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SOME PROPERTIES OF F–FUNCTION OF SET

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Abstract. In this paper we shall introduce the f-function in a set, and give some properties of f-function of a set. In particular, we establish a relation between f-function of a set and fuzzy equivalence relation. We also introduce the notion of f-homomorphism on a semigroup S, and prove the generalized fundamental homomorphism theorem of semigroup.

1. Introduction

After Zadeh introduced fuzzy set, many researchers are engaged in extending the concepts and results of pure algebra to broader framework of the fuzzy setting although not all the results in algebra can be fuzzified. The concept of fuzzy set was applied to the elementary theory of groupoids in Rosenfeld [11] and of semigroups and groups in the author's papers [4]. Fuzzy semigroups were introduced in [11] and discussed further in [8]. N. Kuroki has studied fuzzy ideals and bi-ideals in a semigroup [7]. Many classes of semigroups were studied and discussed further by using fuzzy ideals [6]. The reader may refer to [1], [2], [3], [5] and [10] for the basic theory of semigroup.

Fuzzy relations of a set also have been studied since the concept of fuzzy relations was introduced by Zadeh [13]. Crisp congruence relations and ideals on semigroup play an important role in studying algebraic structures of semigroup.

Recently, fuzzy congruence relation on semigroup and groups appears in [4]. For semigroup S, Samhan obtained that the lattice of fuzzy congruence

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on a semigroup S is complete lattice. In section 3, we introduced a f-function from set X to set Y, which is a kind of fuzzy relation, and proved an important connection between fuzzy equivalence relation and f-function of set. In particular, we attempt to give some study of fuzzy congruence relation generated by f-homomorphism of semigroup S.

2. Definitions and preliminaries

For any nonempty set X, a function θ from X to the unit interval [0, 1] is a *fuzzy subset* of X. Let θ and ψ be two fuzzy subsets of X. Then the inclusion relation $\theta \subset \psi$ is defined by $\theta(x) \leq \psi(x)$ for all $x \in X$. $\theta \cap \psi$ and $\theta \cup \psi$ are fuzzy subsets of X defined by $(\theta \cap \psi)(x) = min\{\theta(x), \psi(x)\}, (\theta \cup \psi)(x) = max\{\theta(x), \psi(x)\}$ for all $x \in X$.

Fuzzy (*binary*) *relation* from X to Y is a fuzzy subset of $X \times Y$. Accordingly, unless otherwise stated, by a fuzzy relation of a set X we mean a fuzzy binary relation R given by a function $R : X \times X \to [0, 1]$.

DEFINITION 2.1. ([9]) *Let X be a nonempty set and R be a fuzzy relation of X. Then R is called a fuzzy equivalence relation of X if*

(1) $R(x, x) = 1$ for all $x \in X$,

(2) $R(x, y) = R(y, x)$ for all $x, y \in X$,

(3) $R(x, y) \geq sup_{z \in X} min\{R(x, z), R(z, y)\}$ for all $x, y \in X$. (transitive)

DEFINITION 2.2. ([5]) *Let A be a fuzzy binary relation from X to Y and B be a fuzzy binary relation from Y to Z. The composition $A \circ B$ is defined as follows;*

$$(A \circ B)(x, z) = sup_{y \in Y} min\{A(x, y), B(y, z)\}.$$

It follows from this definition of composition that the property $(A \circ B) \circ C = A \circ (B \circ C)$ is satisfies for all fuzzy binary relations A, B, C.([5])

It is easy that fuzzy relation R on X satisfies transitive law if and only if $R \circ R \subset R$.

DEFINITION 2.3. *Let R be a fuzzy binary relation of a set X and $x \in X$. Let fuzzy subset $R_x : X \to [0, 1]$ is a function defined by $R_x(a) = R(x, a)$ for every $a \in X$.*

Let R be a fuzzy equivalence relation of X. For each $x \in X$, we shall say that the fuzzy subset R_x of X is the *fuzzy class* corresponding to x and the set $\{R_x | x \in X\}$ of all fuzzy classes of X is called a *fuzzy quotient set* by R and denoted by X/R.

LEMMA 2.4. ([5]) *Let R be a fuzzy equivalence relation of X. Then;*
(1) $R(x, y) = 0$ if and only if $min\{R_x, R_y\} = 0$,
(2) $sup_{x \in X} R_x = 1$,
(3) there exists the surjection $\pi : X \to X/R, \pi(x) = R_x$,
(4) $R_x = R_y$ if and only if $R(x, y) = 1$.

Just as equivalence classes are defined by equivalence relation, fuzzy equivalence classes are defined by fuzzy equivalence relation. Except in the restricted case of equivalence classes themselves, fuzzy equivalence classes are fuzzy set and are therefore not generally disjoint.

DEFINITION 2.5. *For any fuzzy subset θ of X and fuzzy binary relation R from X to Y and for any $A \subset X, B \subset Y$, let sup $\theta(A) = sup_{x \in A} \theta(x)$ and sup $R(A, B) = sup_{a \in A, b \in B} R(a, b)$.*

3. f-function

For a fuzzy subset θ of X and for $A \subset X$ we say θ *has the sup property on* A if, there exists $u \in A$ such that $\theta(u) = sup\ \theta(A)$. We call θ *has the sup property* if θ has the sup property on A for all $A \subset X$. For example, if θ can take on only finitely many values (in particular, if it is characteristic function), it has the sup property. For a fuzzy subset θ of X, $A \subset X, k \in [0, 1]$, we say θ *has* (*unique*) *k-sup property* on A if $sup\ \theta(A) = k$ and there exists (unique) $u \in A$ such that $\theta(u) = k$

DEFINITION 3.1. *Let X, Y are non empty sets. a function $h : X \times Y \to [0, 1]$ is called a f-function from X to Y if h satisfies*
(1) h has unique 1-sup property on $\{x\} \times Y$ for all $x \in X$.
(2) $min\{h(a, s), h(a, t), h(b, s)\} \leq h(b, t)$ for all $a, b \in X, s, t \in Y$.

From above definition we can consider f-function from X to Y is a kind of fuzzy binary relation from X to Y. Following Theorem shows that characteristic function of any crisp function from X to Y is f-function from X to Y.

THEOREM 3.2. *For any function $f : X \to Y$, define $\chi_f : X \times Y \to [0,1]$ by $\chi_f(x,y) = 1$, if $y = f(x)$ and $\chi_f(x,y) = 0$, if $y \neq f(x)$. Then χ_f is a f-function from X to Y.*

Proof. 1) Since f is a function from X to Y, for any $a \in X$, there exist some $b \in Y$ such that $b = f(a)$. Thus $\chi_f(a,b) = 1$ and we have

$$sup\ \chi_f(\{a\} \times Y) = \chi_f(a,b) = 1.$$

If there are some $c \in Y$ such that $\chi_f(a,c) = 1$, then by definition of $\chi_f, f(a) = c$ and so $b = c$. Hence χ_f has unique 1-sup property on $\{a\} \times Y$.

2) Let $a, b \in X, s, t \in Y$. If $t = f(b)$, then $\chi_f(b,t) = 1$ and so

$$\chi_f(b,t) \geq min\{\chi_f(a,s), \chi_f(a,t), \chi_f(b,s)\}$$

and so χ_f is f-function. Assume $t \neq f(b)$. If $s = t$, then $s \neq f(b)$ and so $\chi_f(b,s) = 0$. Thus $min\{\chi_f(a,s), \chi_f(a,t), \chi_f(b,s)\} = 0$. Hence

$$\chi_f(b,t) \geq min\{\chi_f(a,s), \chi_f(a,t), \chi_f(b,s)\} = 0.$$

If $s \neq t$, then since f is a function from X to Y, either $(a,s) \notin f$ or $(a,t) \notin f$. Hence either $\chi_f(a,s) = 0$ or $\chi_f(a,t) = 0$. So,

$$min\{\chi_f(a,s), \chi_f(a,t), \chi_f(b,s)\} = 0.$$

Thus we have

$$\chi_f(b,t) \geq min\{\chi_f(a,s), \chi_f(a,t), \chi_f(b,s)\} = 0$$

and so χ_f is again a f-function. □

THEOREM 3.3. *Let h be a f-function from X to Y. If we define $f^h = \{(x,y) \in X \times Y | h(x,y) = 1\}$, then f^h is a function from X to Y.*

Proof. For any $a \in X$, since h has unique 1-sup property on $\{a\} \times Y$ there exists unique $b \in Y$ such that $h(a,b) = 1$. If $(a,c) \in f^h$ for some $c \in Y$, then $h(a,c) = 1$ and since h has 1-unique sup property on $\{a\} \times Y$, $(a,b) = (a,c)$. Thus f^h is a function from X to Y. □

For any fuzzy binary relation R from X to Y, a fuzzy binary relation $S : Y \times X \to [0,1], S(a,b) = R(b,a)$ is called a *fuzzy inverse relation of* R and is denoted by $S = R^{-1}$. It is important to note that the fuzzy inverse relation of a f-function need not be a f-function.

THEOREM 3.4. *Let h be a f-function from a set X to a set Y. Then $h \circ h^{-1}$ is a fuzzy equivalence relation of X.*

Proof. For any $x, z \in X$,

$$\begin{aligned} h \circ h^{-1}(x,z) &= sup_{y \in Y} min\{h(x,y), h^{-1}(y,z)\} \\ &= sup_{y \in Y} min\{h(x,y), h(z,y)\}. \end{aligned}$$

1) For any $a \in X$,

$$h \circ h^{-1}(a,a) = sup_{y \in Y} min\{h(a,y), h(a,y)\}$$

$$= sup_{y \in Y} h(a,y) = sup\ h(\{a\} \times Y) = 1$$

2) It is easy that for any $a, b \in X, h \circ h^{-1}(a,b) = h \circ h^{-1}(b,a)$.

3) Fuzzy equivalence relation R is transitive if and only if $R \circ R \subset R$. If $h \circ h^{-1}$ is not transitive, there are some $a_0, b_0 \in X$ such that

$$(h \circ h^{-1}) \circ (h \circ h^{-1})(a_0, b_0) > h \circ h^{-1}(a_0, b_0).$$

Since

$$(h \circ h^{-1}) \circ (h \circ h^{-1})(a_0, b_0) = sup_{y \in Y} min\{h \circ h^{-1}(a_0, y), h \circ h^{-1}(y, b_0)\},$$

$h \circ h^{-1}(a_0, b_0) < min\{h \circ h^{-1}(a_0, y_0),\ h \circ h^{-1}(y_0, b_0)\}$ for some $y_0 \in Y$. Hence $h \circ h^{-1}(a_0, b_0) < h \circ h^{-1}(a_0, y_0)$, $h \circ h^{-1}(a_0, b_0) < h \circ h^{-1}(y_0, b_0)$. From $h \circ h^{-1}(a_0, b_0) < h \circ h^{-1}(a_0, y_0)$, we have

$$h \circ h^{-1}(a_0, b_0) < sup_{s \in Y} min\{h(a_0, s), h(y_0, s)\}$$

and so there is some $s_0 \in Y$ such that

$$h \circ h^{-1}(a_0, b_0) < min\{h(a_0, s_0), h(y_0, s_0)\}.$$

Also from $h \circ h^{-1}(a_0, b_0) < h \circ h^{-1}(y_0, b_0)$, we have

$$h \circ h^{-1}(a_0, b_0) < sup_{t \in Y} min\{h(y_0, t), h(b_0, t)\}$$

and so there is some $t_0 \in Y$ such that

$$h \circ h^{-1}(a_0, b_0) < min\{h(y_0, t_0), h(b_0, t_0)\}.$$

So, there are some $y_0, s_0, t_0 \in Y$ such that

$$min\{(h(a_0, s_0), h(y_0, s_0), h(y_0, t_0), h(b_0, t_0)\} > h \circ h^{-1}(a_0, b_0).$$

Since h is a f-function, we have $min\{h(y_0, s_0), h(y_0, t_0), h(b_0, t_0)\} \leq h(b_0, s_0)$ and so

$$\begin{aligned} h \circ h^{-1}(a_0, b_0) &< min\{h(a_0, s_0), h(b_0, s_0)\} \\ &\leq min\{h(a_0, s_0), h^{-1}(s_0, b_0)\} \\ &\leq sup_{s \in Y} min\{h(a_0, s), h^{-1}(s, b_0)\} \\ &= h \circ h^{-1}(a_0, b_0). \end{aligned}$$

We have a contradiction. □

For a f-function h from X to Y, we call the fuzzy equivalence relation $h \circ h^{-1}$ of X as *f-kernel* of h and denoted *ker h*

THEOREM 3.5. *If R is a fuzzy equivalence relation of a set X, then there is a f-function whose f-kernel is R.*

Proof. Define

$$\pi : X \times X/R \to [0, 1], \pi(a, R_x) = R_x(a) = R(x, a).$$

From definition of fuzzy class, $\pi(a, R_x) = R(x, a) = R(a, x)$. We will show that π is f-function and the f-kernel of π is R.

It is easy to see that π is well-defined.

1) For any $\{x\} \times X/R \subset X \times X/R$,

$$\pi(x, R_x) = R(x, x) = 1 \quad \text{and} \quad \pi(x, R_x) \leq sup\ \pi(\{x\} \times X/R).$$

Thus π has 1-sup property on $\{x\} \times X/R$. If $\pi(a, R_x) = \pi(a, R_y) = 1$, then $R(x, a) = R(y, a) = 1$. From transitivity of R,

$$R(x, y) \geq sup_{z\in X} min\{R(x, z), R(z, y)\} \geq min\{R(x, a), R(a, y)\} = 1.$$

From Lemma 2.4.(4), $R_x = R_y$. Thus π satisfies unique 1-sup property on $\{a\} \times X/R$ for all $a \in X$.

2) For any $a, b \in X, R_s, R_t \in X/R$, from transitivity of R,

$$R(b, t) \geq sup_{z\in X} min\{R(b, z), R(z, t)\} \geq min\{R(b, s), R(s, t)\}$$

and from

$$R(s, t) \geq sup_{z\in X} min\{R(s, z), R(z, t)\} \geq min\{R(s, a), R(a, t)\},$$

we have

$$R(b, t) \geq min\{R(b, s), R(s, a), R(a, t)\}.$$

Thus

$$\pi(b, R_t) \geq min\{\pi(b, R_s), \pi(a, R_s), \pi(a, R_t)\}$$

and so π is a f-function. For any $a, b \in X$, since $\pi(a, R_a) = R(a, a) = 1$,

$$\begin{aligned} \pi \circ \pi^{-1}(a, b) &= sup_{R_x\in X/R} min\{\pi(a, R_x), \pi^{-1}(R_x, b)\} \\ &= sup_{R_x\in X/R} min\{\pi(a, R_x), \pi(b, R_x)\} \\ &\geq min\{\pi(a, R_a), \pi(b, R_a)\} \\ &= R(a, b). \end{aligned}$$

Conversely,

$$\begin{aligned} R(a, b) &\geq sup_{z\in X} min\{R(a, z), R(z, b)\} \\ &= sup_{z\in X} min\{\pi(a, R_z), \pi^{-1}(R_z, b)\}. \end{aligned}$$

By Lemma 2.4.(3),

$$\begin{aligned} &sup_{z\in X} min\{\pi(a, R_z), \pi^{-1}(R_z, b)\} \\ &= sup_{R_z\in X/R} min\{\pi(a, R_z), \pi^{-1}(R_z, b)\} = \pi \circ \pi^{-1}(a, b). \end{aligned}$$

So, $\pi \circ \pi^{-1}(a, b) = R(a, b)$ and thus R is a f-kernel of π. $\square$

Definition 3.6. ([4]) *A fuzzy binary relation R of a semigroup S is called fuzzy left (right) compatible if and only if $R(a,b) \leq R(ta,tb)$ for all $t,a,b \in S$ ($R(a,b) \leq R(at,bt)$ for all $t,a,b \in S$).*

If G is a group, it is clear that $R(a,b) = R(ta,tb)(R(a,b) = R(at,bt))$ for all $t,a,b \in G$ for any fuzzy left (right) compatible binary relation R.

Definition 3.7. *A fuzzy binary relation R on a semigroup S is called fuzzy compatible if and only if $min\{R(a,b), R(c,d)\} \leq R(ac,bd)$ for all $a,b,c,d \in S$.*

Definition 3.8. *A fuzzy compatible equivalence relation R on a semigroup S is called a fuzzy congruence relation of S.*

Theorem 3.9. ([4]) *A fuzzy binary relation R of semigroup S is a fuzzy congruence if and only if it is both a fuzzy left and fuzzy right compatible equivalence relation of S.*

On a semigroup S, the product $\theta \circ \psi$ of two fuzzy subsets θ and ψ of S is defined by $\theta \circ \psi(x) = sup_{x=yz} min\{\theta(y), \psi(z)\}$ for $x = yz, y, z \in S, 0$ if x is not expressible as $x = yz$, for all $x \in S$. As is well known ([3]), this operation $\circ$ is associative. a f-function h from a semigroup S to a semigroup T is called a *f-homomorphism* if and only if $min\{h(a,b), h(c,d)\} \leq h(ac,bd)$ for all $a,c \in S, b,d \in T$.

An f-function h from X to Y is called a *f-one to one* if and only if $h(a,c) = h(b,c)$ for all $c \in Y$, then $a = b$. That is if $a \neq b$, then there are some $t \in Y$ such that $h(a,t) \neq h(b,t)$. Above Theorem 3.4 and Theorem 3.5 also hold for a fuzzy congruence relation of a semigroup S and a f-homomorphism from a semigroup S to a semigroup T.

Theorem 3.10. *If h is a f-homomorphism from a semigroup S to a semigroup T, then $h \circ h^{-1}$ is a fuzzy congruence relation of S.*

Proof. Claim $h \circ h^{-1}$ is fuzzy compatible. Let us show

$$min\{h \circ h^{-1}(a,b), h \circ h^{-1}(c,d)\} \leq h \circ h^{-1}(ac,bd)$$

for all $a, b, c, d \in S$. If $h \circ h^{-1}$ is not fuzzy compatible, then there will be some $a_0, b_0, c_0, d_0 \in S$ such that

$$min\{h \circ h^{-1}(a_0, b_0), h \circ h^{-1}(c_0, d_0)\} > h \circ h^{-1}(a_0c_0, b_0d_0).$$

From

$$h \circ h^{-1}(a_0, b_0) > h \circ h^{-1}(a_0c_0, b_0d_0),$$

we have

$$sup_{t \in Y} min\{h(a_0, t), h(b_0, t)\} > h \circ h^{-1}(a_0c_0, b_0d_0)$$

and so

$$h \circ h^{-1}(a_0c_0, b_0d_0) < min\{h(a_0, t_0), h(b_0, t_0)\}$$

for some $t_0 \in Y$. Also from

$$h \circ h^{-1}(c_0, d_0) > h \circ h^{-1}(a_0c_0, b_0d_0),$$

we have

$$sup_{s \in Y} min\{h(c_0, s), h(d_0, s)\} > h \circ h^{-1}(a_0c_0, b_0d_0)$$

and so

$$h \circ h^{-1}(a_0c_0, b_0d_0) < min\{h(c_0, s_0), h(d_0, s_0)\}$$

for some $s_0 \in Y$. Hence

$$\begin{aligned}
&h \circ h^{-1}(a_0c_0, b_0d_0) \\
&< min\{min\{h(a_0, t_0), h(b_0, t_0)\}, min\{h(c_0, s_0), h(d_0, s_0)\}\} \\
&= min\{h(a_0, t_0), h(b_0, t_0), h(c_0, s_0), h(d_0, s_0)\} \\
&= min\{min\{h(a_0, t_0), h(c_0, s_0)\}, min\{h(b_0, t_0), h(d_0, s_0)\}\}.
\end{aligned}$$

Since h is a f-homomorphism,

$$min\{h(a_0, t_0), h(c_0, s_0)\} \leq h(a_0c_0, t_0s_0),$$
$$min\{h(b_0, t_0), h(d_0, s_0)\} \leq h(b_0d_0, t_0s_0).$$

Hence

$$\begin{aligned}
h \circ h^{-1}(a_0c_0, b_0d_0) &< min\{h(a_0c_0, t_0s_0), h(b_0d_0, t_0s_0)\} \\
&\leq sup_{u \in Y} min\{h(a_0c_0, u), h(b_0d_0, u)\} \\
&= h \circ h^{-1}(a_0c_0, b_0d_0).
\end{aligned}$$

This is a contradiction and so $h \circ h^{-1}$ is fuzzy compatible. By Theorem 3.4, we have $h \circ h^{-1}$ is fuzzy congruence relation of S. □

THEOREM 3.11. ([12]) *If R is a fuzzy congruence relation of a semigroup S, then S/R is a semigroup with the operation $R_x * R_y = R_{xy}$ for all $x, y \in S$.*

THEOREM 3.12. *If R is a fuzzy congruence relation of a semigroup S, then there is a f-homomorphism whose f-kernel is R.*

Proof. On above Theorem 3.5, it is sufficient that if R is a fuzzy congruence relation of semigroup S, then

$$\pi : S \times S/R \to [0,1], \pi(a, R_x) = R_x(a) = R(x, a)$$

is a f-homomorphism. Since R is a fuzzy congruence of S, for all $a, b \in S, R_x, R_y \in S/R, min\{R(a,x), R(b,y)\} \leq R(ab, xy)$. Thus

$$\begin{aligned} min\{\pi(a, R_x), \pi(b, R_y)\} &= min\{R(a,x), R(b,y)\} \\ &\leq R(ab, xy) \\ &= \pi(ab, R_{xy}) \\ &= \pi(ab, R_x * R_y). \end{aligned}$$

This means π is f-homomorphism. □

THEOREM 3.13. *If h is a f-function from a semigroup S to a semigroup T, there is a f-one to one f-homomorphism $\bar{h}$ from $S/ker\ h$ to T such that $\pi \circ \bar{h} = h$.*

Proof. Recall that $\pi : S \times S/ker\ h \to [0,1]$,

$$\begin{aligned} \pi(a, (ker\ h)_x) &= (ker\ h)_x(a) = (ker\ h)(x, a) = h \circ h^{-1}(x, a) \\ &= sup_{y \in T} min\{h(x,y), h(a,y)\}. \end{aligned}$$

Define $\bar{h} : S/ker\ h \times T \to [0,1]$ by $\bar{h}((ker\ h)_a, y) = h(a, y)$.

We will show that $\bar{h}$ is f-one to one f-homomorphism such that $\pi \circ \bar{h} = h$. First we will show that $\bar{h}$ is well-defined f-function from $S/ker\ h \times T$ to $[0,1]$

1) Let $(ker\ h)_a = (ker\ h)_b$. From Lemma 2.4 (4),

$$1 = ker\ h(a,b) = sup_{y \in T} min\{h(a,y), h(b,y)\}.$$

For a subset $\{(ker\ h)_a\} \times T$ of $S/ker\ h \times T$, since h has unique 1-sup property, there are unique $u \in T$ such that $h(a,u) = 1$. If $h(b,u) \neq 1$, then $h(b,u) < 1$

and since $h(b,u) < sup_{y\in T} min\{h(a,y),\ h(b,y)\} = 1$, there is some $t_0 \in T$ such that $h(b,u) < min\{h(a,t_0), h(b,t_0)\}$. Since h is a f-function,

$$min\{h(a,t_0), h(b,t_0), h(a,u)\} \le h(b,u) < min\{h(a,t_0), h(b,t_0)\}$$

This means $min\{h(a,t_0), h(b,t_0), h(a,u)\} = h(a,u)$ and it is a contradiction to

$$1 = h(a,u) \le h(b,u) < min\{h(a,t_0), h(b,t_0)\}.$$

Thus $h(b,u) = 1$ and so $h(a,u) = h(b,u)$. For $x \ne u$ in T, since h has unique 1-sup property, $h(a,x) \ne 1$. By the same manner to the above 1),

$$h(a,x) < sup_{y\in T} min\{h(a,y), h(b,y)\} = 1.$$

So,

$$h(a,x) < min\{h(a,s_0), h(b,s_0)\} \quad \text{for some} \quad s_0 \in T.$$

Since h is a f-function,

$$min\{h(a,s_0), h(b,s_0), h(b,x)\} \le h(a,x) < min\{h(a,s_0), h(b,s_0)\}$$

Thus

$$min\{h(a,s_0), h(b,s_0), h(b,x)\} = h(b,x)$$

and so $h(b,x) \le h(a,x) < 1$. Again from $h(b,x) < 1$,

$$h(b,x) < sup_{y\in T} min\{h(a,y), h(b,y)\} = 1$$

and so there is some $k_0 \in T$ such that $h(b,x) < min\{h(a,k_0), h(b,k_0)\}$ and

$$min\{h(a,k_0), h(b,k_0), h(a,x)\} \le h(b,x) < min\{h(a,k_0), h(b,k_0)\}.$$

Thus $min\{h(a,k_0), h(b,k_0), h(a,x)\} = h(a,x)$ and $h(a,x) \le h(b,x)$. We have $h(b,x) = h(a,x)$. So that $\bar{h}$ is well-defined function from $S/ker\ h \times T$ to $[0,1]$.

Next we will show $\bar{h}$ is a f-function.

1) For any $\{(ker\ h)_a\} \times T \subset S/ker\ h \times T$, since h has 1-sup property on $\{a\} \times T$, we can choose $u \in T$ such that $h(a,u) = 1$. Hence $\bar{h}((ker\ h)_a, u)) =$

$h(a,u)=1$. If there is $v\in T$ such that $\bar{h}((ker\ h)_a,v))=1$, then $h(a,v)=h(a,u)=1$ and from unique 1-sup property of $h, u=v$ and thus $\bar{h}$ has unique 1-sup property for any $\{(ker\ h)_a\}\times T\subset S/ker\ h\times T$.

2) For any $((ker\ h)_a,s),((ker\ h)_a,t),((ker\ h)_b,s),((ker\ h)_b,t)\in S/ker\ h\times T$,

$$\begin{aligned}&min\{(\bar{h}(ker\ h)_a,s),\bar{h}((ker\ h)_a,t),\bar{h}((ker\ h)_b,s)\}\\&=min\{h(a,s),h(a,t),h(b,s)\}\le h(b,t)=\bar{h}((ker\ h)_b,t).\end{aligned}$$

So that $\bar{h}$ is a f- function from $S/ker\ h\times T$ to $[0,1]$.

Since h is f-homomorphism and $\bar{h}((ker\ h)_x,y)=h(x,y)$,

$$min\{\bar{h}((ker\ h)_a,x),\bar{h}((ker\ h)_b,y)\}=min\{h(a,x),h(b,y)\}$$

and since h is a f-homomorphism,

$$min\{h(a,x),h(b,y)\}\le h(ab,xy)=\bar{h}((ker\ h)_{ab},xy).$$

By Theorem 3.11, an operation $*$ on $S/ker\ h$ is $(ker\ h)_x*(ker\ h)_y=(ker\ h)_{xy}$ for all $x,y\in S$, we have

$$min\{\bar{h}((ker\ h)_a,x),\bar{h}((ker\ h)_b,y)\}\le\bar{h}((ker\ h)_{ab},xy).$$

Thus $\bar{h}$ is a f-homomorphism.

If $\bar{h}((ker\ h)_x,t)=\bar{h}((ker\ h)_y,t)$ for all $t\in T$, then $h(x,t)=h(y,t)$ for all $t\in T$. From

$$\begin{aligned}(ker\ h)_x(y)=(h\circ h^{-1})(x,y)&=sup_{u\in T}min\{h(x,u),h(y,u)\}\\&=sup_{u\in T}h(x,u)=1.\end{aligned}$$

By Lemma 2.4 (4), $(ker\ h)_x=(ker\ h)_y$. Thus $\bar{h}$ is f-one to one.

Finally, lets show $\pi\circ\bar{h}=h$. For any $a\in S,b\in T$,

$$\begin{aligned}\pi\circ\bar{h}(a,b)&=sup_{(ker\ h)_x\in S/ker\ h}min\{\pi(a,(ker\ h)_x),\bar{h}((ker\ h)_x,b)\}\\&=sup_{(ker\ h)_x\in S/ker\ h}min\{(ker\ h)_x(a),h(x,b)\}\\&\ge min\{h\circ h^{-1}(a,a),h(a,b)\}=h(a,b).\end{aligned}$$

If there is some $a_0\in S,b_0\in T$ such that

$$h(a_0,b_0)<sup_{(ker\ h)_x\in S/ker\ h}min\{(ker\ h)_x(a_0),h(x,b_0)\},$$

then there is some $(ker\ h)_u \in S/ker\ h$ such that

$$h(a_0, b_0) < min\{(ker\ h)_u(a_0), h(u, b_0)\}.$$

From

$$\begin{aligned} h(a_0, b_0) < (ker\ h)_u(a_0) &= (h \circ h^{-1})(u, a_0) \\ &= sup_{t \in T} min\{h(a_0, t), h(u, t)\}, \end{aligned}$$

$h(a_0, b_0) < min\{h(u, t_0), h(a_0, t_0)\}$ for some $t_0 \in T$. Hence

$$h(a_0, b_0) < min\{h(u, t_0), h(a_0, t_0), h(u, b_0)\}.$$

But since h is a f function,

$$min\{h(u, t_0), h(a_0, t_0), h(u, b_0)\} \leq h(a_0, b_0)$$

leads a contradiction. Thus $\pi \circ \bar{h} = h$ □

References

[1] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Math. Surveys No. 7, Am. Math. Soc. Province, RI. **1**, 1961.

[2] J. M. Howie, *An introduction to semigroup theory*, Academic Press, 1976.

[3] J. L. Wang, *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy Set and Systems **8** (1982), 133-139.

[4] J. P. Kim and D. R. Bae, *Fuzzy congruences in groups*, Fuzzy Set and Systems **85** (1997), 115-120.

[5] G. J. Klir and T. A. Folger, *Fuzzy Sets, Uncertainties, and Information*, F Prentice Hall, Englewood Cliffs. NJ, 1988.

[6] T. Kuraoka and N. Kuroki, *On fuzzy quotient rings induced by fuzzy ideals*, Fuzzy Set and Systems **47** (1992), 381-386.

[7] N. Kuroki, *On fuzzy ideals and fuzzy bi-ideals in semigroups*, Fuzzy Set and Systems **5** (1981), 203-251.

[8] N. Kuroki, *Fuzzy semiprime ideals in semigroups*, Fuzzy Set and Systems **8** (1982), 71-79.

[9] V. Murali, *Fuzzy equivalence relations*, Fuzzy Set and Systems **30** (1989), 155-163.

[10] M. Petrich, *Introduction to semigroups*, Charles E. Merrill. Columbus. OH. **5**, 1973.

[11] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512-517.

[12] X. Xie, *Fuzzy Rees congruences on Semigroups*, Fuzzy Set and Systems **102** (1992), 353-359.

[13] L. A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338-353.

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