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ON MODULAR FIBONACCI AND TRIBONACCI TABLES

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Abstract. The work is devoted to study Fibonacci and tribonacci numbers. We study the modular formulas and the periods of the sequences.

1. Introduction

The investigation of Fibonacci sequence $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$, $F_1 = 1$ has been extended to algebraic aspect since D.D. Wall [7] in 1960. In particular researches including [1], [3], [6] were devoted to study Fibonacci sequences by modulo n in connection with order and period. The Fibonacci sequence has been studied in some arithmetic triangle forms, for instance all Fibonacci numbers appear along the diagonal of the Pascal triangle. Instead of triangle, if we display the Fibonacci sequence in rectangle form [2], say a rectangle with three columns, and if we take each numbers by mod $F_3 = 2$ then we have the following tables

1	1	2		1	1	0
3	5	8		1	1	0
13	21	34	and	1	1	0
55	89	$\cdots$		1	1	$\cdots$

We call the left table the 3 columns Fibonacci table. It shows $(2\cdot 2)34 + 8 = 144$ and $(2\cdot 2)55 + 13 = 233$, where these can be expressed by

$$2F_3F_9 + F_6 = F_{12} \text{ and } 2F_3F_{10} + F_7 = F_{13}.$$

And the right table, called the 3 columns modular table, shows a repetition of modular Fibonacci numbers. Similarly the 4 columns Fibonacci and its modular table by mod $T_4 = 3$

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1	1	2	3		1	1	2	0
5	8	13	21		2	2	1	0
34	55	89	144		1	1	2	0
233	377	610	$\cdots$		2	2	1	$\cdots$

show that $(2(3)+1)233 - 34 = 1597$, i.e., $(2F_4 + F_1)F_{13} - F_9 = F_{17}$. Thus for instance, the 25th Fibonacci number F_{25} can be obtained by

$$(2F_4 + F_1)F_{21} + (-1)^3 F_{17} = (7)10946 - 1597 = 75025 = F_{25}.$$

When we say tribonacci sequence T_n, we mean a sequence like F_n, but instead of two initial 0 and 1, the tribonacci sequence starts with three values 0, 0 and 1 and each term afterwards is the sum of the preceding three terms. Hence $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ with $T_0 = 0$, $T_1 = T_2 = 1$, and the first a few tribonacci numbers are $\{0, 0, 1, 1, 2, 4, 7, 13, 24, 44, \cdots\}$.

In this work we study Fibonacci and tribonacci sequence by displaying in rectangle form. By taking modular, we will find periods of the sequences.

2. Fibonacci table and modular Fibonacci table

The Fibonacci number F_n can be extended to negative n such that $F_{-1} = 1$, $F_{-2} = -1$ and $F_{-3} = 2$, and $F_{-n} = (-1)^{n+1}F_n$ for all $n \in \mathbb{Z}$.

Lemma 2.1. *Let $n, t \in \mathbb{Z}$.*

(1) $F_{n+3} = 2F_3F_n + F_{n-3}$. *If $n = 3t + r$ $(1 \le r \le 3)$ then* $F_n = 2F_3F_{3(t-1)+r} + F_{3(t-2)+r}$. *So* $F_{3t+r} \equiv F_{3(t-2)+r} (mod F_3)$.
(2) $F_{n+4} = (2F_4 + F_1)F_n - F_{n-4}$.
(3) *If* $n = 4t+r$ $(1 \le r \le 4)$ *then* $F_n = (2F_4+F_1)F_{4(t-1)+r} - F_{4(t-2)+r}$. *So,* $F_{4t+r} \equiv F_{4(t-1)+r} - F_{4(t-2)+r} (mod F_4)$.

Proof. We have seen that $F_{n+3} = 2F_3F_n + F_{n-3}$ for $n = 1, 2$. Assume $F_{i+3} = 2F_3F_i + F_{i-3}$ for all $i \le n$. Then (1) is clear that

$$\begin{aligned} F_{(n+1)+3} &= F_{n+3} + F_{(n-1)+3} \\ &= 2F_3F_n + F_{n-3} + 2F_3F_{n-1} + F_{n-4} \\ &= 2F_3(F_n + F_{n-1}) + F_{n-3} + F_{n-4} = 2F_3F_{n+1} + F_{(n+1)-3}. \end{aligned}$$

The rest can be proved similarly. □

This can be generalized as follows.

Theorem 2.2. *If* $n \in \mathbb{Z}$ *then* $F_{n+k} = (2F_k + F_{k-3})F_n + (-1)^{k-1}F_{n-k}$ *for all* $k \geq 3$. *If we write* $n = kt + r$ $(t, r \in \mathbb{Z},\ 1 \leq r \leq k)$ *then*

$$F_n = F_{kt+r} = (2F_k + F_{k-3})F_{k(t-1)+r} + (-1)^{k-1}F_{k(t-2)+r}$$

and $F_{kt+r} \equiv F_{k-3}F_{k(t-1)+r} + (-1)^{k-1}F_{k(t-2)+r} \pmod{F_k}$.

Proof. The cases of $k = 3$ or 4 are due to Lemma 2.1. We now will consider the 5 columns Fibonacci table:

1	1	2	3	5
8	13	21	34	55
89	144	233	377	610
987	1597	2584	4181	$\cdots$

It shows $(2(5)+1)(377)+34 = 4181$, i.e., $(2F_5 + F_2)F_{16} + F_{11} = F_{21}$. So

$$F_{n+5} = (2F_5 + F_2)F_5 + F_{n-5},$$

thus

$$F_{n+i} = (2F_i + F_{i-3})F_n + (-1)^{i-1}F_{n-i} \quad \text{for } 3 \leq i \leq 5.$$

Assume it is true for $i \leq k-1$. Then in the k columns Fibonacci table,

$$\begin{aligned}
&F_{n+k} \\
&= F_{n+(k-1)} + F_{n+(k-2)} \\
&= (2F_{k-1} + F_{k-4})F_n + (-1)^{k-2}F_{n-(k-1)} + (2F_{k-2} + F_{k-5})F_n + \\
&\quad (-1)^{k-1}F_{n-(k-2)} \\
&= (2(F_{k-1} + F_{k-2}) + F_{k-4} + F_{k-5})F_n + (-1)^{k-3}(-F_{n-(k-1)} + F_{n-(k-2)}) \\
&= (2F_k + F_{k-3})F_n + (-1)^{k-1}F_{n-k},
\end{aligned}$$

since $F_{n-k} + F_{n-(k-1)} = F_{n-(k-2)}$. Moreover for $n = kt + r$ $(1 \leq r \leq k)$,

$$\begin{aligned}
F_{kt+r} = F_{(n-k)+k} &= (2F_k + F_{k-3})F_{n-k} + (-1)^{k-1}F_{n-2k} \\
&= (2F_k + F_{k-3})F_{k(t-1)+r} + (-1)^{k-1}F_{k(t-2)+r}.
\end{aligned}$$

Thus $F_{kt+r} \equiv F_{k-3}F_{k(t-1)+r} + (-1)^{k-1}F_{k(t-2)+r} \pmod{F_k}$. □

It shows that F_{kt+r} is a combination of $F_{k(t-1)+r}$ and $F_{k(t-2)+r}$ with coefficient $2F_k + F_{k-3}$ and $(-1)^{k-1}$. Inductively we have the following.

Theorem 2.3. *Let* $n = kt + r$ $(1 \leq r \leq k)$. *Then every* F_n *can be written by only four Fibonacci numbers* F_k, F_{k-3}, F_r *and* F_{k+r}. *Moreover if* $F_{kt+r} = \theta_1 F_{k+r} + \theta_2 F_r$ *with* $\theta_1, \theta_2 \in \mathbb{Z}$ *then* $F_{k(t+1)+r} = ((2F_k + F_{k-3})\theta_1 + \theta_2)F_{k+r} + \theta_1 F_r$.

Proof. Theorem 2.2 implies that $F_{kt+r} = \mu F_{k(t-1)+r} + (-1)^{k-1}F_{k(t-2)+r}$ with $\mu = 2F_k + F_{k-3}$. We first assume k is odd. Then

$F_{2k+r} = \mu F_{k+r} + F_r$

$F_{3k+r} = \mu F_{2k+r} + F_{k+r} = \mu(\mu F_{k+r} + F_r) + F_{k+r} = (\mu^2+1)F_{k+r} + \mu F_r$

$F_{4k+r} = \mu F_{3k+r} + F_{2k+r} = (\mu^3 + 2\mu)F_{k+r} + (\mu^2+1)\mu F_r$

$F_{5k+r} = \mu F_{4k+r} + F_{3k+r} = (\mu^4 + 3\mu^2 + 1)F_{k+r} + (\mu^3 + 2\mu)\mu F_r.$

The first coefficient in this stage is μ times the first coefficient in previous step added to the second coefficient in previous step, while the second coefficient in this stage is the first coefficient in the previous step.

Now suppose that this pattern is true for all jth stages $(1 \leq j < t)$. That is, we assume that if $F_{k(j-1)+r} = \chi_1 F_{k+r} + \chi_2 F_r$ then $F_{kj+r} = \theta_1 F_{k+r} + \theta_2 F_r$ where $\theta_1 = \mu\chi_1 + \chi_2$ and $\theta_2 = \chi_1$ for $\chi_1, \chi_2 \in \mathbb{Z}$. Due to Theorem 2.2,

$$\begin{aligned} F_{k(j+1)+r} &= \mu F_{kj+r} + F_{k(j-1)+r} \\ &= \mu(\theta_1 F_{k+r} + \theta_2 F_r) + \chi_1 F_{k+r} + \chi_2 F_r \\ &= (\mu\theta_1 + \chi_1)F_{k+r} + (\mu\theta_2 + \chi_2)F_r \\ &= (\mu\theta_1 + \theta_2)F_{k+r} + (\mu\chi_1 + \chi_2)F_r = (\mu\theta_1 + \theta_2)F_{k+r} + \theta_1 F_r. \end{aligned}$$

The case when k is even can be prove similarly. □

It gives a good way to compute F_n by knowing only a few information about F_k, F_{k-3}, F_r and F_{k+r}. The first three are in the first row while the last one is in the second row of the k columns Fibonacci table.

Example 2.4. *For 50th Fibonacci F_{50}, take $k = 7$ for instance, then*

$$\begin{aligned} F_{50} &= F_{7\cdot7+1} = \mu F_{7\cdot6+1} + F_{7\cdot5+1} = (\mu^2+1)F_{7\cdot5+1} + \mu F_{7\cdot4+1} \\ &= (\mu(\mu^2+1) + \mu)F_{7\cdot4+1} + (\mu^2+1)F_{7\cdot3+1} \\ &= (\mu(\mu^3+2\mu) + \mu^2+1)F_{7\cdot3+1} + (\mu^3+2\mu)F_{7\cdot2+1} \\ &= (\mu(\mu^4+3\mu^2+1) + \mu^3 + 2\mu F_{7\cdot2+1} + (\mu^4+3\mu^2+1)F_{7+1} \\ &= (\mu(\mu^5+4\mu^3+3\mu) + \mu^4+3\mu^2+1)F_{7+1} + (\mu^5+4\mu^3+3\mu)F_1 \\ &= 12,586,269,025 \end{aligned}$$

by plugging $F_7 = 13$, $F_4 = 3$, $F_1 = 1$, $F_8 = 21$ and $\mu = 2F_7 + F_4 = 29$.

Corollary 2.5.

(1) *Every $F_{kt} \equiv 0 \pmod{F_k}$. If $n|m$ then $F_n|F_m$ for every $n, m \in \mathbb{Z}$.*
(2) *If k is even, every (t)th row is congruent to $(t\pm2)$th row by mod F_k in the k columns modular table. The first two rows are repeated in order, so the modular Fibonacci sequence by mod F_k is periodic of length $2k$.*

(3) *If k is odd, every (t)th row is congruent to $(t \pm 2)$th row with negative sign by mod F_k in the k columns modular table. The first four rows are repeated in order, so the modular Fibonacci sequence by mod F_k is periodic of length $4k$.*

Proof. Since F_{kt+r} is written by F_{k+r} and F_r, $F_{kt} = F_{k(t-1)+k}$ is a linear combination of F_{k+k} and F_k, and again by F_k and F_0. But since both F_k and F_0 are 0 by mod F_k, it follows $F_{kt} \equiv 0 (mod F_k)$. The rest are due to Theorem 2.3. □

Note that $F_k|F_{kt}$ in (1) has been proved by various ways. One way is due to show $\mathrm{per}_F(n) = lcm(\mathrm{per}_F(p_1), \cdots, \mathrm{per}_F(p_s))$ for all primes $p_i|n$ [7]. The other method is to use the fact $\gcd(F_k, F_t) = F_{\gcd(k,t)}$ in [4]. Of course $F_k|F_{kt}$ can be proved by induction on t after fixing k. However it seems that the proof using the k columns modulo table is more convenient than any other methods. Owing to Corollary 2.4, we can construct the modular Fibonacci tables for $5 \leq k \leq 8$:

mod ($F_5 = 5$)							mod ($F_6 = 8$)						
1	1	2	3	0			1	1	2	3	5	0	
3	3	1	4	0			5	5	2	7	1	0	
−1	−1	−2	−3	0			1	1	2	3	5	0	
−3	−3	−1	−4	⋯			5	5	2	7	1	⋯	
mod ($F_7 = 13$)							**mod ($F_8 = 21$)**						
1	1	2	3	5	8	0	1	1	2	3	5	8	13	0
8	−5	3	−2	1	−1	0	13	13	5	18	2	20	1	0
−1	−1	−2	−3	−5	−8	0	1	1	2	3	5	8	13	0
−8	5	−3	2	−1	1	⋯	13	13	5	18	2	20	1	⋯

3. Tribonacci table and modular tribonacci table

In this section we deal with tribonacci sequence $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ with $T_0 = 0$ and $T_1 = T_2 = 1$. Similar to Fibonacci numbers, T_n can be extended to negative n such that $T_{-1} = 0$, $T_{-2} = 1$, $T_{-3} = -1$ and $T_{-4} = 0$, etc. Let us consider the 4 columns tribonacci table

1	1	2	4
7	13	24	44
81	149	274	504
927	1705	3136	⋯

It is clear to see that
$T_{16} = (11)504 + (5)44 + 4 = (3T_4 - 1)T_{12} + (T_4 + 1)T_8 + T_4 = 5768$
$T_{19} = (11)3136 + (5)274 + 24 = (3T_4 - 1)T_{15} + (T_4 + 1)T_{11} + T_7 = 35890$

THEOREM 3.1. *Let $n = kt + r$ $(1 \le r \le k)$. Then for $4 \le k \le 6$,*

$$T_{kt+r} = \mu_1 T_{k(t-1)+r} + \mu_2 T_{k(t-2)+r} + T_{k(t-3)+r},$$

that is, $T_n = \mu_1 T_{n-k} + \mu_2 T_{n-2k} + \mu_3 T_{n-3k}$, where the coefficients (μ_1, μ_2, μ_3) depending on k are as follows

	$k = 4$	$k = 5$	$k = 6$
(μ_1, μ_2, μ_3)	$(3T_4 - 1, T_4 + 1, 1)$	$(3T_5, 1, 1)$	$(3T_6, -T_6 + 2, 1)$

Proof. When $k = 4$ we will prove

$$T_n = (3T_4 - 1)T_{n-4} + (T_4 + 1)T_{n-8} + T_{n-12}.$$

If $n = 12$ then $(3T_4 - 1)T_8 + (T_4 + 1)T_4 + T_0 = 504 = T_{12}$. Assume that

$$T_i = \mu_1 T_{i-4} + \mu_2 T_{i-8} + T_{i-12} \quad \text{for all} \quad 12 \le i \le n$$

with $\mu_1 = 3T_4 - 1$ and $\mu_2 = T_4 + 1$. Then

$$\begin{aligned}
&\mu_1 T_{(n+1)-4} + \mu_2 T_{(n+1)-8} + T_{(n+1)-12} \\
&= \mu_1(T_{n-4} + T_{(n-4)-1} + T_{(n-4)-2}) + \mu_2(T_{n-8} + T_{(n-8)-1} + T_{(n-8)-2}) \\
&\quad + (T_{n-12} + T_{(n-12)-1} + T_{(n-12)-2}) \\
&= (\mu_1 T_{n-4} + \mu_2 T_{n-8} + T_{n-12}) + (\mu_1 T_{(n-1)-4} + \mu_2 T_{(n-1)-8} + T_{(n-1)-12}) \\
&\quad + (\mu_1 T_{(n-2)-4} + \mu_2 T_{(n-2)-8} + T_{(n-2)-12}) \\
&= T_n + T_{n-1} + T_{n-2} = T_{n+1}.
\end{aligned}$$

If $n < 12$ then by considering negative tribonaccis $T_{-1} = 0$, $T_{-2} = 1$, etc., without loss of generality we have
$T_n = (3T_4 - 1)T_{n-4} + (T_4 + 1)T_{n-8} + T_{n-12}$ for all n.

Similarly from the 5 columns tribonacci table

1	1	2	4	7
13	24	44	81	149
274	504	927	1705	3136
5768	10609	19513	35890	$\cdots$

we can find that

$$\begin{cases} T_{17} = 10690 = (21)504 + 24 + 1 = (3T_5)T_{12} + T_7 + T_2 \\ T_{23} = 410744 = (21)19513 + 927 + 44 = (3T_5)T_{18} + T_{13} + T_8 \end{cases}$$

Moreover from the 6 columns tribonacci table

1	1	2	4	7	13
24	44	81	149	274	504
927	1705	3136	5768	10609	19513
35890	66012	121415	223317	410744	$\cdots$

it can be seen that

$$\begin{cases} T_{20} = 66012 = (39)1705 - (13-2)44 + 1 = (3T_6)T_{14} - (T_6 - 2)T_8 + T_2 \\ T_{22} = 223317 = (39)5768 - (13-2)149 + 4 = (3T_6)T_{16} - (T_6 - 2)T_{10} + T_4 \end{cases}$$

Now we assume that, for $k = 5$ or 6 the equality

$$T_{n+ki} = \mu_1 T_{n+k(i-1)} + \mu_2 T_{n+k(i-2)} + \mu_3 T_{n+k(i-3)}$$

with $(\mu_1, \mu_2, \mu_3) = (3T_5, 1, 1)$ or $(3T_6, -T_6 + 2, 1)$ hold for all $1 \le i < v$. Then

$$\begin{aligned} T_{n+kv} &= T_{(n+k)+k(v-1)} \\ &= \mu_1 T_{(n+k)+k(v-2)} + \mu_2 T_{(n+k)+k(v-3)} + \mu_3 T_{(n+k)+k(v-4)} \\ &= \mu_1 T_{n+k(v-1)} + \mu_2 T_{n+k(v-2)} + \mu_3 T_{n+k(v-3)}, \end{aligned}$$

it proves the theorem. □

THEOREM 3.2. *Let $n = kt + r$ $(1 \le r \le k)$. Then for $7 \le k \le 10$,*

$$T_{kt+r} = \mu_1 T_{k(t-1)+r} + \mu_2 T_{k(t-2)+r} + \mu_3 T_{k(t-3)+r}$$

where the coefficients (μ_1, μ_2, μ_3) are determined as follows.

$k = 7$	8	9	10
$(3T_7 - 1, 15, 1)$	$(3T_8 - 1, -1, 1)$	$(3T_9 - 2, -23, 1)$	$(3T_{10} - 4, 41, 1)$

Proof. The 7 columns tribonacci tables

1	1	2	4	7	13	24
44	81	149	274	504	927	1705
3136	5768	10609	19513	35890	66012	121415
223317	410744	755476	1389537	2555757	4700770	$\cdots$

shows that

$$\begin{cases} T_{22} = 223317 = (71)3136 + (15)44 + 1 = (3T_7 - 1)T_{15} + 15T_8 + T_1 \\ T_{27} = 4700770 = (71)66012 + (15)927 + 13 = (3T_7 - 1)T_{20} + 15T_{13} + T_6. \end{cases}$$

Thus similar to the proof of Theorem 3.1, it can be proved

$$T_{7t+r} = (3T_7 - 1)T_{7(t-1)+r} + 15T_{7(t-2)+r} + T_{7(t-3)+r} \ (1 \le r \le 7).$$

From the 8 columns tribonacci table

1	1	2	4	7	13	24	44
81	149	274	504	927	1705	3136	5768
10609	19513	35890	66012	121415	223317	$\cdots$	
1389537	2555757	4700770	8646064	15902591	$\cdots$		

we find that

$$\begin{cases} T_{25} = 1389537 = (131)10609 - (3)81 + = (3T_8 - 1)T_{17} - 3T_9 + T_1 \\ T_{29} = 15902591 = (131)121415 - (3)927 + 7 = (3T_8 - 1)T_{21} - 3T_{13} + T_5 \end{cases}$$

hence $T_{8t+r} = (3T_8 - 1)T_{8(t-1)+r} - T_{8(t-2)+r} + T_{8(t-3)+r}$ $(1 \le r \le 8)$.

The 9 and 10 columns tribonacci tables show that, for instance

$$\begin{cases} T_{28} = (241)35890 - (23)149 + 1 = (3T_9 - 2)T_{19} - (23)T_{10} + T_1 \\ T_{33} = (241)755476 - (23)3136 + 13 = (3T_9 - 2)T_{24} - (23)T_{15} + T_6 \end{cases}$$

hence $T_{9t+r} = (3T_9 - 2)T_{9(t-1)+r} - 23T_{9(t-2)+r} + T_{9(t-3)+r}$ $(1 \le r \le 9)$. And

$$\begin{cases} T_{31} = (443)121415 + (41)274 + 1 = (3T_{10} - 4)T_{21} + (41)T_{11} + T_1 \\ T_{34} = (443)410744 + (41)927 + 2 = (3T_{10} - 4)T_{24} + (41)T_{14} + T_4 \end{cases}$$

so $T_{10t+r} = (3T_{10} - 4)T_{10(t-1)+r} + 41T_{10(t-2)+r} + T_{10(t-3)+r}$ $(1 \le r \le 10)$.

Analogue to the proof of Theorem 3.1, the induction yields the identity $T_{kt+r} = \mu_1 T_{k(t-1)+r} + \mu_2 T_{k(t-2)+r} + \mu_3 T_{k(t-3)+r}$. □

We note that Theorem 3.1 and 3.2 can be extended to negative n of T_n by taking $T_{-1} = 0$, $T_{-2} = 1$, $T_{-3} = -1$, $\cdots$. The following theorem provides an efficient method for T_n with $n < 0$.

THEOREM 3.3. *Let $-n = k(-t) + r < 0$ $(1 \le r \le k,\ t > 0)$. Then*

$$T_{-n} = T_{k(-t)+r} = -\mu_2 T_{k(-t+1)+r} - \mu_1 T_{k(-t+2)+r} + T_{k(-t+3)+r}$$

for $4 \le k \le 10$, where the coefficients μ_1 and μ_2 (depending on k) are as in Theorem 3.1 and 3.2.

Proof. Due to Theorem 3.1 and 3.2,

$$\mu_1 T_{k(-t+2)+r} + \mu_2 T_{k(-t+1)+r} + \mu_3 T_{k(-t)+r} = T_{k(-t+3)+r}.$$

Since $\mu_3 = 1$ for all $4 \le k \le 10$,

$$T_{k(-t)+r} = -\mu_1 T_{k(-t+2)+r} - \mu_2 T_{k(-t+1)+r} + T_{k(-t+3)+r}.$$

□

For instance, $T_{-16} == -T_{5(-3)+4} - 21T_{5(-2)+4} + T_{5(-1)+4} = 56$.

THEOREM 3.4. *T_{kt+r} $(4 \le k \le 10)$ is a linear combination of T_{2k+r}, T_{k+r} and T_r.*

Proof. Due to Theorem 3.1 and 3.2, we have

$$
\begin{aligned}
&T_{kt+r}\\
&= \mu_1 T_{k(t-1)+r} + \mu_2 T_{k(t-2)+r} + \mu_3 T_{k(t-3)+r}\\
&= \mu_1(\mu_1 T_{k(t-2)+r} + \mu_2 T_{k(t-3)+r} + \mu_3 T_{k(t-4)+r}) + \mu_2 T_{k(t-2)+r}\\
&\quad + \mu_3 T_{k(t-3)+r}\\
&= (\mu_1^2 + \mu_2) T_{k(t-2)+r} + (\mu_1\mu_2 + \mu_3) T_{k(t-3)+r} + \mu_1 T_{k(t-4)+r}\\
&= (\mu_1^2 + \mu_2)(\mu_1 T_{k(t-3)+r} + \mu_2 T_{k(t-4)+r} + \mu_3 T_{k(t-5)+r})\\
&\quad + (\mu_1\mu_2 + \mu_3) T_{k(t-3)+r} + \mu_1 T_{k(t-4)+r}\\
&= (\mu_1(\mu_1^2 + \mu_2) + (\mu_1\mu_2 + \mu_3)) T_{k(t-3)+r} + (\mu_2(\mu_1^2 + \mu_2) + \mu_3) T_{k(t-4)+r}\\
&\quad + (\mu_1^2 + \mu_2)\mu_3 T_{k(t-5)+r}
\end{aligned}
$$

Hence after some steps, if we write

$$T_{kt+r} = \theta_1 T_{k(t-i-1)+r} + \theta_2 T_{k(t-i-2)+r} + \theta_3 T_{k(t-i-3)+r}$$

for some $i \in \mathbb{Z}$, then the next stage should be

$$T_{kt+r} = (\mu_1\theta_1 + \theta_2) T_{k(t-i-2)+r} + (\mu_2\theta_1 + \theta_3) T_{k(t-i-3)+r} + \mu_3\theta_1 T_{k(t-i-4)+r}.$$

Thus if $i = t-4$ then T_{kt+r} is a combination of T_{2k+r}, T_{k+r} and T_r. □

Example 3.5. *For T_{50}, take $k = 7$ for instance, then*

$$T_{50} = T_{7(7)+1} = \mu_1 T_{7(6)+1} + \mu_2 T_{7(5)+1} + \mu_3 T_{7(4)+1}$$

with $(\mu_1, \mu_2, \mu_3) = (3T_7 - 1,\ 15,\ 1) = (71, 15, 1)$. So we have

$$
\begin{aligned}
&T_{50}\\
&= 71 T_{7(6)+1} + 15 T_{7(5)+1} + T_{7(4)+1}\\
&= (71 \cdot 71 + 15) T_{7(5)+1} + (15 \cdot 71 + 1) T_{7(4)+1} + 71 T_{7(3)+1}\\
&= 5056 T_{7(5)+1} + 1066 T_{7(4)+1} + 71 T_{7(3)+1}\\
&= 360042 T_{7(4)+1} + 75911 T_{7(3)+1} + 5056 T_{7(2)+1}\\
&= 25638893 T_{7(3)+1} + 5405686 T_{7(2)+1} + 360042 T_{7(1)+1}\\
&= 1825767089 T_{7(2)+1} + 384943437 T_{7+1} + 25638893 T_1\\
&= 5,742,568,741,225,
\end{aligned}
$$

by plugging $T_{7(2)+1} = 3136$, $T_{7+1} = 44$ and $T_1 = 1$.

We note that, unlike the Fibonacci case in Theorem 2.2, the coefficients (μ_1, μ_2, μ_3) for tribonacci numbers in Theorem 3.1 and 3.2 depend on k. Now taking modular by tribonacci number T_k, the next corollary follows immediately.

Corollary 3.6. *Let $n = kt + r$ ($1 \le r \le k$). For $4 \le k \le 10$,*

$$T_{kt+r} \equiv \nu_1 T_{k(t-1)+r} + \nu_2 T_{k(t-2)+r} + \nu_3 T_{k(t-3)+r} \pmod{T_k}$$

where the coefficients (ν_1, ν_2, ν_3) are

k	(ν_1,ν_2,ν_3)	k	(ν_1,ν_2,ν_3)	k	(ν_1,ν_2,ν_3)	k	(ν_1,ν_2,ν_3)
4	$(-1,1,1)$	5	$(0,1,1)$	6	$(0,2,1)$	7	$(-1,15,1)$
8	$(-1,-1,1)$	9	$(-2,-23,1)$	10	$(-4,41,1)$		

Example 3.7. *For T_{50}, take $k = 5$ and by mod $T_5 = 7$ for instance,*
$T_{50} = T_{5\cdot 9+5} \equiv T_{5\cdot 7+5} + T_{5\cdot 6+5} \equiv (T_{5\cdot 5+5} + T_{5\cdot 4+5}) + T_{5\cdot 6+5}$
$\equiv T_{5\cdot 6+5} + T_{5\cdot 5+5} + T_{5\cdot 4+5} \equiv (T_{5\cdot 4+5} + T_{5\cdot 3+5}) + T_{5\cdot 5+5} + T_{5\cdot 4+5}$
$\equiv T_{5\cdot 5+5} + 2T_{5\cdot 4+5} + T_{5\cdot 3+5} \equiv 2T_{5\cdot 4+5} + 2T_{5\cdot 3+5} + T_{5\cdot 2+5}$
$\equiv 2T_{5\cdot 3+5} + 3T_{5\cdot 2+5} + 2T_{5+5} \equiv 3T_{5\cdot 2+5} + 4T_{5+5} + 2T_5 \equiv 1.$
On the other hand, by taking different $k = 10$, we have
$T_{50} = T_{10\cdot 4+10} \equiv -4T_{10\cdot 3+10} + 41T_{10\cdot 2+10} + T_{10\cdot 1+10}$
$\equiv (56)T_{10+10} + 98T_{10} + 57T_0 \equiv 56 \cdot 5 \equiv 131 \pmod{T_{10} = 149}$.

Corollary 3.5 yields k columns modular tribonacci tables, for instance

mod ($T_4 = 4$)				mod ($T_5 = 7$)					mod ($T_6 = 13$)					
1	1	2	0	1	1	2	4	0	1	1	2	4	7	0
3	1	0	0	6	3	2	4	2	11	5	3	6	1	10
1	1	2	0	1	0	3	4	0	4	2	3	9	1	0
3	1	0	0	0	4	4	1	2	10	11	8	3	9	7
1	1	2	0	0	3	5	1	2	6	9	9	11	3	10
3	1	0	$\cdots$	1	4	0	5	$\cdots$	11	11	6	2	6	$\cdots$

Theorem 3.8.

(1) *In the 4 columns modular tribonacci table*
  (i) $T_{4t+r} + T_{4(t-1)+r} \equiv T_{4(t-2)+r} + T_{4(t-3)+r}$ *(mod $T_4 = 4$).*
  (ii) $T_{4t+4} \equiv 0$ *and* $T_{4t+2} \equiv 1$ *for every t*
  (iii) $T_{4t+1} \equiv 1$ *and* $T_{4t+3} \equiv 2$ *if t is even*
  (iv) $T_{4t+1} \equiv 3$ *and* $T_{4t+3} \equiv 0$ *if t is odd*
  (v) *(t)th row is congruent to $(t \pm 2)$th row, i.e., $T_{k(t+2)+r} \equiv T_{kt+r}$.*

(2) *In the 5 columns modular tribonacci table*
  (i) $T_{5t+r} \equiv T_{5(t-2)+r} + T_{5(t-3)+r}$ *(mod $T_5 = 7$)*
  (ii) *(t)th row is congruent to the sum of $(t-2)$th and $(t-3)$th rows.*

Proof. In the 4 columns tribonacci table, Corollary 3.5 yields (i) that

$$T_{4t+r} \equiv -T_{4(t-1)+r} + T_{4(t-2)+r} + T_{4(t-3)+r} \pmod{T_4 = 4}.$$

We will only show (iv), and the rest can be proved similarly. Clearly $T_{4t+1} \equiv 3$ if $t = 1,\ 3$. Assume t is odd and $T_{4i+1} \equiv 3 \pmod{T_4}$ for all odd $i \leq t$. Then

$$\begin{aligned} T_{4(t+2)+1} &\equiv -T_{4(t+1)+r} + T_{4t+r} + T_{4(t-1)+r} \\ &\equiv -(-T_{4t+r} + T_{4(t-1)+r} + T_{4(t-2)+r}) + T_{4t+r} + T_{4(t-1)+r} \\ &\equiv 2 \cdot 3 - T_{4(t-2)+r} \equiv 2 - (-T_{4(t-3)+r} + 3 + T_{4(t-5)+r}) \\ &\equiv -1 + T_{4(t-3)+r} - T_{4(t-5)+r} \equiv 3. \end{aligned}$$

□

We remark that in [5], the $4n$ subscripted tribonacci numbers was proved that

$$T_{4(n+1)} = 11T_{4n} + 5T_{4(n-1)} + T_{4(n-2)}$$

by mathematical induction. This is the case for $k = 4$ in Theorem 3.1. In this sense Theorem 3.1 and 3.2 dealt with the kn subscript tribonacci numbers for $4 \leq k \leq 10$. The identity $\sum_{t=0}^{n} T_{4t} = (T_{4n+4}+6T_{4n}+T_{4n-4}-T_4)/T_4^2$ was proved in [5] using matrix calculations. But Theorem 3.1 shows the identity easily.

Corollary 3.9. $T_4^2 \sum_{t=0}^{n} T_{4t} = T_{4n+4} + 6T_{4n} + T_{4n-4} - T_4$.

Proof. Since $T_{4(3)+4} = (3T_4-1)T_{4(2)+4}+(T_4+1)T_{4+4}+T_4$ by Theorem 3.1, $T_4^2 \sum_{t=0}^{i} T_{4t} = T_{4i+4}+6T_{4i}+T_{4i-4}-T_4$ is true if $i = 3$. By induction we assume the equality holds for all $1 \leq i \leq n$. Then since $T_4 = 4$, it follows that

$$\begin{aligned} &T_{4(n+1)+4} + 6T_{4(n+1)} + T_{4(n+1)-4} - T_4 \\ &= (3T_4-1)T_{4n+4}+(T_4+1)T_{4(n-1)+4}+T_{4(n-2)+4}+6T_{4(n+1)}+T_{4(n+1)-4}-T_4 \\ &= T_4^2T_{4n+4} - (T_4+1)T_{4n+4} + (T_4+1)T_{4(n-1)+4} + T_{4(n-2)+4} \\ &\quad + 6T_{4(n+1)} + T_{4(n+1)-4} - T_4 \\ &= T_4^2T_{4n+4} + T_{4(n+1)} + (T_4+2)T_{4n} + T_{4n-4} - T_4 \\ &= T_4^2T_{4n+4} + T_{4n+4} + 6T_{4n} + T_{4n-4} - T_4 \\ &= T_4^2T_{4n+4} + T_4^2 \sum_{t=0}^{n} T_{4t} = T_4^2 \sum_{t=0}^{n+1} T_{4t}. \end{aligned}$$

□

4. Matrix for modular Fibonacci sequence

It is sometimes convenient to consider the k columns Fibonacci table as the k columns Fibonacci matrix. Then F_{kt+r} can be regarded as the $(t+1)$th row and (r)th column entry $e_{(t+1,r)}$, so Theorem 2.2 implies that

$$F_{kt+r} = e_{(t+1,r)} = (2e_{(1,k)} + e_{(1,k-3)})e_{(t,r)} + (-1)^{k-1}e_{(t-1,r)}.$$

Hence F_{kt+r} is a linear sum of three entries $e_{(1,k)}$, $e_{(1,k-3)}$ and $e_{(1,r)}$ in the 1st row, and $e_{(2,r)}$ in the 2nd row of k columns Fibonacci matrix. Moreover F_{kt+r} is expressed by two previous entries $e_{(t,r)}$ and $e_{(t-1,r)}$ in the same (r)th column.

THEOREM 4.1. *Any Fibonacci number* $F_n = F_{kt+r}$ *is*

$$F_{kt+r} \equiv XM^{t-2} \begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \end{bmatrix} \pmod{F_k}$$

where $X = [(-1)^{k-1} \quad e_{(1,k-3)}]$ *and* $M = \begin{bmatrix} 0 & 1 \\ (-1)^{k-1} & e_{(1,k-3)} \end{bmatrix}$. *Moreover if let* a *and* b *be roots of* $x^2 - e_{(1,k-3)}x + (-1)^k = 0$ *then*

$$F_{kt+r} \equiv \frac{1}{a-b} X \begin{bmatrix} (-1)^{k-1}(a^{t-3} - b^{t-3}) & a^{t-2} - b^{t-2} \\ (-1)^{k-1}(a^{t-2} - b^{t-2}) & a^{t-1} - b^{t-1} \end{bmatrix} \begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \end{bmatrix}$$

Proof. In the k columns Fibonacci matrix, by mod $F_k = e_{(1,k)}$,

$$\begin{aligned}
&F_{kt+r} \\
&= e_{(t+1,r)} \\
&\equiv e_{(1,k-3)}e_{(t,r)} + (-1)^{k-1}e_{(t-1,r)} \\
&\equiv e_{(1,k-3)}(e_{(1,k-3)}e_{(t-1,r)} + (-1)^{k-1}e_{(t-2,r)}) + (-1)^{k-1}e_{(t-1,r)} \\
&\equiv [e^2_{(1,k-3)} + (-1)^{k-1}]e_{(t-1,r)} + (-1)^{k-1}e_{(1,k-3)}e_{(t-2,r)} \\
&\equiv [e^3_{(1,k-3)} + 2(-1)^{k-1}e_{(1,k-3)}]e_{(t-2,r)} + (-1)^{k-1}[e^2_{(1,k-3)} \\
&\quad + (-1)^{k-1}]e_{(t-3,r)} \\
&\equiv [e^4_{(1,k-3)} + 3(-1)^{k-1}e^2_{(1,k-3)} + (-1)^{2(k-1)}]e_{(t-3,r)} \\
&\quad + (-1)^{k-1}[e^3_{(1,k-3)} + 2(-1)^{k-1}e_{(1,k-3)}]e_{(t-4,r)}
\end{aligned}$$

Continuing this process, F_{kt+r} is expressed by means of matrices that

$$\begin{aligned}
F_{kt+r} &\equiv [(-1)^{k-1} \quad e_{(1,k-3)}] \begin{bmatrix} e_{(t-1,r)} \\ e_{(t,r)} \end{bmatrix} \equiv XM \begin{bmatrix} e_{(t-2,r)} \\ e_{(t-1,r)} \end{bmatrix} \\
&\equiv XM^2 \begin{bmatrix} e_{(t-3,r)} \\ e_{(t-2,r)} \end{bmatrix} \equiv XM^3 \begin{bmatrix} e_{(t-4,r)} \\ e_{(t-3,r)} \end{bmatrix} \equiv \cdots \\
&\equiv XM^u \begin{bmatrix} e_{(t-u-1,r)} \\ e_{(t-u,r)} \end{bmatrix} (\text{ for } u \le t-2) \equiv XM^{t-2} \begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \end{bmatrix}
\end{aligned}$$

where $M = \begin{bmatrix} 0 & 1 \\ (-1)^{k-1} & e_{(1,k-3)} \end{bmatrix}$. Observe that $M = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix}$, $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, with roots a, b of $x^2 - e_{(1,k-3)}x + (-1)^k = 0$. Thus $a + b = e_{(1,k-3)}$ and $ab = (-1)^k$, so

$$M^u = PD^uP^{-1} = \frac{1}{a-b}\begin{bmatrix} -ab(a^{u-1} - b^{u-1}) & a^u - b^u \\ -ab(a^u - b^u) & a^{u+1} - b^{u+1} \end{bmatrix}$$

and it proves the Theorem that

$$F_{kt+r} \equiv \frac{1}{a-b}X\begin{bmatrix} (-1)^{k-1}(a^{t-3} - b^{t-3}) & a^{t-2} - b^{t-2} \\ (-1)^{k-1}(a^{t-2} - b^{t-2}) & a^{t-1} - b^{t-1} \end{bmatrix}\begin{bmatrix} e_{(1,r)} \\ e_{(2,r)} \end{bmatrix}.$$

□

Thus any F_{kt+r} is obtained by $e_{(1,k-3)}$, $e_{(1,r)}$, $e_{(1,k)}$ and $e_{(2,r)}$, where the first three are in the 1st row and the last one is in the 2nd row in the k columns Fibonacci matrix.

EXAMPLE 4.2. *For F_{99}, consider $k = 7$ for instance. Write a and b be roots of $x^2 - e_{(1,4)}x - 1 = x^2 - 3x - 1 = 0$. Due to Theorem 4.1,*

$$\begin{aligned} F_{99} &= F_{7(14)+1} \equiv \frac{1}{a-b}[1 \;\; e_{(1,4)}]\begin{bmatrix} 0 & 1 \\ 1 & e_{(1,4)} \end{bmatrix}^{12}\begin{bmatrix} e_{(1,1)} \\ e_{(2,1)} \end{bmatrix} \\ &= \frac{1}{a-b}[1 \;\; 3]\begin{bmatrix} a^{11} - b^{11} & a^{12} - b^{-12} \\ a^{12} - b^{12} & a^{13} - b^{13} \end{bmatrix}\begin{bmatrix} 1 \\ 8 \end{bmatrix}. \end{aligned}$$

But since $a^2 = 3a+1$, $a^3 = 3(3a+1)+a = 10a+3$, we have $a^{11} = 2a+2$, $a^{12} = 8a + 2$ and $a^{13} = 8$. Hence

$$a^{11} - b^{11} \equiv 2(a-b), \quad a^{12} - b^{12} \equiv 8(a-b), \quad a^{13} - b^{13} \equiv 0 (mod F_7 = 13),$$

and so F_{99} is congruent to

$$\frac{1}{a-b}[1 \; 3]\begin{bmatrix} 2(a-b) & 8(a-b) \\ 8(a-b) & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 8 \end{bmatrix} = [1 \; 3]\begin{bmatrix} 2 & 8 \\ 8 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 8 \end{bmatrix} \equiv 12.$$

In fact, $F_{99} = 218,922,995,834,555,169,026 \equiv 12 \pmod{13}$.

The smallest integer $h > 0$ satisfying $F_h \equiv 0$ and $F_{h+1} \equiv 1 \pmod n$ is called the period of Fibonacci sequence by mod n. We write $h = \mathrm{per}_F(n)$. Investigating the period of Fibonacci have been studied since Wall [7], so the period is usually called the Wall number by many researchers ([1]). A theorem about the period by mod Fibonacci numbers is as follows.

Theorem 4.3. *$per_F(F_k) = \begin{cases} 2k & \text{if } k : \text{even} \\ 4k & \text{if } k : \text{odd} \end{cases}$. In particular we have the table.*

k	F_k	$per_F(F_k)$	k	F_k	$per_F(F_k)$
4	3	$per_F(3) = 8 = 2 \cdot 4$	5	5	$per_F(5) = 20 = 4 \cdot 5$
6	8	$per_F(8) = 12 = 2 \cdot 6$	7	13	$per_F(13) = 28 = 4 \cdot 7$
8	21	$per_F(21) = 16 = 2 \cdot 8$	9	34	$per_F(34) = 36 = 4 \cdot 9$
10	55	$per_F(55) = 20 = 2 \cdot 10$	11	89	$per_F(89) = 44 = 4 \cdot 11$
12	144	$per_F(144) = 24 = 2 \cdot 12$	13	233	$per_F(233) = 52 = 4 \cdot 13$

The proof is due to Theorem 2.3 and Corollary 2.4. And Theorem 4.2 shows that period $\mathrm{per}_F(F_k)$ depends on only k not on F_k, and is relatively short period comparing to the other $\mathrm{per}_F(n)$. For example,

$$\mathrm{per}_F(987) = \mathrm{per}_F(F_{16}) = 32, \quad \mathrm{per}_F(1597) = \mathrm{per}_F(F_{17}) = 68,$$

however $\mathrm{per}_F(n)$ for $970 \leq n \leq 985$ is equal to

$2940, 970, 648, 368, 2928, 1400, 120, 652, 984,$
$220, 1680, 216, 1470, 1968, 120, 1980$

which show very long periods.

References

[1] A. Andreassian, *Fibonacci sequence moduol M*, Fibonacci Quarterly **12** (1974), no. 1, 51-64.

[2] H. W. Austin, *Columns of Fibonacci or Lucas Numbers*, Mathematical Spectrum **37** (2005), 67-72.

[3] A. Ehrlich, *On the periods of the Fibonacci sequence modulo M*, Fibonacci Quarterly **27** (1989), no. 1, 11-13.

[4] J. H. Halton, *On the divisibility properties of Fibonacci Number*, Fibonacci Quarterly **4** (1966), no. 3, 217-240.

[5] E. Kilic, *Tribonacci sequences with certain indices and their sums*, Ars. Combinatorics **86** (2008), 13-22.

[6] A. Vince, *The Fibonacci sequence modulo N*, Fibonacci Quarterly **16** (1978), no. 4, 403-407.

[7] D. D. Wall, *Fibonacci series modulo m*, Amer. Math. Monthly **67** (1960), 525-532.

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