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ON A GENERALIZED E-KKM THEOREM

Won Kyu Kim*

Abstract. In this paper, using the E-convexity, we introduce the generalized E-KKM map, and next prove the generalized E-KKM theorem which generalizes the KKM theorem due to Fan [4]. As an application, a fixed point theorem in E-convex sets is given.

1. Introduction

The concept of convexity and its various generalizations are very important to quantitative and qualitative studies of nonlinear analysis and convex analysis. In 1929, Knaster et al. [7] first established the famous KKM theorem in finite dimensional spaces which is a basic result for combinatorial mathematics. As we know, the KKM theorem is equivalent to many important theorems such as Sperner's lemma, Brouwer's fixed point theorem, and Fan's minimax inequality. In 1961, Fan [4] extended the KKM theorem to infinite dimensional topological vector spaces and gave applications in several directions. Since then, many authors have made important contributions to the progress of the KKM theorem as in [1,2,5,6].

On the other hand, Youness [9] introduced a class of sets and a class of functions called E-convex sets and E-convex functions by relaxing the definitions of convex sets and convex functions. This new concept inspired a great deal of subsequent researches which have greatly expanded the role of E-convexity in various branches of mathematical sciences as we can see in [3,8,10]. However, we can not find any generalization of the KKM theorem using the E-convexity in the development of the KKM theory.

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In this paper, using the E-convexity, we first introduce the generalized E-KKM map which generalizes the classical KKM map in the E-convex set. Next, we prove a generalized E-KKM theorem which is a generalization of the classical KKM Theorem. As an application, a fixed point theorem in E-convex sets is given.

2. Preliminaries

We begin with some notations and definitions. Let X be a nonempty subset of a Hausdorff topological vector space Y. We shall denote by 2^X the family of all subsets of X, and for any nonempty subset A of Y, by $co\,A$ the convex hull of A in Y. When a multimap $T : X \to 2^Y$ is given, we shall denote $T^{-1}(y) := \{x \in X \mid y \in T(x)\}$ for each $y \in Y$. Denote by $[0,1]^n$ the Cartesian product of n unit intervals $[0,1] \times \cdots \times [0,1]$, and denote the unit simplex in $[0,1]^n$ by Δ_{n-1}, and simply denote $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta_{n-1}$ with $\Sigma_{i=1}^n \lambda_i = 1$.

In [9], Youness first introduced the following which generalizes the convex condition:

Definition 2.1. Let X be a nonempty subset of a vector space Y. A set X is said to be *E-convex* with respect to a map $E : Y \to Y$ if there is a mapping $E : Y \to Y$ such that $\lambda E(x) + (1-\lambda)E(y) \in X$ for each $x, y \in X$ and $\lambda \in [0,1]$.

Every E-convex set is convex when $E : Y \to Y$ is the identity map on Y. From the definition, we note that if X is E-convex, then

$$E(X) \subseteq co\,E(X) \subseteq X$$

holds. Also, if X_1, X_2 are E-convex subsets of Y, then it is easy to see that $X_1 \cap X_2$ is E-convex. As we can see Example 2.1 in [9], there is an E-convex set in $\mathbb{R}^2$ but not convex so that the E-convexity is a genuine generalization of the convex condition.

From now on, we shall assume that X is a nonempty subset of a Hausdorff topological vector space Y equipped with a given map $E : Y \to Y$.

Next, we will introduce the generalized E-KKM map which includes the KKM map and its generalizations as follows:

Definition 2.2. Let X be a nonempty subset of a vector space Y. A multimap $T : X \to 2^Y$ is called a *generalized E-KKM map* on X if for any finite subset $\{x_1, \ldots, x_n\} \subseteq X$, there exists a finite subset $\{y_1, \ldots, y_n\} \subseteq Y$ such that

$$co\big(\{E(y_{i_1}),\ldots,E(y_{i_k})\}\big)\subseteq\bigcup_{j=1}^{k}T(x_{i_j})$$

for any subset $\{y_{i_1},\ldots,y_{i_k}\}\subseteq\{y_1,\ldots,y_n\}$ $(1\le k\le n)$.

If T is a generalized E-KKM map on X, each $T(x)$ is clearly nonempty, and $co\big(\{E(y_1),\ldots,E(y_n)\}\big)\subseteq\bigcup_{i=1}^{n}T(x_i)$. When $E:Y\to Y$ is the identity map on Y, then the generalized E-KKM map is exactly the same as a generalized KKM map due to Chang-Zhang in [2].

In Definition 2.2, for each finite subset $\{x_1,\ldots,x_n\}\subseteq X$, if we take $y_i=x_i$ for each $i=1,\ldots,n$, then we simply call T an *E-KKM map.* When $E:Y\to Y$ is the identity map on Y, then the E-KKM map is exactly the same as the KKM map in [5]. It is clear that an E-KKM map is a generalized E-KKM map; however any generalized E-KKM map need not be an E-KKM map in general. Now we shall give a simple example to illustrate the converse does not hold:

EXAMPLE 2.3. *Let $Y=\mathbb{R}$, $X=[0,2]$, and $E:Y\to Y$ is the identity map on Y. Let $T:X\to 2^Y$ be defined by $T(x):=[0,\frac{1}{5}x^2+1]$ for each $x\in X$. Then $x\notin T(x)$ for $x\in[\frac{9}{5},2]$ so that T is not an E-KKM nor a KKM map on X. Now we show that T is a generalized E-KKM map on X. Indeed, for any finite set $\{x_1,\ldots,x_n\}\subseteq X$, if we take any finite set $\{y_1,\ldots,y_n\}\subseteq[0,1]\subseteq Y$, then for any subset $\{y_{i_1},\ldots,y_{i_k}\}\subseteq\{y_1,\ldots,y_n\}$ $(i\le k\le n)$, we have $co(\{y_{i_1},\ldots,y_{i_k}\})\subseteq[0,1]\subseteq\bigcup_{j=1}^{k}T(x_{i_j})$ so that T is a generalized E-KKM map on X.*

3. A generalized E-KKM theorem and its application

Now we begin with the following:

THEOREM 3.1. *Let X be a nonempty subset of a Hausdorff topological vector space Y, and $T:X\to 2^Y$ be a generalized E-KKM map. If $T(x)$ is finitely closed (i.e., for each finite dimensional subspace L in Y, $T(x)\cap L$ is closed in the Euclidean topology in L) for each $x\in X$. Then the family of sets $\{T(x)\mid x\in X\}$ has the finite intersection property.*

Proof. For any finite subset $\{x_1,\ldots,x_n\}\subseteq X$, we shall show that $\bigcap_{i=1}^{n}T(x_i)\neq\emptyset$. Since T is a generalized E-KKM map on X, there exists a finite subset $\{y_1,\ldots,y_n\}\subseteq Y$ such that for any subset $\{y_{i_1},\ldots,y_{i_k}\}\subseteq$

$\{y_1, \dots, y_n\}$ $(i \le k \le n)$, we have

$$co(\{E(y_{i_1}), \dots, E(y_{i_k})\}) \subseteq \bigcup_{j=1}^{k} T(x_{i_j}),$$

and in particular, $co\big(\{E(y_1), \dots, E(y_n)\}\big) \subseteq \bigcup_{i=1}^{n} T(x_i)$. Now we consider the $(n-1)$-simplex Δ_{n-1} with the vertices $e_1 = (1, 0, \cdots, 0), e_2 = (0, 1, 0, \cdots, 0), \cdots, e_n = (0, 0, \cdots, 1)$; and define a continuous map $f : \Delta_{n-1} \to Y$ by

$$f(\Sigma_{i=1}^{n} \lambda_i e_i) := \Sigma_{i=1}^{n} \lambda_i E(y_i), \quad \text{for each } (\lambda_1, \dots, \lambda_n) \in \Delta_{n-1}.$$

Since $f(\Delta_{n-1}) = co(\{E(y_1), \dots, E(y_n)\})$ is a finite dimensional subset of Y and $T(x_i)$ is nonempty finitely closed in Y, we have that each $f^{-1}(T(x_i))$ is a nonempty closed subset of Δ_{n-1}. Indeed, $f(e_i) = E(y_i) \in T(x_i)$, that is, $e_i \in f^{-1}(T(x_i))$ so that $f^{-1}(T(x_i))$ is nonempty, and

$$f^{-1}(T(x_i)) = f^{-1}(T(x_i)) \cap \Delta_{n-1} = f^{-1}\big(T(x_i) \cap f(\Delta_{n-1})\big)$$

is closed in Δ_{n-1} since f is a continuous map. Therefore, we consider the family of nonempty n closed subsets $\{G_i := f^{-1}(T(x_i)) \mid i = 1, 2, \dots, n\}$ of Δ_{n-1}, and now we will show $\bigcap_{i=1}^{n} G_i \neq \emptyset$. Since T is a generalized E-KKM map, for any indices $1 \le i_1 < i_2 < \cdots < i_k \le n$,

$$f(\Sigma_{j=1}^{k} \lambda_{i_j} e_{i_j}) = \Sigma_{j=1}^{k} \lambda_{i_j} E(y_{i_j}) \subseteq \bigcup_{j=1}^{k} T(x_{i_j})$$

so that

$$\begin{aligned} \Sigma_{j=1}^{k} \lambda_{i_j} e_{i_j} \in f^{-1}(\bigcup_{j=1}^{k} T(x_{i_j})) &= \bigcup_{j=1}^{k} f^{-1}(T(x_{i_j})) \\ &= \bigcup_{j=1}^{k} G_{i_j} \subseteq \Delta_{n-1}. \end{aligned}$$

Therefore, we can apply the KKM theorem [4] to the family of closed subsets $\{G_i \mid 1 \le i \le n\}$ of Δ_{n-1} so that we have $\bigcap_{i=1}^{n} G_i \neq \emptyset$. Hence

$$\emptyset \neq \bigcap_{i=1}^{n} G_i = \bigcap_{i=1}^{n} f^{-1}(T(x_i)) = f^{-1}\big(\bigcap_{i=1}^{n} T(x_i)\big)$$

so that we have $\bigcap_{i=1}^{n} T(x_i) \neq \emptyset$. This completes the proof. □

Remark 3.2. In Theorem 3.1, if we replace the finitely closed assumption on $T(x)$ with the finitely open assumption on $T(x)$, then the same conclusion holds by replacing the KKM theorem with Theorem 1 in [6] in the above proof.

As a consequence of Theorem 3.1, we can obtain the following which is a generalization of the KKM theorem in E-convex settings:

Theorem 3.3. *Let X be a nonempty subset of a Hausdorff topological vector space Y, and $T : X \to 2^Y$ be a generalized E-KKM map. If $T(x)$ is closed for each $x \in X$, and $T(x_o)$ is compact for some $x_o \in X$, then we have*

$$\bigcap_{x \in X} T(x) \neq \emptyset.$$

Proof. Since T is a generalized E-KKM map and $T(x)$ is nonempty finitely closed for each $x \in X$, by Theorem 3.1, the family of sets $\{T(x) \mid x \in X\}$ has the finite intersection property so that $\{T(x) \cap T(x_o) \mid x \in X\}$ has also the finite intersection property. Since $T(x) \cap T(x_o)$ is a compact subset of $T(x_o)$ for each $x \in X$, we have

$$\emptyset \neq \bigcap_{x \in X} \big(T(x) \cap T(x_o)\big) = \big(\bigcap_{x \in X} T(x)\big) \cap T(x_o) = \bigcap_{x \in X} T(x)$$

which completes the proof. □

The following fixed point theorem, which generalizes the Fan-Browder fixed point theorem [1,4] in E-convex sets, can be a basic tool in proving many variational inequalities and intersection theorems in E-convex settings:

Theorem 3.4. *Let X be a nonempty subset of a Hausdorff topological vector space Y, and let $T : X \to 2^X$ be a multimap satisfying the following:*

(1) *for each $x \in X$, $T(x)$ is open in X;*
(2) *for each $y \in X$, $T^{-1}(y)$ is a nonempty E-convex subset of X;*
(3) *there exists an $y_o \in X$ such that $X \setminus T(y_o)$ is compact.*

Then T has a fixed point $\hat{x} \in X$, i.e., $\hat{x} \in T(\hat{x})$.

Proof. In case $T(x) = X$ for some $x \in X$, then we have done. Suppose that each $T(x)$ is a proper subset of X. Consider the multimap $S : X \to 2^X$ defined by

$$S(x) := X \setminus T(x) \quad \text{for each } x \in X.$$

By the assumption (1), each $S(x)$ is nonempty closed in X, and by the assumption (3), $S(y_o)$ is nonempty compact. Note that $X = \bigcup_{x\in X} T(E(x))$. In fact, for each $y \in X$, by the assumption (2), choose $x \in T^{-1}(y)$; then $y \in T(x)$. Since $T^{-1}(y)$ is E-convex, $E(x) \in T^{-1}(y)$ so that $y \in T(E(x))$. Therefore, we have

$$X = \bigcup_{x\in X} T(x) = \bigcup_{x\in X} T(E(x));$$

so that we have

$$\bigcap_{x\in X} S(x) = \bigcap_{x\in X} \big(X \setminus T(x)\big) = X \setminus \bigcup_{x\in X} T(x) = \emptyset.$$

Therefore, by Theorem 3.3, S should not be a (generalized) E-KKM map on X. Therefore, there exist a finite subset $\{x_1, \ldots, x_n\} \subseteq X$ and a point

$$\hat{x} = \Sigma_{i=1}^n \lambda_i E(x_i) \in co(\{E(x_1), \ldots, E(x_n)\}) \subseteq X$$

with $(\lambda_1, \ldots, \lambda_n) \in \Delta_{n-1}$ such that

$$\hat{x} = \Sigma_{i=1}^n \lambda_i E(x_i) \notin \bigcup_{i=1}^n S(x_i) = \bigcup_{i=1}^n \big(X \setminus T(x_i)\big) = X \setminus \bigcap_{i=1}^n T(x_i);$$

so that $\hat{x} \in \bigcap_{i=1}^n T(x_i)$. Therefore, $x_i \in T^{-1}(\hat{x})$ for each $i = 1, \ldots, n$. Since $T^{-1}(\hat{x})$ is a nonempty E-convex subset of X, we have

$$\{E(x_1), \ldots, E(x_n)\} \subseteq E\big(T^{-1}(\hat{x})\big) \subseteq co\{E\big(T^{-1}(\hat{x})\big)\} \subseteq T^{-1}(\hat{x}).$$

Therefore, we obtain that $\hat{x} = \Sigma_{i=1}^n \lambda_i E(x_i) \in T^{-1}(\hat{x})$ so that $\hat{x} \in T(\hat{x})$ which completes the proof. □

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Department of Mathematics Education
Chungbuk National University
Cheongju 361-763, Republic of Korea
E-mail: wkkim@chungbuk.ac.kr