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WEAK LAW OF LARGE NUMBERS FOR WEIGHTED SUMS IN NONCOMMUTATIVE LORENTZ SPACE

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Abstract. In this paper, we prove the weak law of large numbers for weighted sums of noncommutative random variables in noncommutative Lorentz space under weaker conditions than the conditions in [7].

1. Introduction

Let $\{X_i\}$ be a sequence of (classical) random variables. The law of large numbers plays an important role in probability theory, which is concerned with the convergence of $(S_n - b_n)/n$, where $S_n = \sum_{i=1}^n X_i$ and $\{b_n\}$ is a sequence of real numbers. If $(S_n - b_n)/n$ converges in probability (measure), then the convergence theorem is referred to the weak law of large numbers (WLLN). The WLLN in a classical (commutative) probability space has been extended to the WLLN in a noncommutative probability space by several authors, e.g. Batty [2], Jajte [10], Łuczak [14], Bercovici & Pata [3, 4], Lindsay & Pata [13], Stoica [21] and the references cited therein.

A weighted sum of a sequence $\{X_i\}$ of random variables is of the form

$$\sum_{i=1}^{n} a_{ni} X_i, \tag{1.1}$$

where the weighted sequence $\{a_{ni} \,|\, 1 \le i \le n\}$ is a triangular array.

In classical probability theory, the law of large numbers for weighted sums of classical random variables with scalar valued weights has been studied by many authors [18, 19, 5, 23, 6, 22, 16, 8, 11], etc. Especially,

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in [6], the authors proved that if the weighted sequence $\{a_{ni}\}$ satisfies the following condition

$$\max_{1\leq i\leq n} |a_{ni}| = O(1/n)$$

then we have $\sum_{i=1}^{n} a_{ni}X_i \longrightarrow 0$ a.s. as $n \to \infty$, where $\{X_i\}$ is independent, identically distributed random variables with mean 0.

The WLLN for weighted sums of noncommutative random variables with scalar valued weighted sequences has been studied by Pata [17], Balan & Stoica [1]. On the other hand, in [7], the authors proved the WLLN for weighted sums of noncommutative random variables with von Neumann algebra valued weighted sequences.

The main purpose of this paper is to prove the WLLN for weighted sums of the forms (1.1) of noncommutative random variables $\{X_i\}$ in noncommutative Lorentz space under weaker conditions than the conditions in [7].

This paper is organized as follows. In Section 2, we recall elementary notions in noncommutative probability theory. In Section 3, we study the WLLN for weighted sums of random variables in noncommutative Lorentz space, and prove the main result (Theorem 3.2).

2. Noncommutative probability space

Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space (or noncommutative probability space) with a von Neumann algebra $\mathcal{M}$ (with unit $\mathbf{1}$) and a normal faithful tracial state τ on $\mathcal{M}$.

Now, we recall the *measure topology* [15] of $\mathcal{M}$ given by the fundamental system of neighborhoods of 0: for any $\epsilon > 0$ and $\delta > 0$

$$\begin{aligned} N(\epsilon, \delta) = \{X \in \mathcal{M} \,|\, &\text{there exists a projection } P \in \mathcal{M} \\ &\text{with } \tau(\mathbf{1} - P) \leq \delta \text{ such that } \|XP\| \leq \epsilon\}. \end{aligned}$$

We denote by $\widetilde{\mathcal{M}}$ the completion of $\mathcal{M}$ with respect to the measure topology. Then the mappings

$$\mathcal{M} \times \mathcal{M} \ni (X, Y) \mapsto X + Y,\ XY \in \mathcal{M},$$
$$\mathcal{M} \ni X \mapsto X^* \in \mathcal{M}$$

have unique continuous extensions as mappings of $\widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}$, $\widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}$, respectively, with which $\widetilde{\mathcal{M}}$ becomes a topological *-algebra (see [15]). For notational consistency, we denote by $L^0(\mathcal{M}, \tau)$ for $\widetilde{\mathcal{M}}$.

Then we have natural inclusions:

$$\mathcal{M} \equiv L^{\infty}(\mathcal{M},\tau) \subset L^{q}(\mathcal{M},\tau) \subset L^{p}(\mathcal{M},\tau) \subset \cdots \subset L^{0}(\mathcal{M},\tau) = \widetilde{\mathcal{M}}$$

for $1 \leq p \leq q < \infty$, where $L^p(\mathcal{M},\tau)$ is a Banach space of all elements in $L^0(\mathcal{M},\tau)$ satisfying

$$\|X\|_p = [\tau(|X|^p)]^{1/p} \left(= \left(\int_0^{\infty} [\mu_\lambda(X)]^p \, d\lambda \right)^{1/p} \right) < \infty, \tag{2.1}$$

where $\mu_\lambda(X)$ is the generalized singular number of X which is defined as in (3.1) (see [20, 15, 9]).

An element of $L^0(\mathcal{M},\tau)$ is called a *random variable* (or *τ-measurable operator*). A densely defined closed operator X in H is said to be *affiliated* with the von Neumann algebra $\mathcal{M} \subset \mathcal{B}(H)$ if U and the spectral projections of $|X|$ belong to $\mathcal{M}$, where $X = U|X|$ is the polar decomposition of X and $|X| = (X^*X)^{1/2}$. In fact, X is affiliated with the von Neumann algebra $\mathcal{M}$ if and only if $U^*XU = X$ for any unitary operator U commuting with $\mathcal{M}$. Note that $\widetilde{\mathcal{M}}$ is the set of all such operators X. For a set $\mathcal{S}$ of densely defined closed operators in H, $W^*(\mathcal{S})$ denotes the smallest von Neumann algebra to which each element of $\mathcal{S}$ is affiliated. For the case of $\mathcal{S} = \{X\}$ with a densely defined closed operator X, we write $W^*(X) \equiv W^*(\mathcal{S})$ for simple notation. If $\mathcal{S}$ consists of bounded operators on H, then $W^*(\mathcal{S}) = (\mathcal{S} \cup \mathcal{S}^*)''$ (double commutant), and so, if X is a self-adjoint operator, then $W^*(X)$ is a commutative von Neumann algebra.

A sequence $\{X_n\}$ of random variables in $\mathcal{M}$ (or $\widetilde{\mathcal{M}}$) is said to be *pairwise τ-independent* [13] (or simply, *pairwise independent*) if

$$\tau(XY) = \tau(X)\tau(Y), \qquad X \in W^*(X_i), \quad Y \in W^*(X_j)$$

for any pair (i,j) of distinct numbers.

For a sequence $\{X_n\} \subseteq L^0(\mathcal{M},\tau)$ and $X \in L^0(\mathcal{M},\tau)$, we say that X_n converges to X *in measure* if X_n converges to X in the measure topology, in this case, we write $X_n \xrightarrow{\text{m}} X$ as $n \to \infty$. We now recall a useful equivalent condition to the convergence in measure.

Theorem 2.1 ([24, 10]). *Let $\{X_n\}$ be a sequence in $L^0(\mathcal{M},\tau)$. The following conditions are equivalent:*

(i) $X_n \xrightarrow{\text{m}} 0$ *as* $n \to \infty$,
(ii) *for any* $\epsilon > 0$, $\tau\left(e_{|X_n|}([\epsilon,\infty))\right) \to 0$ *as* $n \to \infty$,

where $e_X(B)$ is the spectral projection of a self-adjoint operator X corresponding to the Borel subset B of $\mathbf{R}$.

3. WLLN for weighted sums in Lorentz space

In this section, we study the WLLN for weighted sums of random variables in noncommutative Lorentz spaces. Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space and X an element in $L^0(\mathcal{M}, \tau)$. For each $\lambda \geq 0$, we define the *generalized singular number* of X by

$$\mu_\lambda(X) = \inf\{u > 0 \mid \tau\left(e_{|X|}((u, \infty))\right) \leq \lambda\}. \tag{3.1}$$

For more detailed study of generalized singular numbers of τ-measurable operators in the sense of Nelson [15], we refer to [9].

Let f be a concave C^1-function from $[0, \infty)$ into itself such that $f(0) = 0$ and $f(\infty) = \infty$. For $X \in L^0(\mathcal{M}, \tau)$, put

$$\|X\|_{f,q} = \left(\int_0^\infty f'(\lambda)\,[\mu_\lambda(X)]^q\, d\lambda\right)^{1/q}, \qquad 1 \leq q < \infty,$$
$$\|X\|_{f,\infty} = \sup_{\lambda > 0}\left\{\frac{1}{f(\lambda)}\int_0^\lambda \mu_u(X) du\right\}, \qquad q = \infty.$$

For each $1 \leq q \leq \infty$, let $L_{f,q}(\mathcal{M}, \tau)$, called a *noncommutative Ciach space* (more generally, *noncommutative Lorentz space*), be the space of all random variables $X \in L^0(\mathcal{M}, \tau)$ with $\|X\|_{f,q} < \infty$. For notational convenience, for each $0 < \lambda < \infty$, we put

$$\|X\|_{f,q,\lambda} = \left(\int_0^\lambda f'(s)\,[\mu_s(X)]^q\, ds\right)^{1/q},$$
$$\|X\|_{f,\infty,\lambda} = \sup\left\{\frac{1}{f(u)}\int_0^u \mu_s(X) ds \,\middle|\, 0 < u \leq \lambda\right\}.$$

THEOREM 3.1 ([7]). *Let $\beta > 1/2$ be given. Let $\{X_i\}$ be a sequence of self-adjoint random variables in $L^0(\mathcal{M}, \tau)$ and $\{a_{ni} \mid 1 \leq i \leq n\}$ be self-adjoint elements of $\mathcal{M}$ such that $\max_{1 \leq i \leq n} \|a_{ni}\| = O(1/n^\beta)$. If $\{X_i\}$ and $\{a_{ni}\}$ satisfy the following conditions:*

(i) *for any $n \geq 1$, $\{a_{ni}X_i \mid 1 \leq i \leq n\}$ is a pairwise independent sequence,*

(ii) $\lim_{\lambda \to \infty} \lambda^{1/\beta}\left(\sup_i \tau\left(e_{|X_i|}([\lambda, \infty))\right)\right) = 0,$

then $\sum_{i=1}^n \left[a_{ni}X_i - \tau\left(a_{ni}X_i e_{|X_i|}([0, n^\beta))\right)\right] \xrightarrow{\text{m}} 0$ *as* $n \longrightarrow \infty$.

With above setting we have the following main theorem.

THEOREM 3.2. *Let $\beta > 1/2$ be given. Let $\{X_i\}$ be a sequence of self-adjoint random variables in $L_{f,q}(\mathcal{M},\tau)$. For each $n \geq 1$, let $\{a_{ni} \,|\, 1 \leq i \leq n\}$ be a finite sequence of self-adjoint elements of $\mathcal{M}$ such that $\max_{1\leq i\leq n} \|a_{ni}\| = O(1/n^{\beta})$ and $\{a_{ni}X_i \,|\, 1 \leq i \leq n\}$ are pairwise independent. If one of the following two conditions is satisfied:*

(i) *in case of $1 \leq q < \infty$, there exist $C_0, C_1 > 0$, $\alpha > \max\{1/\beta, q\}$ and $0 \leq r < 1$ such that $\left(r + \frac{1}{\alpha}\right) q \leq 1$ and*

$$M_{q,\lambda} \equiv \sup_i \|X_i\|_{f,q,\lambda} \leq C_0\lambda^r, \quad f(\lambda) \geq C_1\lambda^{\left(r+\frac{1}{\alpha}\right)q}, \quad \lambda > 0,$$

(ii) *in case of $q = \infty$, there exist $C_2, C_3 > 0$, $\alpha > \max\{1/\beta, 1\}$ and $0 \leq r < 1$ such that $r + \frac{1}{\alpha} \leq 1$ and*

$$M_{\infty,\lambda} \equiv \sup_i \|X_i\|_{f,\infty,\lambda} \leq C_2\lambda^r, \quad f(\lambda) \leq C_3\lambda^{1-\left(r+\frac{1}{\alpha}\right)}, \quad \lambda > 0,$$

then $\displaystyle\sum_{i=1}^{n} \left[a_{ni}X_i - \tau\left(a_{ni}X_i e_{|X_i|}([0, n^{\beta}))\right)\right] \xrightarrow{\text{m}} 0$ *as* $n \longrightarrow \infty$.

Proof. Since $\{X_i\} \subset L_{f,q}(\mathcal{M},\tau) \subset L^0(\mathcal{M},\tau)$, we only prove that $\{X_i\}$ satisfies the condition (ii) in Theorem 3.1. First, we consider the case of $1 \leq q < \infty$. Then for any $i \geq 1$ and $\lambda > 0$, since $\lambda \mapsto \mu_\lambda(X_i)$ is decreasing for each i, by condition (i) we obtain that

$$\begin{aligned}(3.2)\qquad C_0^q\lambda^{rq} \geq M_{q,\lambda}^q &\geq \int_0^\lambda f'(s)\,[\mu_s(X_i)]^q\,ds \geq [\mu_\lambda(X_i)]^q \int_0^\lambda f'(s)ds \\ &= f(\lambda)\,[\mu_\lambda(X_i)]^q \geq C_1\lambda^{\left(r+\frac{1}{\alpha}\right)q}\,[\mu_\lambda(X_i)]^q,\end{aligned}$$

where for the third inequality, we used the fact that $[\mu_s(X_i)]^q$ is decreasing in the variable s. In the case of $q = \infty$, for any $i \geq 1$ and $\lambda > 0$, by condition (ii) we obtain that

$$\begin{aligned}(3.3)\qquad C_2\lambda^r \geq M_{\infty,\lambda} &\geq \frac{1}{f(\lambda)}\int_0^\lambda \mu_s(X_i)ds \geq \frac{1}{f(\lambda)}\lambda\mu_\lambda(X_i) \\ &\geq \frac{1}{C_3}\lambda^{\left(r+\frac{1}{\alpha}\right)}\mu_\lambda(X_i).\end{aligned}$$

Therefore, by (3.2) and (3.3), for any $1 \leq q \leq \infty$ we have

$$(3.4)\qquad \mu_\lambda(X_i) \leq C\lambda^{-1/\alpha}, \quad i \geq 1, \quad \lambda > 0$$

for some constants $C > 0$. Then, by the definition of $\mu_\lambda(X_i)$, the inequality (3.4) implies that

$$\tau\left(e_{|X_i|}([\lambda,\infty))\right) \leq (C/\lambda)^{\alpha}, \quad i \geq 1, \quad \lambda > 0$$

(see [21]). Consequently, for any $1 \le q \le \infty$ we prove that

$$\lim_{\lambda\to\infty} \lambda^{1/\beta} \left(\sup_i \tau\left(e_{|X_i|}([\lambda,\infty))\right)\right) = 0$$

and so the proof follows from Theorem 3.1. □

Note that in Theorem 3.2, if for any $\lambda > 0$ we take $r = 0$, then $\{X_i\}$ are uniformly bounded, i.e., $\sup_i \|X_i\|_{f,q} < \infty$, $1 \le q \le \infty$. Therefore, we have the following Corollary.

COROLLARY 3.3. *Let $\beta > 1/2$ be given. Let $\{X_i\}$ be a sequence of self-adjoint random variables in $L_{f,q}(\mathcal{M},\tau)$. For each $n \ge 1$, let $\{a_{ni} \,|\, 1 \le i \le n\}$ be a finite sequence of self-adjoint elements of $\mathcal{M}$ such that $\max_{1\le i\le n} \|a_{ni}\| = O(1/n^{\beta})$ and $\{a_{ni}X_i \,|\, 1 \le i \le n\}$ are pairwise independent. If one of the following two conditions is satisfied:*

(i) *$\{X_i\}$ are uniformly bounded in $L_{f,q}(\mathcal{M},\tau)$ and there exist $C_1 > 0$ and $\alpha > \max\{1/\beta, q\}$ such that*

$$f(\lambda) \ge C_1 \lambda^{\frac{q}{\alpha}} \qquad \textit{for} \quad \lambda > 0, \tag{3.5}$$

(ii) *$\{X_i\}$ are uniformly bounded in $L_{f,\infty}(\mathcal{M},\tau)$ and there exist $C_2 > 0$ and $\alpha > \max\{1/\beta, 1\}$ such that*

$$f(\lambda) \le C_2 \lambda^{1-\frac{1}{\alpha}} \qquad \textit{for} \quad \lambda > 0,$$

then $\sum_{i=1}^{n} \left[a_{ni}X_i - \tau\left(a_{ni}X_i e_{|X_i|}([0,n^{\beta}))\right)\right] \xrightarrow{\ \mathrm{m}\ } 0$ *as* $n \longrightarrow \infty$.

REMARK 3.4. In Corollary 3.3, since the indefinite integrals of non-negative functions preserve inequality, the condition (3.5) is weaker than

$$f'(\lambda) \ge \lambda^{-1+\frac{q}{\alpha}} \qquad \text{for} \quad \lambda > 0,$$

which is the condition (i) in Theorem 4.1 in [7].

For each $1 \le p < \infty$ and $1 \le q \le \infty$, let $\mathcal{L}_{p,q}(\mathcal{M},\tau)$ be the space of all random variables $X \in L^0(\mathcal{M},\tau)$ such that $\|X\|_{p,q} < \infty$, where

$$\|X\|_{p,q} = \left(\int_0^{\infty} \lambda^{-1+\frac{q}{p}} [\mu_\lambda(X)]^q d\lambda\right)^{1/q}, \quad 1 \le q < \infty,$$

$$\|X\|_{p,\infty} = \sup_{\lambda>0} \left\{\lambda^{\frac{1}{p}} \mu_\lambda(X)\right\}, \quad q = \infty. \tag{3.6}$$

REMARK 3.5. If $p = q$, then the noncommutative Lorentz space $\mathcal{L}_{p,p}(\mathcal{M},\tau)$ coincides with the noncommutative L^p space (see Corollary 1.5 in [12]).

REMARK 3.6. Let $1 \le p < \infty$ and $\{X_i\}$ be a uniformly bounded sequence of self-adjoint random variables in $L^p(\mathcal{M}, \tau)$, i.e., $\sup_i \|X_i\|_p < \infty$. Then, for any $i \ge 1$ and $\lambda > 0$, we obtain that

$$C^p \ge \|X_i\|_p^p = \int_0^\infty [\mu_\lambda(X_i)]^p \, d\lambda \ge \int_0^\lambda [\mu_t(X_i)]^p \, dt \ge \lambda \, [\mu_\lambda(X_i)]^p \, ,$$

where $C \equiv \sup_i \|X_i\|_p < \infty$. Therefore, we have

$$\mu_\lambda(X_i) \le C\lambda^{-1/p}, \qquad i \ge 1, \quad \lambda > 0,$$

which implies that

$$\tau\left(e_{|X_i|}([\lambda, \infty))\right) \le (C/\lambda)^p, \qquad i \ge 1, \quad \lambda > 0.$$

Therefore, for given $\beta > 1/2$, if $p > 1/\beta$ then we have

$$\lim_{\lambda \to \infty} \lambda^{1/\beta} \left(\sup_i \tau\left(e_{|X_i|}([\lambda, \infty))\right) \right) = 0,$$

which is the condition (ii) in Theorem 3.1.

The following theorem generalizes Theorem 4.3 in [7] and Proposition 2 in [21].

THEOREM 3.7. *Let $\beta > 1/2$ be given. Let $\{X_i\}$ be a sequence of self-adjoint, uniformly bounded random variables in $\mathcal{L}_{p,q}(\mathcal{M}, \tau)$. For each $n \ge 1$, let $\{a_{ni} \,|\, 1 \le i \le n\}$ be a finite sequence of self-adjoint elements of $\mathcal{M}$ such that $\max_{1 \le i \le n} \|a_{ni}\| = O(1/n^\beta)$ and $\{a_{ni}X_i \,|\, 1 \le i \le n\}$ are pairwise independent. If $p > 1/\beta$, then*

$$\sum_{i=1}^{n} \left[a_{ni}X_i - \tau\left(a_{ni}X_i e_{|X_i|}([0, n^\beta)) \right) \right] \xrightarrow{\ \mathrm{m}\ } 0$$

as $n \longrightarrow \infty$.

Proof. We first consider the case $1 \le q \le p < \infty$. If $p \ne q$, put $f(\lambda) = (p/q)\lambda^{q/p}$, then the concave function f satisfies the condition (i) in Corollary 3.3, in fact, we can choose $\alpha = p$ and $C_1 = 1$ in Corollary 3.3. Therefore, the proof is immediate from Corollary 3.3. If $p = q$, then the proof follows from Remark 3.5, Remark 3.6 and Theorem 3.1.

Secondly, we consider the case $1 \le p < q \le \infty$. If $q < \infty$, then by direct computation, for $\lambda > 0$ we have

$$C_1^q = \sup_i \|X_i\|_{p,q}^q \ge \int_0^\lambda s^{-1+\frac{q}{p}} [\mu_s(X_i)]^q ds \ge \frac{p}{q} \lambda^{\frac{q}{p}} \, [\mu_\lambda(X_i)]^q \, ,$$

which implies that

$$\mu_\lambda(X_i) \le C_1 \left(\frac{q}{p}\right)^{1/q} \lambda^{-\frac{1}{p}}, \qquad \lambda > 0.$$

In the case of $q = \infty$, the equation (3.4) is immediate from (3.6). In fact, since

$$C_2 = \sup_i \|X_i\|_{p,\infty} \ge \lambda^{\frac{1}{p}} \mu_\lambda(X_i), \qquad \lambda > 0,$$

we obtain that

$$\mu_\lambda(X_i) \le C_2 \lambda^{-\frac{1}{p}}, \qquad \lambda > 0.$$

Therefore, the rest of the proof is same with the proof of Theorem 3.2. □

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