j is $\delta-$fine and

$$\begin{aligned} f^*(\mathcal{P}_j) &= f(a_{k_j})(t_{q_j} - a_{k_j}) + f(b_{k_j})(b_{k_j} - t_{q_j}) \\ &= f(a_{k_j})(b_{k_j} - a_{k_j}) + (f(b_{k_j}) - f(a_{k_j}))(b_{k_j} - t_{q_j}) \\ &\ge f(a_{k_j})(b_{k_j} - a_{k_j}) - (|f(b_{k_j})| + |f(a_{k_j})|)(b_{k_j} - t_{q_j}) \\ &> f(Q_j) - \frac{\epsilon}{2^{k_j}}. \end{aligned}$$

Case 2. $t_{p_j-1} = a_{k_j}$ and $b_{k_j} < t_{q_{j+1}}$.
Let

$$Q_j = \begin{cases} \{(\xi_{q_j+1}, [a_{k_j}, t_{q_j+1}])\} & \text{if } f(a_{k_j}) \ge f(\xi_{q_j+1}) \\ \{(\xi_{p_j}, [a_{k_j}, b_{k_j}]), (\xi_{q_j+1}, [b_{k_j}, t_{q_j+1}])\} & \text{if } f(a_{k_j}) > f(\xi_{q_j+1}). \end{cases}$$

Then Q_j is $\delta-$fine and $f^*(\mathcal{P}_j) \ge f(Q_j)$.

Case 3. $t_{p_j-1} < a_{k_j}$ and $b_{k_j} = t_{q_j+1}$.
In this case, we have

$$\begin{aligned} f^*(\mathcal{P}_j) &= f(\xi_{p_j})(t_{p_j} - t_{p_j-1}) + f(b_{k_j})(b_{k_j} - t_{q_j}) + f(a_{k_j})(t_{q_j} - t_{p_j}) \\ &\ge f(\xi_{p_j})(t_{p_j} - t_{p_j-1}) + f(a_{k_j})(b_{k_j} - t_{p_j}) - \frac{\epsilon}{2^{k_j}}. \end{aligned}$$

Let

$$Q_j = \begin{cases} \{(\xi_{p_j}), [t_{p_j-1}, b_{k_j}]\} & \text{if } f(\xi_{p_j}) \le f(a_{k_j}) \\ \{(\xi_{p_j}, [t_{p_j-1}, a_{k_j}]), (a_{k_j}, [a_{k_j}, b_{k_j}])\} & \text{if } f(\xi_{p_j}) > f(a_k). \end{cases}$$

Then $f^*(\mathcal{P}_j) \ge f(Q_j) - \frac{\epsilon}{2^{k_j}}$.

Case 4. $t_{p_j-1} < a_{k_j}$ and $b_{k_j} < t_{q_j+1}$.

Let $m = \min\{f(\xi_{p_j}), f(a_{k_j}), f(\xi_{q_j+1})\}$. Define

$$Q_i = \begin{cases} \{(\xi_{p_j}, [t_{p_j-1}, b_{k_j}]), (\xi_{q_j+1}, [b_{k_j}, t_{q_j+1}])\} \text{ if } f(\xi_{p_j}) = m \\ \{(\xi_{p_j}, [t_{p_j-1}, a_{k_j}]),(\xi_{q_j+1}, [a_{k_j}, t_{q_j+1}])\} \\ \qquad \text{if } f(\xi_{p_j}) = m \text{ and } f(a_{k_j}) \geq f(\xi_{q_j+1}) \\ \{(\xi_{p_j}, [t_{p_j-1}, a_k]), (a_{k_j}, [a_{k_j}, b_{k_j}]), (\xi_{q_j+1}, [b_{k_j}, t_{q_j+1}])\} \\ \qquad \text{if } f(\xi_{p_j}) \neq m \text{ and } f(a_{k_j}) < f(\xi_{q_j+1}) \end{cases}$$

Then $f^*(\mathcal{P}_j) \geq f(Q_j)$. Now let $Q = Q_0 \cup Q_1 \cup Q_2 \cup \cdots \cup Q_v$. Then Q is a $\delta-$fine partition of $[a, b]_{\mathbb{T}}$ and

$$(H_\Delta) \int_a^b f \Delta t \geq f^*(\mathcal{P}) \geq f(Q) - \sum_{j=1}^{v} \frac{\epsilon}{2^{k_j}} > (H_\Delta) \int_a^b f \Delta t - 2\epsilon.$$

Similarly, in the case $f^*(\mathcal{P}) > (H_\Delta) \int_a^b f \Delta t$ we can construct a $\delta-$fine partition Q' of $[a, b]_{\mathbb{T}}$ such that

$$f^*(\mathcal{P}) \leq f(Q') + \epsilon < (H_\Delta) \int_a^b f \Delta t + 2\epsilon.$$

Hence, f^* is $H-$integrable and

$$(H) \int_a^b f^* = (H_\Delta) \int_a^b f \Delta t.$$

□

From Theorem 2.5 and 2.6, we get the following theorem.

Theorem 2.7. *A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is $H_\Delta-$ integrable if and only if $f^* : [a, b] \to \mathbb{R}$ is Henstock integrable. In this case,*

$$(H_\Delta) \int_a^b f \Delta t = (H) \int_a^b f^*.$$

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