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DEFINING EQUATIONS OF CERTAIN MODULAR CURVES

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Abstract. In this paper, we explain how to get defining equations of the modular curves $X_1(2, 2N)$ which show the moduli problems and present defining equations of $X_1(2, 2N)$ for $N = 2, 3, \ldots, 8$.

1. Introduction

For positive integers $M|N$, consider the congruence subgroup $\Gamma_1(M, N)$ of $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\Gamma_1(M,N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N,\ M \mid b \right\}.$$

Then the modular curve $X_1(M, N)$ corresponding to $\Gamma_1(M, N)$ is related to moduli problems of elliptic curves containing a subgroup which is isomorphic to $\mathbb{Z}/M\mathbb{Z} \bigoplus \mathbb{Z}/N\mathbb{Z}$. If $M = 1$, then $X_1(M, N)$ is the same as $X_1(N)$ which is the coarse moduli space of elliptic curves with N-torsion points.

It is not much known for the defining equations of $X_1(M, N)$, in particular, the equations which show the moduli problems of $X_1(M, N)$. But recently, the author with C. H. Kim and Y. Lee [2] found a defining equation of $X_1(2, 14)$ which enables us to construct a family of elliptic curves over cubic number fields with the torsion subgroups $\mathbb{Z}/2\mathbb{Z} \bigoplus \mathbb{Z}/14\mathbb{Z}$.

In this paper, we explain how to get defining equations of $X_1(2, 2N)$ which show the moduli problems and present defining equations of $X_1(2, 2N)$ for some N by following the method in [2].

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2. Preliminaries

The Tate normal form of an elliptic curve with $P = (0,0)$ is given as follows:

$$E(b,c) : y^2 + (1-c)xy - by = x^3 - bx^2,$$

and this is nonsingular if and only if $b \neq 0$. In this case, P is not of order 2 or 3(cf. [1]). On the curve $E(b,c)$ we have the following by the chord-tangent method(cf. [3]):

(2.1)

$$\begin{aligned}
P &= (0,0),\\
2P &= (b, bc),\\
3P &= (c, b-c),\\
4P &= \big(r(r-1), r^2(c-r+1)\big)\,; \quad b = cr,\\
5P &= \big(rs(s-1), rs^2(r-s)\big)\,; \quad c = s(r-1),\\
6P &= \big(-mt, m^2(m+2t-1)\big)\,; \quad m(1-s) = s(1-r), r-s = t(1-s).
\end{aligned}$$

The condition $NP = O$ in $E(b,c)$ gives a defining equation for $X_1(N)$. For example, $10P = O$ implies $4P = -6P$, so

$$x_{4P} = x_{-6P} = x_{6P},$$

where x_{nP} denote the x-coordinate of the n-multiple nP of P. Eq. (2.1) implies that

$$r(r-1) = -mt. \tag{2.2}$$

Without loss of generality, the cases $r = 1$ and $s = 1$ may be excluded. Reversing the substitutions made for calculating $6P$: $m = \frac{s(1-r)}{1-s}$, $t = \frac{r-s}{1-s}$, Eq. (2.2) becomes as follows:

$$r - 3rs + rs^2 + s^2 = 0,$$

which is one of the equations $X_1(10)$.

By using the above method and Sutherland's calculation [4] we have the following defining equations of the modular curves $X_1(2N)$ for $N = 2, 3, \ldots, 8$ as follows:

Theorem 2.1. *For $N = 2, 3, ..., 8$ the modular curves $X_1(2N)$ are given by the following equations:*

(1) $X_1(4) :\ v - u = 0,$
(2) $X_1(6) :\ v - u^2 - u = 0,$
(3) $X_1(8) :\ uv - 2u + 1 = 0,$

TABLE 1. The relations between b, c and u, v

$2N$	The relations between b, c and u, v
4	$\begin{cases} b = v \\ c = u \end{cases}$
6	$\begin{cases} b = v \\ c = u \end{cases}$
8	$\begin{cases} b = u(u-1)v \\ c = (u-1)v \end{cases}$
10	$\begin{cases} b = v(v-1)u \\ c = (v-1)u \end{cases}$
12	$\begin{cases} b = \frac{v(v+1)(v+u)}{u^2} \\ c = \frac{v(v+1)}{u} \end{cases}$
14	$\begin{cases} b = \frac{(u-1)(v+u)(v^2+uv+v+1)}{(v+1)^3(v+u+1)^2} \\ c = \frac{(u-1)(v+u)}{(v+1)^2(v+u+1)} \end{cases}$
16	$\begin{cases} b = -\frac{(v+1)(v-u)(v-u+1)(v^2-uv+v+u^2)}{(u+1)(v-u^2-u+1)^2} \\ c = -\frac{(v+1)(v-u)(v-u+1)}{(u+1)(v-u^2-u+1)} \end{cases}$

(4) $X_1(10): \ (u^2-3u+1)v+u^2=0,$
(5) $X_1(12): \ v-u^2+3u-2=0,$
(6) $X_1(14): \ v^2+(u^2+u)v-u=0,$
(7) $X_1(16): \ v^2+(u^3+u^2-u+1)v+u^2=0.$

In the above theorem, for each point (u, v) satisfying the defining equation $f_{2N}(u, v) = 0$ of $X_1(2N)$, the corresponding elliptic curve $E(b, c)$ is defined over the number field $K = \mathbb{Q}(u, v)$ and has the torsion subgroup containing $\mathbb{Z}/2N\mathbb{Z}$. In Table 1, we list the relations between b, c and u, v.

3. Defining equations of $X_1(2, 2N)$

There are *forgetful* maps from $X_1(2, 2N)$ to $X_1(2N)$ which send (E, P, R) to (E, P) where P (resp. R) is a torsion point of order $2N$ (resp. 2) of E. In order to find the defining equations of $X_1(2, 2N)$, we use forgetful maps from $X_1(2, 2N)$ to $X_1(2N)$.

Let $f_{2N}(u, v) = 0$ be a defining equation of $X_1(2N)$. Each point (u, v) on $X_1(2N)$ corresponds to the elliptic curve $E(b, c)$ with a torsion point $P = (0, 0)$ of order $2N$ where b, c can be expressed by u, v. By replacing

TABLE 2. Defining equations of $X_1(2, 2N)$

$X_1(2,2N)$	Defining equations of $X_1(2,2N)$
$X_1(2,4)$	$\begin{cases} w^2 = (u-1)(u^3-19u^2-13u-1) \\ v-u=0 \end{cases}$
$X_1(2,6)$	$\begin{cases} w^2 = (u+1)(9u+1) \\ v-u^2-u=0 \end{cases}$
$X_1(2,8)$	$\begin{cases} w^2 = (8u^2-8u+1) \\ uv-2u+1=0 \end{cases}$
$X_1(2,10)$	$\begin{cases} w^2 = (2u-1)(4u^2-2u-1) \\ (u^2-3u+1)v-u^2=0 \end{cases}$
$X_1(2,12)$	$\begin{cases} w^2 = (u^2-6u+6)(u^2-2u+2) \\ v-u^2+3u-2=0 \end{cases}$
$X_1(2,14)$	$\begin{cases} w^2 = -(u-1)(u+1)(u^8v+7u^7v+16u^6v+10u^5v \\ -18u^4v-26u^3v+12uv+u^7+6u^6+10u^5+u^4 \\ -14u^3-7u^2+4u+1) \\ v^2+(u^2+u)v-u=0 \end{cases}$
$X_1(2,16)$	$\begin{cases} w^2 = -(u^2+2u-1)(u^2-2u-1)(u^{13}v+7u^{12}v+18u^{11}v \\ +20u^{10}v+5u^9v-26u^7v-15u^8v-18u^6v-u^5v+9u^4v \\ +8u^3v+2u^2v-uv-v+u^{12}+6u^{11}+13u^{10}+12u^9+u^8 \\ -12u^7-16u^6-8u^5+2u^4+6u^3+3u^2-1) \\ v^2+(u^3+u^2-u+1)v+u^2=0 \end{cases}$

y by $y + \frac{(c-1)}{2}x + \frac{b}{2}$ in the equation of $E(b,c)$, we have the following form:

$$E:\ y^2 = x^3 + \frac{1}{4}(c^2-2c+1-4b)x^2 + \frac{1}{2}b(c-1)x + \frac{b^2}{4}. \tag{3.1}$$

Note that NP is of order 2. The cubic polynomial in the right hand side of Eq. (3.1) is divisible by $x - x_{NP}$, and we have a quadratic factor $q(x)$. Then the torsion subgroup of the elliptic curve E defined over the field $K = \mathbb{Q}(u,v)$ contains the group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}$ if and only if the quadratic factor $q(x)$ splits over K, and it holds if and only if the discriminant $d_{2N}(u,v)$ of $q(x)$ is a square in K. Therefore we have the following result:

THEOREM 3.1. *A defining equation of the modular curve $X_1(2,2N)$ is given by*

$$\begin{cases} w^2 = d_{2N}(u,v), \\ f_{2N}(u,v) = 0. \end{cases} \tag{3.2}$$

In Table 2, we list defining equations of $X_1(2,2N)$ only for $N = 2,3,\dots,8$ because those are very complicated for $N \geq 9$. We omit a defining equation of $X_1(2,2)$ for it has no model obtaining from the Tate normal form. We note that $d_{2N}(u,v)$ in Table 2 is not the exact discriminant but the same as a multiple by a square factor.

Example 3.2. *A defining equation of* $X_1(10)$ *is*

$$(u^2-3u+1)v+u^2=0,$$

and

$$d_{10}(u,v)=\frac{(4u^2-2u-1)(2u-1)^5}{16(u^2-3u+1)^4}.$$

Therefore a defining equation of $X_1(2,10)$ *is as follows:*

$$X_1(2,10):\begin{cases} w^2=(2u-1)(4u^2-2u-1),\\ (u^2-3u+1)v-u^2=0.\end{cases}$$

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