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SPHERICAL CAPS IN A CONVEX CONE

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Abstract. We show that a compact embedded hypersurface with constant ratio of mean curvature functions in a convex cone $C \subset \mathbb{R}^{n+1}$ is part of a hypersphere if it has a point where all the principal curvatures are positive and if it is perpendicular to ∂C.

1. Introduction

Let S be a hypersurface in the $n+1$ dimensional Euclidean space $\mathbb{R}^{n+1}$. Its rth mean curvature function H_r is the rth elementary symmetric function of principal curvature function of M divided by $\begin{pmatrix} n \\ r \end{pmatrix}$. Hence the Gauss-Kronecker curvature is H_n and the usual mean curvature function is H_1. H_0 is defined to be one. It is well known that an embedded closed hypersurface in $\mathbb{R}^{n+1}$ with nonzero constant mean curvature function H_r is a round sphere [1, 6]. A closed embedded hypersurface in $\mathbb{R}^{n+1}$ with constant ratio of mean curvature functions, $H_k/H_r = c$, is also a round sphere [4, 5].

Among embedded compact hypersurfaces with nonempty boundary, it is known that compact embedded hypersurface in $\mathbb{R}^{n+1}$ with nonzero constant $H_r, r \geq 2$ and spherical boundary are spherical caps, that is, part of a round hypersphere [2]. It is also known recently in [3] that a compact embedded hypersurface with constant H_r in a convex piecewise smooth cone C which is perpendicular to ∂C is part of a spherical cap. In this paper, we generalize this in the following theorem:

Theorem 1.1. *Let C be a domain in $\mathbb{R}^{n+1}$ which is a convex cone with piecewise smooth boundary ∂C and with the vertex at the origin. Let $S \subset C$ be an embedded compact hypersurface with boundary in ∂C such that S is perpendicular to ∂C along ∂S. If there is a point on S where all the principal curvatures are positive and if the ratio H_k/H_l is*

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a nonzero constant for some $k, l = 1, 2, \cdots, n$, $k \neq l$, S is a spherical cap.

2. Proof

Let C be a domain in $\mathbb{R}^{n+1}$ which is a convex cone with piecewise smooth boundary ∂C and with the vertex at the origin. Let $S \subset C$ be an embedded compact hypersurface with boundary in ∂C such that S is perpendicular to ∂C along ∂S. Let η be the unit normal vector field of the embedding $X : S \to \mathbb{R}^{n+1}$.

The following Lemma is given in [3].

LEMMA 2.1. *The following holds for $k = 1, 2, \cdots, n$:*

$$\int_S (H_{k-1} - H_k \langle X, \eta \rangle) = 0.$$

The following lemma is given in [4].

LEMMA 2.2. *Suppose $H_k > 0$ for some $k \geq 2$. Then the followings hold:*

(i) *For any $j = 1, 2, \cdots, k, H_j > 0$. Moreover, $H_k^{\frac{k-1}{k}} \leq H_{k-1}$.*
(ii) *$H_k / H_{k-1} \leq H_{k-1} / H_{k-2}$.*
(iii) *For every $l < k$, $H_k / H_l \leq H_{k-1} / H_{l-1}$.*

Now, since S is compact, one can find a point in S where all the principal curvatures are positive. Without loss of generality, we may assume that $1 \leq l < k \leq n$. Then all H_k's are positive at that point. Since H_k / H_l is constant on S and since H_l does not vanish on S by assumption, H_k and H_l are both positive on S. Then from Lemma 2.2 (iii), we have

$$0 < \alpha := H_k / H_l \leq H_{k-1} / H_{l-1}. \tag{2.1}$$

Since $H_k = \alpha H_l$, we have by Lemma 2.1

$$\begin{aligned} 0 &= \int_S (H_{k-1} - H_k \langle X, \eta \rangle) \\ &= \int_S (H_{k-1} - \alpha H_l \langle X, \eta \rangle), \end{aligned}$$

that is, we have

$$\int_S H_{k-1} = \int_S \alpha H_l \langle X, \eta \rangle. \tag{2.2}$$

On the other hand, since α is constant, we also have by Lemma 2.1

$$\int_S \alpha(H_{l-1} - H_l\langle X, \eta\rangle) = 0,$$

that is, we have

$$\int_S \alpha H_{l-1} = \int_S \alpha H_l \langle X, \eta\rangle. \tag{2.3}$$

From (2.2) and (2.3), we have

$$\int_S (H_{k-1} - \alpha H_{l-1}) = 0.$$

Since we have from (2.1) and Lemma 2.2 (i), $H_{k-1} - \alpha H_{l-1} \geq 0$, it follows that

$$H_{k-1}/H_{l-1} = \alpha = H_k/H_l$$

everywhere on S. Now proceeding inductively, we have finally

$$H_{k-l} = H_{k-l}/H_0 = \alpha$$

everywhere on S. Thus by the aforementioned result of [3], S is a spherical cap.

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