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ANALYSIS OF THE VLASOV-POISSON EQUATION BY USING A VISCOSITY TERM

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Abstract. The well-known Vlasov-Poisson equation describes plasma physics as nonlinear first-order partial differential equations. Because of the nonlinear condition from the self consistency of the Vlasov-Poisson equation, many problems occur: the existence, the numerical solution, the convergence of the numerical solution, and so on. To solve the problems, a viscosity term (a second-order partial differential equation) is added. In a viscosity term, the Vlasov-Poisson equation changes into a parabolic equation like the Fokker-Planck equation. Therefore, the Schauder fixed point theorem and the classical results on parabolic equations can be used for analyzing the Vlasov-Poisson equation. The sequence and the convergence results are obtained from linearizing the Vlasove-Poisson equation by using a fixed point theorem and Gronwall's inequality. In numerical experiments, an implicit first-order scheme is used. The numerical results are tested using the changed viscosity terms.

1. Introduction and preparation

Plasma physics is a fascinating scientific domain where an applied mathematician and more specifically, a numerical analyst can find a variety of beautiful and difficult nonlinear problems of great practical interest. More details can be found elsewhere in [1, 3, 5, 6, 7, 9, 11, 12, 13, 14, 16, 17, 18, 19, 23, 20, 21, 22]. However, because plasmas are hot, ionized gases that are composed of ions, electrons, and neutral atoms, it is difficult to experiment on neutral atoms for realtime and to understand the behavior of particles(ionized gases composed of ions,

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electrons, and neutral atoms). A very efficient tool for experiments is the computer simulation of the mathematical model of a plasma using a numerical method. The equation that describes the idealized plasma in a mathematical model is the Vlasov equation, which describes the electron distribution u,

$$\frac{\partial u}{\partial t} + v \cdot \nabla_x u + F \cdot \nabla_y u = 0, \quad x, y \in R^3, \quad t > 0, \tag{1.1}$$

as coupled to the Poisson equation

$$\Delta_x \Phi = \rho = \int_{\Omega_y} u dy - 1, \tag{1.2}$$

$$\nabla_x \Phi = F \tag{1.3}$$

where ρ is the charge density. In an electrostatic case, $\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, $\nabla_y = (\partial/\partial y_1, \partial/\partial y_2, \partial/\partial y_3)$, $v = (v_1, v_2, v_3)$, $F = (F_1, F_2, F_3)$ and $\Omega = \Omega_x \times \Omega_y$. To simplify and generalize the presentation of the method, the Vlasov equation is presented as follows:

$$u_t + \zeta(u) \cdot \nabla u = 0. \tag{1.4}$$

In this paper, let us assume that

$$\zeta(u) = (v, F) = (v_1, v_2, v_3, F_1, F_2, F_3),$$

and the particles are in a tokamak, where $\partial v_i/\partial x_j = 0$ and $\partial F_i/\partial y_j = 0$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. More details can be found elsewhere in [5, 6].

The tokamak is characterized by azimuthal (rotational) symmetry, and the use of the plasma-borne electric current to generate the helical component of the magnetic field necessary for stable equilibrium. This can be contrasted to another toroidal magnetic confinement device, the stellarator, which has a discrete (e.g. fivefold) rotational symmetry, and in which all of the confining magnetic fields are produced by external coils with a negligible electric current flowing through the plasma. Hence the domain Ω is $S^1 \times D^5$ in R^6, which is obtained by thickening the S^1 that is embedded in R^6. Its boundary is equal to $S^1 \times S^4$. From (1.4) and $\zeta(u)$, the solution space has a compact support, where $\Omega_x = S^1 \times D^2$, $\Omega_y = D^3$.

In the above equations, u is the electron distribution function and F is the electrostatic field. The charge density is known as ρ. Assuming a periodic plasma of period L, the functions u and F satisfy the periodic boundary conditions:

$$\begin{aligned} u(t,0,x_2,x_3,y) &= u(t,L,x_2,x_3,y) \\ F(t,0,x_2,x_3,y) &= F(t,L,x_2,x_3,y), \end{aligned}$$

where $(x_2, x_3, y) \in D^5$ for all $t \geq 0$. The periodic boundary condition is equivalent to

$$\int_{\Omega_x}\int_{\Omega_y} u dy dx = |\Omega_x|$$

$$\Leftrightarrow \int_{\partial\Omega_x} F \cdot n = 0,$$

where n is a normal vector. To completely define the problem, a zero-mean electrostatic field condition is added as follows

$$\int_{\Omega_x} F = \vec{0},$$

where $\vec{0} = (0,0,0)$ with an initial condition of $u(0,x,y) = u_0(x,y)$. More details can be found elsewhere in [9]. From the above conditions, and with Φ being defined up to an additive constant, the following equation is obtained

$$\begin{aligned} \Delta_x\Phi &= \rho = \int_{\Omega_y} u dy - 1, \\ &\Phi|_{\partial\Omega_x} = 0. \end{aligned}$$

By Poisson equation and Friedrichs' inequality,

$$\begin{aligned} \|\Delta_x\Phi\|_{L^2(\Omega_x)} &= \|\rho\|_{L^2(\Omega_x)} \leq C_0\|u\|_{L^2(\Omega)} \\ \|\nabla\Phi\|_{L^2(\Omega_x)} &\leq C_1\|u\|_{L^2(\Omega)}, \end{aligned}$$

where C_0, C_1 are constants depending on Ω. More details can be found elsewhere in [4, 10].

To show the existence of the solution of the Vlasov-Poisson system the second-order parabolic equation is used where $\|\zeta(u)\|_{L^\infty(\Omega)}$ is bounded. The Sobolev imbedding theorem is that if $mp > n$, then $W^{m,p}(\Omega) \subset L^\infty(\Omega)$, where n is the dimension of Ω and W is the Sobolev space. Since the dimension of Ω_x is 3 and p is 2, it is enough to make $m = 2$. Then, $H^2(\Omega_x) \subset L^\infty(\Omega_x)$. Therefore, if $u \in H^1(\Omega)$, then $\rho \in H^1(\Omega_x)$. So, $\Phi \in H^3(\Omega_x)$ by Poisson's equation and $F_i \in H^2(\Omega_x)$ for $i = 1,2,3$. $F_i \in L^\infty(\Omega_x)$ for $i = 1,2,3$ are obtained. More details can be found elsewhere in [2].

In the following, let Ω denote an open subset of R^6, with boundary $\partial\Omega$ and $H^k(\Omega)$ as the Hilbert space for the norm

$$\|u\|_{H^k(\Omega)} = \left(\sum_{|s|\leq k} \int_\Omega |D^s u(x)|^2 \right).$$

The space $L^p(0,T;H^k(\Omega))$ consists of all functions u, such that for almost every t in $(0,T)$, $u(t)$ belongs to $H^k(\Omega)$. $L^p(0,T;H^k(\Omega))$ is a normed space for the norm

$$\|u\|_{L^p(0,T;H^k(\Omega))} = \left(\int_0^T \|u(t)\|^p_{H^k(\Omega)} \right)^{1/p},$$

where $p > 1$ and k is a positive integer. $L^\infty(0,T;L^2(\Omega))$ is a normed space for the norm

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} = ess \sup_{0\leq t\leq T} \|u(t)\|_{L^2(\Omega)}.$$

Some other spaces that appear below can be defined analogously. More details can be found elsewhere in [8, 10, 27].

The space $W(0,T)$ is introduced as follows:

$$W(0,T) = \left\{ w \in L^2\left(0,T;H^1_0(\Omega)\right),\ \ \frac{dw}{dt} \in L^2\left(0,T;H^1(\Omega)'\right) \right\},$$

where $H^1(\Omega)'$ is the dual space of $H^1_0(\Omega)$. $W(0,T)$ is a Hilbert space with the graph norm.

If $w \in W(0,T)$, then

$$\rho_w = \int_{\Omega_y} w dy - 1 \in H^1(\Omega_x).$$

By using the classical results of Poisson's equation and integration, the following equations are obtained

$$\zeta(w) \in \prod^6 L^\infty(\Omega) = \underbrace{L^\infty(\Omega) \times \ldots \times L^\infty(\Omega)}_{6}$$

and

$$\|\zeta(w)\|_{L^\infty(\Omega)} \leq M = \max\left\{ \max_{\Omega_y} \|v\|_\infty, C_0\|w\|_{L^2(\Omega)} \right\},$$

where C_0 is a constant depending on Ω.

For all $u, w \in W(0,T)$, Φ_u and Φ_w can be obtained by Poisson's equation as follows: $\Delta_x \Phi_u = \rho_u$ and $\Delta_x \Phi_w = \rho_w$.

$$\begin{aligned} \|\Delta_x \Phi_u - \Delta_x \Phi_w\|_{L^2(\Omega_x)} &= \|\rho_u - \rho_w\|_{L^2(\Omega_x)} \\ \|\nabla_x \Phi_u - \nabla_x \Phi_w\|_{L^2(\Omega_x)} &\leq K\|\rho_u - \rho_w\|_{L^2(\Omega_x)} \end{aligned}$$

are obtained by Friedrichs' inequality, where K is a constant. More details can be found elsewhere in [4, 10]. By using the Corollary of the Sobolev imbedding theorem, the following equation is obtained

$$\begin{aligned} \|\nabla_x \Phi_u - \nabla_x \Phi_w\|_{L^\infty(\Omega_x)} &\leq K_1\|\nabla_x \Phi_u - \nabla_x \Phi_w\|_{L^2(\Omega_x)} \\ &\leq K_2\|\rho_u - \rho_w\|_{L^2(\Omega_x)} \leq K_3\|u - w\|_{L^2(\Omega)}, \end{aligned}$$

where K_1, K_2 and K_3 are the constants.

Therefore, since $v = y$, $\|\zeta(u) - \zeta(w)\|_{L^\infty(\Omega)} \leq C_1\|u - w\|_{L^2(\Omega)}$ for all u, w in $W(0,T)$ is obtained, where C_1 is a constant.

However, it is difficult to show the existence of the solution of (1.4), which is a nonlinear first-order partial differential equation. One of the methods for solving the problem is to add a second-order partial differential equation because the results (e.g. existence, uniqueness and regularity) of second-order partial differential equations are well known. Therefore, the (1.4) changes into

$$u_t + \zeta(u) \cdot \nabla u - \nabla \cdot (\epsilon \nabla u) = 0.$$

More details can be found elsewhere in [10, 27].

The paper is organized as follows: In Section 2, the existence and uniqueness of the proposed model Eq. (1.4) are shown. In Section 3, an iteration scheme, which converges to the solution of (1.4), is described. In section 4, numerical experiments are shown.

2. Existence and uniqueness results

The standard notations will be used throughout this paper. More details can be found elsewhere in [8, 10, 27].

THEOREM 2.1. *Let $u_0 \in H_0^1(\Omega)$. Then, there is the unique weak solution u, such that $u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$, with $du/dt \in L^2(0,T; H^1(\Omega)')$, which satisfies*

$$\begin{aligned} u_t + \zeta(u) \cdot \nabla u - \nabla \cdot (\epsilon \nabla u) &= 0, \\ u|_{\partial\Omega} &= 0, \\ u(0) &= u_0. \end{aligned} \tag{2.1}$$

Proof. The proof consists of three parts:

1) the existence of a weak solution,
2) the regularity of the solution,
3) the uniqueness of the solution,

1) *The existence of a weak solution.* Unfortunately, the existence of the solution (1.4) does not always occur. To overcome this problem, the second-order parabolic equation theorem can be used by adding a second-order term, $-\nabla \cdot (\epsilon \nabla u)$ to (1.4), as follows:

$$u_t + \zeta(u) \cdot \nabla u - \nabla \cdot (\epsilon \nabla u) = 0. \tag{2.2}$$

Since $\|\zeta(w)\|_\infty < M$, let $w \in W(0,T) \cap L^\infty(0,T;L^2(\Omega))$, such that

$$\|w\|_{L^\infty(0,T;L^2(\Omega))} \leq \|u_0\|_{L^2(\Omega)}.$$

From the linearization of (2.2) and the weak formulation, the following linear problem $[Ew]$ is obtained: Find $u \in W(0,T)$, such that

(2.3)

$$\begin{aligned} (u_t, v) + (\zeta(w) \cdot \nabla u, v) + \epsilon(\nabla u, \nabla v) &= 0, \quad \forall v \in H_0^1(\Omega) \ \text{ a.e. in } \ [0,T] \\ u(0) &= u_0. \end{aligned}$$

Since $\zeta(w) \in \prod^6 L^\infty(\Omega)$, there is a unique solution $U(w)$ of (2.3), which depends on w and ϵ. This follows from the classical results of the parabolic equation [8, 10, 27]. In fact, $U(w)$ is in the nonempty, convex, and weakly compact subset W_0 of $W(0,T)$ which is defined by

$$W_0 = \Big\{ w \in W(0,T) : \ \ \|w\|_{L^\infty(0,T;L^2(\Omega))} \leq \|u_0\|_{L^2(\Omega)}, \ w(0) = u_0, \\ \|w\|_{L^2(0,T;H^1(\Omega))} + \|w_t\|_{L^2(0,T;H^1(\Omega)')} \leq C\|u_0\|_{L^2(\Omega)} \Big\},$$

where C is a constant which only depends on Ω, T, $\zeta(w)$ and ϵ. Thus, U is mapped from W_0 into W_0.
Since $W(0,T)$ is compactly imbedded into $L^2(0,T;L^2(\Omega))$, W_0 is a closed, convex, bounded subset of $L^2(0,T;L^2(\Omega))$. In order to apply the Schauder fixed point theorem, it must be shown that $U(w)$ is a compact continuous mapping from W_0 into W_0. Let $\{w_k\}$ be a sequence in W_0 which converges weakly to some w in W_0 and $u_k = U(w_k)$. The sequence $\{w_k\}$ contains a subsequence such that $u_k \to u$ weakly in $L^2(0,T;H^2(\Omega))$, $du_k/dt \to du/dt$ weakly in $L^2(0,T;H^2(\Omega))$. More details can be found elsewhere in [8, 15, 27]. By passing to the limit in the

relation

$$\begin{aligned}\left(\frac{dU(w_k)}{dt}(t), v\right) + (\zeta(w_k)\cdot\nabla U(w_k), v) + \epsilon(\nabla U(w_k), \nabla v) &= 0,\\ U(w_k)(0) &= u_0,\end{aligned}$$

$U(w_k) \to U(w)$ and $u = U(w)$ are obtained. Due to the uniqueness of the solution of $[Ew]$, the whole sequence $u_k = U(w_k)$ converges weakly in $W(0,T)$ to $u = U(w)$. Hence, the mapping of U is weakly continuous from W_0 into W_0. This, in turn, shows that the mapping of U is compact. A similar argument shows that U is continuously mapped. More details can be found elsewhere in [8, 15, 27]. By applying the Schauder fixed point theorem, u exists such that $u = U(u)$, which consequently solves (2.3).

2) *The regularity of the solution.* Using the general theory of parabolic equations and the bootstrap argument, u is a strong solution of (2.3) and $u \in C^\infty((0,T)\times\Omega)$. More details can be found elsewhere in [8, 10, 27].

3) *The uniqueness of the solution.* Let u and w be the two solutions for (2.2). For almost every t in $[0,T]$,

$$\begin{aligned} u_t + \zeta(u)\cdot\nabla u - \nabla\cdot(\epsilon\nabla u) = 0,\\ w_t + \zeta(w)\cdot\nabla w - \nabla\cdot(\epsilon\nabla w) = 0,\\ \text{and}\ \ u(0) = w(0) = u_0.\end{aligned}$$

By subtracting, multiplying and integrating the parts,

$$\begin{aligned}&((u-w)_t, v) + (\zeta(u)\cdot\nabla u - \zeta(w)\cdot\nabla w, v) + \epsilon(\nabla u - \nabla w, \nabla v) = 0\\ &((u-w)_t, v) + \epsilon(\nabla u - \nabla w, \nabla v)\\ &= -\left((\zeta(u)-\zeta(w))\nabla u, v\right) - \left(\zeta(w)(\nabla u - \nabla w), v\right).\end{aligned}$$

By taking $v = u - w$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u-w\|^2_{L^2(\Omega)} + \epsilon\|\nabla u - \nabla w\|^2_{L^2(\Omega)}\\ &\le \|\zeta(u)-\zeta(w)\|_{L^\infty(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|u-w\|_{L^2(\Omega)}\\ &\quad + \|\zeta(w)\|_{L^\infty(\Omega)}\|\nabla u - \nabla w\|_{L^2(\Omega)}\|u-w\|_{L^2(\Omega)}.\end{aligned}\tag{2.4}$$

Moreover $\|\zeta(u)-\zeta(w)\|_{L^\infty(\Omega)} \le C_1\|u-w\|_{L^2(\Omega)}$ and $\|\zeta(w)\|_{L^\infty(\Omega)} \le M$. By combining these inequalities and using Schwarz's inequality, (2.4) changes into

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u-w\|^2_{L^2(\Omega)} + \epsilon\|\nabla u - \nabla w\|^2_{L^2(\Omega)} \\
&\le C_1\|\nabla u\|_{L^2(\Omega)}\|u-w\|^2_{L^2(\Omega)} \\
&\quad + \epsilon_1 M\|\nabla u - \nabla w\|^2_{L^2(\Omega)} + \frac{4M}{\epsilon_1}\|u-w\|^2_{L^2(\Omega)}.
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u-w\|^2_{L^2(\Omega)} + (\epsilon - \epsilon_1 M)\|\nabla u - \nabla w\|^2_{L^2(\Omega)} \\
&\le \left(C_1\|\nabla u\|_{L^2(\Omega)} + \frac{4M}{\epsilon_1}\right)\|u-w\|^2_{L^2(\Omega)}.
\end{aligned} \tag{2.5}$$

Assuming $0 \le \epsilon - \epsilon_1 M$, (2.5) is changed into

$$\frac{1}{2}\frac{d}{dt}\|u-w\|^2_{L^2(\Omega)} \le C\|u-w\|^2_{L^2(\Omega)}, \tag{2.6}$$

where $C = \big(C_1\|\nabla u\|_{L^2(\Omega)} + 4M/\epsilon_1\big)$ is bounded by $u \in L^2(0,T;H^1_0(\Omega))$.

Since $u(0) = w(0) = u_0$, (2.6) and Gronwall's inequality yield $u(t,x) = w(t,x)$ on $[0,T]\times\Omega$, which establishes the uniqueness. □

3. Convergent iterative scheme

Discretised difference schemes will be used for the numerical experiments.

THEOREM 3.1. *Let $u_0 \in H^1_0(\Omega)$. The sequence $\{u^n\}$ is defined by*

$$\begin{aligned}
u_t^{n+1} + \zeta(u^n)\cdot\nabla u^{n+1} - \nabla\cdot(\epsilon\nabla u^{n+1}) &= 0, \\
u^{n+1}|_{\partial\Omega} &= 0, \\
u^{n+1}(0) &= u_0,
\end{aligned} \tag{3.1}$$

which converges in $C([0,T];L^2(\Omega))$ to the strong solution of (2.2).

Proof. From (2.3), and assuming $u^n \in H^1_0(\Omega)$, the linear problem E_{u^n} has a unique solution u^{n+1}. It is clear that $u^{n+1} \in H^1_0(\Omega)$ by Theorem 2.1 and the classical results of the parabolic equations.

By using (2.4) in Theorem 2.1,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|u-u^{n+1}\right\|^2_{L^2(\Omega)}+\epsilon\|\nabla u-\nabla u^{n+1}\|^2_{L^2(\Omega)}\\&\le\|\zeta(u)-\zeta(u^n)\|_{L^\infty_0(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|u-u^{n+1}\|_{L^2(\Omega)}\\&\quad+\|\zeta(u^n)\|_{L^\infty_0(\Omega)}\|\nabla u-\nabla u^{n+1}\|_{L^2(\Omega)}\|u-u^{n+1}\|_{L^2(\Omega)}.\end{aligned}$$

By using Schwarz's inequality, Poincare's inequality, ϵ and conditions of ζ,

$$\frac{1}{2}\frac{d}{dt}\left\|u-u^{n+1}\right\|^2_{L^2(\Omega)}\le\|\nabla u\|^2_{L^2(\Omega)}\|u-u^n\|^2_{L^2(\Omega)}.$$

Moreover

$$\|u-u^0\|^2_{L^2(\Omega)}\le C^*\quad\forall t\in[0,T],$$

where C^* is a constant which only depends on ζ, ϵ and u_0. Then, Gronwall's inequality yields, for any $t\in[0,T]$,

$$\|u-u^1\|^2_{L^2(\Omega)}\le C^*\left(\int_0^T\|\nabla u(s)\|^2_{L^2(\Omega)}ds\right),$$

and, by iteration,

$$\|u-u^{n+1}\|^2_{L^2(\Omega)}\le\frac{C^*}{(n+1)!}\left(\int_0^T\|\nabla u(s)\|^2_{L^2(\Omega)}ds\right)^{n+1}.$$

The sequence $\{u^n\}$ converges in $C([0,T];L^2(\Omega))$ to the strong solution of (2.2). □

4. Numerical experiments

In order to simplify the presentation of the method for the numerical experiments, the 1D electrostatic case is presented as follows:

$$\begin{aligned}u_t+v\partial_x u&+F\partial_y u=0,\\ \Delta_x\Phi=\rho&=\int_{\Omega_y}udy-1,\\ \Phi|_{\partial\Omega_x}&=0.\end{aligned}$$

By adding a second-order term and by using a standard finite differential method (an implicit first-order scheme), the 1D electrostatic case changes into

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\triangle t}+v_j^n\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}}{2\triangle x}+F_i^n\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n+1}}{2\triangle y}$$
$$-\epsilon\left(\frac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{(\triangle x)^2}+\frac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{(\triangle y)^2}\right)=0,$$
$$v_j^n=y_j.$$

The discrete problem can now be written

$$\frac{u^{n+1}-u^n}{\triangle t}+A_h(\epsilon,u^n)u^{n+1}=0,$$

where the matrix A_h is positively defined. By classical arguments, $I+\triangle tA_h(\epsilon,u^n)$ is invertible. More details can be found elsewhere in [4, 10, 23].

It is hard to confirm the superiority of the methods. One verifiable means for confirmation is Landau damping. Landau damping occurs when the energy exchanges between a wave with phase velocity Vph and particles in the plasma with velocity are approximately equal to Vph, which can interact strongly with the wave. Those particles that have velocities slightly less than Vph will be accelerated by the wave electric field in order to move with the wave phase velocity, while those particles with velocities slightly greater than Vph will be decelerated by the wave electric field, losing energy to the wave. In a collisionless plasma, the particle velocities are often taken to be approximately a Maxwellian distribution function. If the slope of the function is negative, the number of particles with velocities slightly less than the wave phase velocity is larger than the number of particles with velocities slightly greater. Hence, there are more particles that gain energy from the wave than lose energy to the wave, which leads to wave damping. If, however, the slope of the function is positive, the number of particles with velocities slightly less than the wave phase velocity is smaller than the number of particles that have velocities that are slightly greater. Hence, there are more particles that lose energy to the wave than gain energy from the wave, which leads to an increase in the wave energy.

To confirm this method, Finite Difference Method is used, where the initial data are given by

$$u_0(x,y)=\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\left(1+\alpha\cos kx\right),$$

where $k = 0.5$, $-5 \leq y \leq 5$, and $0 \leq x \leq 4\pi$. The linear Landau damping theory is valid as long as $t < \alpha^{1/2}$: For longer times, the problem is inherently nonlinear. Here, the Landau theory can not be applied because nonlinear effects are too important, despite the fact that this test has been studied numerically by many authors. The electric energy $\log \sum |F|$ first decays linearly and then periodically oscillates. More details can be found elsewhere in [11, 12, 13, 14, 16, 17, 18, 19, 23, 22].

In this case, $\alpha = 0.01$, $\triangle t = 0.01$, and $T =$ a number of iteration $\times \triangle t$. A number of cells ($N_x = 32$ in the $x-$direction and $N_y = 32$ in the $y-$direction) are used.

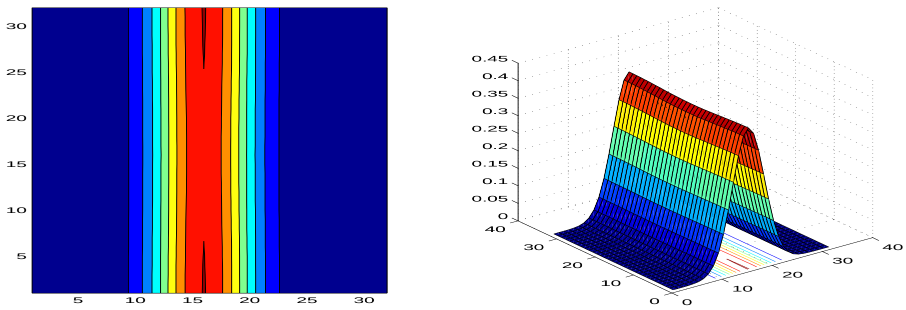

Figure 1.Initial condition $\alpha = 0.01$

Figure 1 shows the initial function u_0 on $\alpha = 0.01$. In Figure 1, the left image is a contour image and the right image is a 3-D plot image.

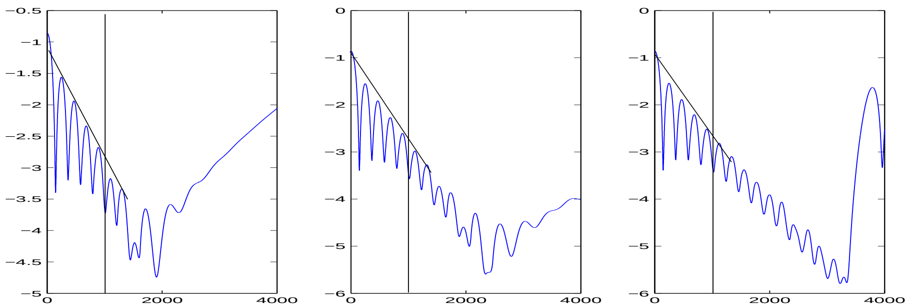

Figure 2. $\epsilon = 0.01,\ 0.005,\ 0$

Figure 2 shows the electric energy $\log \sum |F|$ for ϵ ($\epsilon = 0.01,\ 0.005,\ 0$). More details can be found elsewhere in [7, 11].

In this case, $\alpha = 0.5$, $\triangle t = 0.01$, and $T =$ a number of iteration $\times \triangle t$. A number of cells ($N_x = 32$ in the $x-$direction and $N_y = 32$ in the $y-$direction) are used.

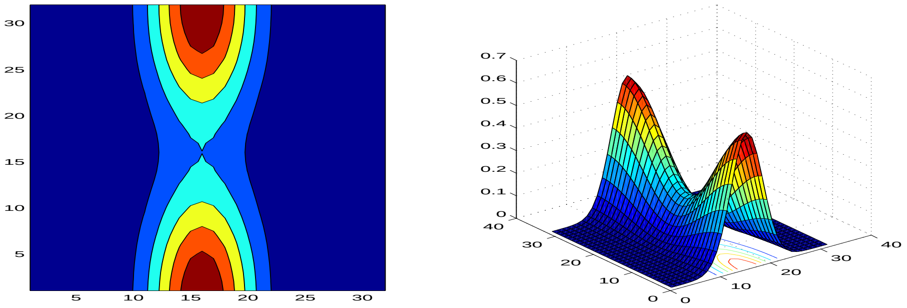

Figure 3. Initial condition $\alpha = 0.5$

Figure 3 shows the initial function u_0 on $\alpha = 0.5$. In Figure 3, the left image is a contour image and the right image is a 3-D plot image.

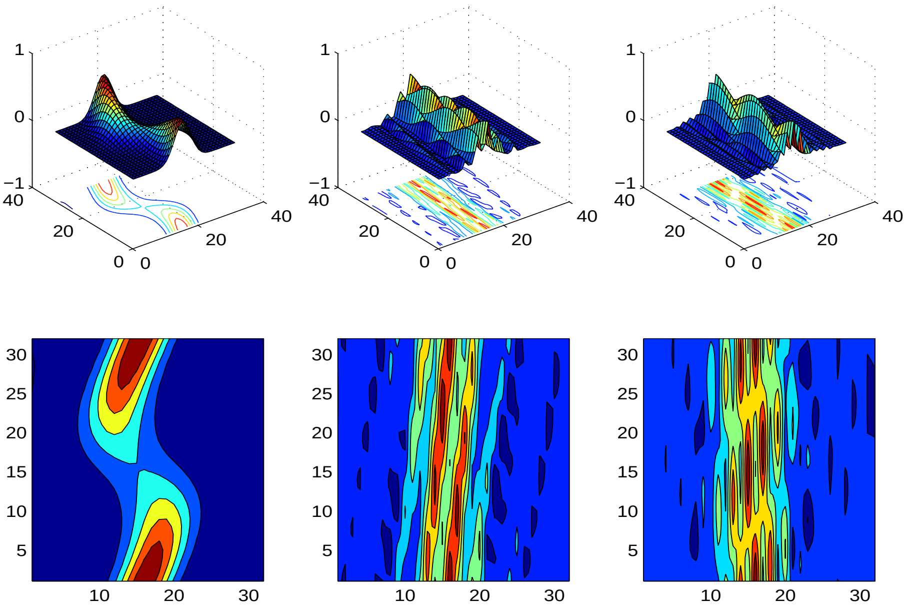

Figure 4. $T = 1,\ 10,\ 20$

In Figure 4, the upper images are 3-D plot images and the lower images are contour images for each time $(T = 1, 10, 20)$ on $\epsilon = 0$.

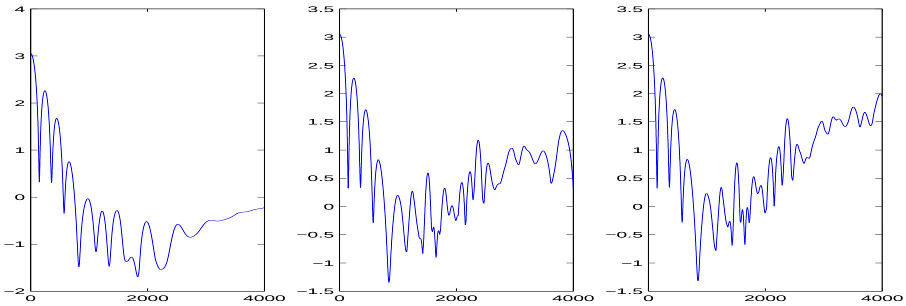

Figure 5. $\epsilon = 0.01,\ 0.001,\ 0$

Figure 5 shows the electric energy $\log \sum |F|$ of results on $\alpha = 0.5$ for ϵ ($\epsilon = 0.01,\ 0.001,\ 0$).

In this case, the initial data

$$u_0(x, y) = \frac{y^2}{\sqrt{2\pi}} e^{-y^2/2} \left(1 + \alpha \cos kx\right)$$

are given, where $\triangle t = 0.01$ and $T =$ a number of iteration $\times \triangle t$. A number of cells ($N_x = 32$ in the $x-$direction and $N_y = 32$ in the $y-$direction) are used.

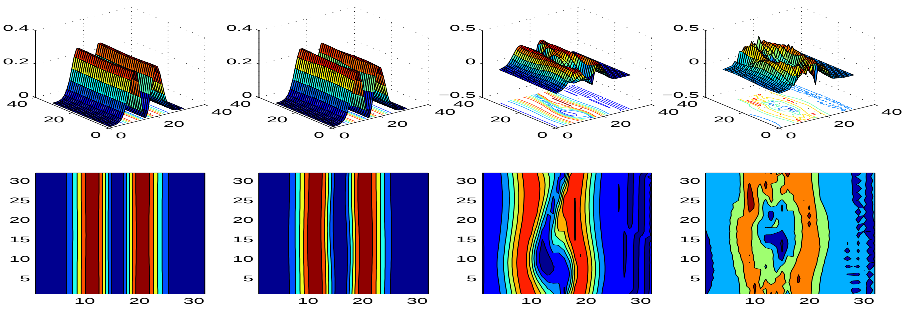

Figure 6. T= 0, 10, 20, 30

Figure 6 shows the results of condition $\alpha = 0.01$. In Figure 6, the upper images are 3-D plot images and the lower images are contour images for each time ($T = 0, 10, 20, 30$) on $\epsilon = 0$.

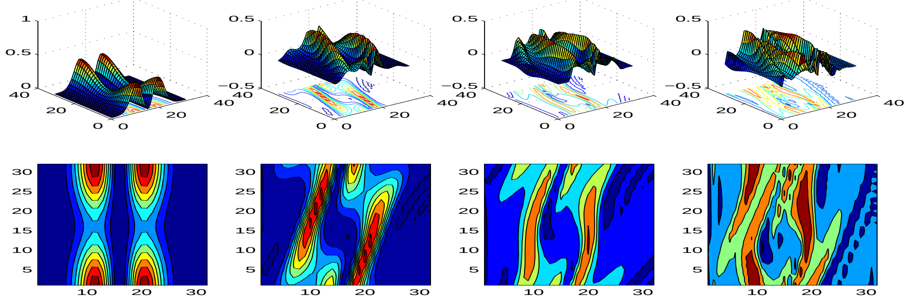

Figure 7. T= 0, 3, 4, 6

Figure 7 shows the results of condition $\alpha = 0.5$. In Figure 7, the upper images are 3-D plot images and the lower images are contour images for each time $(T = 0, 3, 4, 6)$ on $\epsilon = 0$.

Conclusion

In this paper, the nonlinear first-order partial differential equation containing self consistence condition (the Vlasov-Poisson equation) was studied with a viscosity term. By using the viscosity term, this equation changes into a parabolic equation. Therefore, the well-known classical results of parabolic equations can be used. Our results have the viscosity effects. However, using small viscosity terms or without the viscosity term $(\epsilon = 0)$ in numerical experiments, the results of proposed schemes are similar to the numerical results of the Vlasov-Poisson equation. Moreover, we analyzed the tokamak structure and obtained the confidential results of the long-time numerical solution, using the implicit scheme.

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