# BOUNDEDNESS AND COMPACTNESS OF SOME TOEPLITZ OPERATORS 

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#### Abstract

We consider the problem to determine when a Toeplitz operator is bounded on weighted Bergman spaces. We introduce some set $C G$ of symbols and we prove that Toeplitz operators induced by elements of $C G$ are bounded and characterize when Toeplitz operators are compact and show that each element of $C G$ is related with a Carleson measure.


## 1. Introduction

Let $d A$ denote normalized Lebesgue area measure on the unit disk $\mathbb{D}$. For $\alpha>-1$, the weighted Bergman space $A_{\alpha}^{p}$ consists of the analytic functions in $L^{p}\left(\mathbb{D}, d A_{\alpha}\right)$, where $d A_{\alpha}(z)=(\alpha+1)\left(1-|z|^{2}\right)^{\alpha} d A(z)$. Since $A_{\alpha}^{2}$ is a closed subspace of $L^{2}\left(\mathbb{D}, d A_{\alpha}\right)$, for any $z \in \mathbb{D}$, there is a unique function $K_{z}^{\alpha}$ in $A_{\alpha}^{2}$ such that $f(z)=<f, K_{z}^{\alpha}>$ for all $f \in A_{\alpha}^{2}$, in fact, $K_{z}^{\alpha}(w)=\frac{1}{(1-\bar{z} w)^{2+\alpha}}$ and the normalized reproducing kernel $k_{z}^{\alpha}$ is the function $\frac{K_{z}^{\alpha}(w)}{\left\|K_{z}^{\alpha}\right\|_{2, \alpha}}=\frac{\left(1-|z|^{2}\right)^{1+\frac{\alpha}{2}}}{(1-\bar{z} w)^{2+\alpha}}$, where the norm $\|\cdot\|_{p, \alpha}$ and the inner product are taken in the space $L^{p}\left(\mathbb{D}, d A_{\alpha}\right)$ and $L^{2}\left(\mathbb{D}, d A_{\alpha}\right)$, respectively.

For a linear operator $S$ on $A_{\alpha}^{2}, S$ induces a functions $\widetilde{S}$ on $\mathbb{D}$ given by $\widetilde{S}(z)=<S k_{z}^{\alpha}, k_{z}^{\alpha}>, z \in \mathbb{D}$. The function $\widetilde{S}$ is called the Berezin transform of $S$.

[^0]For $u \in L^{1}\left(\mathbb{D}, d A_{\alpha}\right)$, the Toeplitz operator $T_{u}^{\alpha}$ with symbol $u$ is the operator on $A_{\alpha}^{2}$ defined by $T_{u}^{\alpha}(f)=P_{\alpha}(u f), f \in A_{\alpha}^{2}$, where $P_{\alpha}$ is the orthogonal projection from $L^{2}\left(\mathbb{D}, d A_{\alpha}\right)$ onto $A_{\alpha}^{2}$ and let $\widetilde{u}$ denote $\widetilde{T_{u}^{\alpha}}$. Many mathematicians working in operator theory are interested in the boundedness and compactness of Toeplitz operators on the Bergman spaces. It is well-known that the Toeplitz operator $T_{u}^{\alpha}$ induced by any element of $L^{\infty}\left(\mathbb{D}, d A_{\alpha}\right)$ is bounded. Since $L^{\infty}\left(\mathbb{D}, d A_{\alpha}\right)$ is dense in $L^{1}\left(\mathbb{D}, d A_{\alpha}\right)$, for any $u \in L^{1}\left(\mathbb{D}, d A_{\alpha}\right), T_{u}^{\alpha}$ is densely defined on $A_{\alpha}^{2}$ but in general, $T_{u}^{\alpha}$ is not bounded. We note that Berezin transforms and Carleson measures are useful tools in the syudy of Toeplitz operators ([2], [4], [5]). Using those tools, many mathematicians working in the operator theory characterized the boundedness and compactness of Toeplitz operators.

In this paper, we introduce some set $C G$ and prove that Toeplitz operators induced by elements of $C G$ are bounded and $\|u\|_{G}$ having vanishing property implies the compactness of Toeplitz operators $T_{u}^{\alpha}$ and $T_{\bar{u}}^{\alpha}$.

Sections 3 contains some upper bounds of Toeplitz operators induced by elements of $C G$ and relationship between elements of $C G$ and Carleson measures and we deal with the compactness of appropriate products of Toeplitz operators and Hankel operators.

Throughout this paper, we use the symbol $A \preceq B$ for nonnegative constants $A$ and $B$ to indicate that $A$ is dominated by $B$ time some positive constant and $p^{\prime}$ to denote the conjugate of $p$, that is, $\frac{1}{p}+\frac{1}{p^{\prime}}=1$.

## 2. Some linear operators

A nice survey of previously known results connecting Toeplitz operators with bounded symbol can be found in [2].

For $z \in \mathbb{D}$, let $\varphi_{z}(w)=\frac{z-w}{1-\bar{z} w}$. Then $\varphi_{z}$ is an element of $A u t(\mathbb{D})$ which is the set of all bianalytic map of $\mathbb{D}$ onto $\mathbb{D}$. Moreover, $\varphi_{z} \circ \varphi_{z}$ is the identity map on $\mathbb{D}$ and $A u t(\mathbb{D})$ is the Möbius group under composition.

For $\alpha>-1$ and $z \in \mathbb{D}$, let $U_{z}^{\alpha}: L^{2}\left(\mathbb{D}, d A_{\alpha}\right) \rightarrow L^{2}\left(\mathbb{D}, d A_{\alpha}\right)$ be an isometry operator defined by

$$
U_{z}^{\alpha} f(w)=f \circ \varphi_{z}(w) \frac{\left(1-|z|^{2}\right)^{1+\frac{\alpha}{2}}}{(1-\bar{z} w)^{2+\alpha}}
$$

$f \in L^{2}\left(\mathbb{D}, d A_{\alpha}\right)$ and $w \in \mathbb{D}$.

Since $\left(1-\bar{z} \varphi_{z}(w)\right)^{2+\alpha}=\left(\frac{1-|z|^{2}}{1-\bar{z} w}\right)^{2+\alpha},\left(U_{z}^{\alpha}\right)^{-1}=U_{z}^{\alpha}$ and hence $U_{z}^{\alpha}$ is a self-adjoint unitary operator on $A_{\alpha}^{2}$ and $U_{z}^{\alpha} 1=k_{z}^{\alpha}(w)$.

For a linear operator $S$ on $A_{\alpha}^{2}$, define $S_{z}$ by $U_{z}^{\alpha} S U_{z}^{\alpha}$. Since $U_{z}^{\alpha}$ is a self-inverse operator, $S_{z}$ is the operator given by conjugation with $U_{z}^{\alpha}$.

Now we are ready to state useful properties.
Lemma 2.1. For $u \in L^{1}\left(\mathbb{D}, d A_{\alpha}\right)$ and $z \in \mathbb{D},\left(T_{u}^{\alpha}\right)_{z}=T_{u \circ \varphi_{z}}^{\alpha}$.
Proof. Take any $f$ in $A_{\alpha}^{2}$ and any $w$ in $\mathbb{D}$. Since $U_{z}^{\alpha}$ is self-adjiont,

$$
\begin{aligned}
U_{z}^{\alpha} T_{u}^{\alpha}(f)(w) & =<U_{z}^{\alpha} T_{u}^{\alpha}(f), K_{w}^{\alpha}> \\
& =<U_{z}^{\alpha}(u f), K_{w}^{\alpha}> \\
& =<\left(u \circ \varphi_{z}\right)\left(f \circ \varphi_{z}\right) \frac{\left(1-|z|^{2}\right)^{1+\frac{\alpha}{2}}}{(1-\bar{z} w)^{2+\alpha}}, K_{w}^{\alpha}> \\
& =<T_{u \circ \varphi_{z}}^{\alpha}\left(U_{z}^{\alpha} f\right), K_{w}^{\alpha}> \\
& =T_{u \circ \varphi_{z}}^{\alpha}\left(U_{z}^{\alpha} f\right)(w)
\end{aligned}
$$

Thus $\left(T_{u}^{\alpha}\right)_{z}=T_{u \circ \varphi_{z}}^{\alpha}$.
Corollary 2.2. For $u_{1}, u_{2}, \cdots, u_{n} \in L^{1}\left(\mathbb{D}, d A_{\alpha}\right)$ and $z \in \mathbb{D}$,

$$
U_{z}^{\alpha} T_{u_{1}}^{\alpha} T_{u_{2}}^{\alpha} \cdots T_{u_{n}}^{\alpha} U_{z}^{\alpha}=T_{u_{1} \circ \varphi_{z}}^{\alpha} \cdots T_{u_{n} \circ \varphi_{z}}^{\alpha}
$$

Proof. If follows immediately from the fact that $\left(U_{z}^{\alpha}\right)^{-1}=U_{z}^{\alpha}$ and Lemma 2.1.

Proposition 2.3. For $u \in L^{1}\left(\mathbb{D}, d A_{\alpha}\right)$ and $z \in \mathbb{D}, \widetilde{T_{u \circ \varphi_{z}}^{\alpha}}=\widetilde{T_{u}^{\alpha}} \circ \varphi_{z}$ and hence $\left(\widetilde{T_{u}^{\alpha}}\right)_{z}=\widetilde{T_{u \circ \varphi_{z}}^{\alpha}}=\widetilde{T_{u}^{\alpha}} \circ \varphi_{z}$.

Proof. Take any $w$ in $\mathbb{D}$. Since $<u \circ \varphi_{z} k_{w}^{\alpha}, k_{w}^{\alpha}>=<u k_{\varphi_{z}(w)}^{\alpha}, k_{\varphi_{z}(w)}^{\alpha}>$,

$$
\begin{aligned}
\widetilde{T_{u \circ \varphi_{z}}^{\alpha}}(w) & =<T_{u \circ \varphi_{z}}^{\alpha} k_{w}^{\alpha}, k_{w}^{\alpha}> \\
& =<u \circ \varphi_{z} k_{w}^{\alpha}, k_{w}^{\alpha}> \\
& =<u k_{\varphi_{z}(w)}^{\alpha}, k_{\varphi_{z}(w)}^{\alpha}> \\
& =<P_{\alpha}\left(u k_{\varphi_{z}(w)}^{\alpha}\right), k_{\varphi_{z}(w)}^{\alpha}> \\
& =\widetilde{T_{u}^{\alpha}}\left(\varphi_{z}(w)\right) \\
& =\widetilde{T_{u}^{\alpha}} \circ \varphi_{z}(w)
\end{aligned}
$$

This completes the proof.
Proposition 2.4. If $S: A_{\alpha}^{2} \rightarrow A_{\alpha}^{2}$ is a bounded linear operator then $\widetilde{S}$ and $S_{z} 1$ are in $L^{2}\left(\mathbb{D}, d A_{\alpha}\right)$.

Proof. Since $\left\|S_{z} 1\right\|_{2, \alpha}=\left\|S U_{z}^{\alpha} 1\right\|_{2, \alpha} \leq\|S\|$ and

$$
\|\widetilde{S}\|_{2, \alpha}=\int_{\mathbb{D}}|\widetilde{S}(z)|^{2} d A_{\alpha}(z) \leq \int_{\mathbb{D}}\|S\|^{2} d A_{\alpha}(z)=\|S\|^{2},
$$

$\widetilde{S}$ and $S_{z} 1$ are in $L^{2}\left(\mathbb{D}, d A_{\alpha}\right)$.
We notice that $P_{\alpha}: L^{2}\left(\mathbb{D}, d A_{\alpha}\right) \rightarrow A_{\alpha}^{2}$ is bounded linear operator and hence for any $u \in L^{\infty}\left(\mathbb{D}, d A_{\alpha}\right),\left\|P_{\alpha}(u f)\right\|_{2, \alpha} \leq\|u\|_{\infty}\|f\|_{2, \alpha}$. Thus $T_{u}^{\alpha}$ is a bounded linear operator. Moreover, we extend the domain of $P_{\alpha}$ to $L^{1}\left(\mathbb{D}, d A_{\alpha}\right)$ and for $f \in A_{\alpha}^{1}$ and $z \in \mathbb{D}, f(z)=\int_{\mathbb{D}} f(w) \overline{K_{z}^{\alpha}(w)} d A_{\alpha}(w)$.

We define $f(z)=\sum_{k=1}^{\infty} k \chi_{\left(\frac{1}{2^{k}}-\frac{1}{2^{k+1}}, \frac{1}{2^{k}}\right)}(|z|)$ for all $z \in \mathbb{D}$. Then $f$ is a radial function and $f \notin L^{\infty}\left(\mathbb{D}, d A_{\alpha}\right)$. Since

$$
\begin{aligned}
\|f\|_{1, \alpha} & =\int_{\mathbb{D}}|f(z)|\left(1-|z|^{2}\right)^{\alpha} d A(z) \\
& \leq\left\{\begin{array}{ll}
\left(1-\frac{1}{4}\right)^{\alpha} \sum_{k=1}^{\infty} \frac{k}{2^{k+1}}, & , \alpha<0 \\
\sum_{k=1}^{\infty} \frac{k}{2^{k+1}} & , \alpha \geq 0
\end{array}, f \in L^{1}\left(\mathbb{D}, d A_{\alpha}\right) .\right.
\end{aligned}
$$

For $p>2$,

$$
\left\|\left(T_{f}^{\alpha}\right)_{z} 1\right\|\left\|_{p, \alpha}=\right\| U_{z}^{\alpha} T_{f}^{\alpha} U_{z}^{\alpha} 1\left\|_{p, \alpha} \leq\right\| f k_{z}^{\alpha} \|_{p, \alpha}<\infty
$$

because $\sup \left\{\left|k_{z}^{\alpha}(w)\right|:|w| \leq \frac{1}{2}\right\} \leq 2^{2+\alpha}$. Since for each $z \in \mathbb{D}$,

$$
\left.\widetilde{|f|}\left|(z)=\int_{\mathbb{D}}\right| k_{z}^{\alpha}(w)\right|^{2}|f(w)| d A_{\alpha}(w) \leq 2^{4+2 \alpha} c \sum_{k=1}^{\infty} \frac{k}{2^{k+1}}
$$

for some constant $c,|f| d A_{\alpha}$ is a Carleson measure and hence $T_{f}^{\alpha}$ is a bounded linear operator. But every element of $L^{1}\left(\mathbb{D}, d A_{\alpha}\right)$ does not imply a bounded Toeplitz operator. Let $C G=\left\{u \in L^{1}\left(\mathbb{D}, d A_{\alpha}\right)\right.$ : $\sup _{z}\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{p, \alpha}<\infty$ and $\sup _{z}\left\|\left(T_{u}^{\alpha *}\right)_{z} 1\right\|_{p, \alpha}<\infty$ for some $\left.p \in(2, \infty)\right\}$. Suppose $f, g \in A_{\alpha}^{2}$. Since $<T_{u}^{\alpha} f, g>=<u f, g>=<f, \bar{u} g>=<$ $f, T_{\bar{u}}^{\alpha} g>,\left(T_{u}^{\alpha}\right)^{*}=T_{\bar{u}}^{\alpha}$. If $\left\|\left(T_{u}\right)_{z} 1\right\|_{p, \alpha}<\infty$ then $\left\|\left(T_{u}^{\alpha *}\right)_{z} 1\right\|_{p, \alpha}=\left\|\left(T_{\bar{u}}^{\alpha}\right)_{z} 1\right\|_{p, \alpha}$ $<\infty$ and clearly $C G$ is closed under the formation of conjugation and hence $\left\{T_{u}^{\alpha}: u \in C G\right\}$ is self-adjoint in $\mathcal{L}\left(A_{\alpha}^{2}\right)$ which is the set of all bounded linear operators on $A_{\alpha}^{2}$. Moreover, $C G$ is a vector space over $\mathbb{C}$ and we definde $\|u\|_{G}=\max \left\{\sup _{z}\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{p, \alpha}, \sup _{z}\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{p, \alpha}\right\}$.

By the above observation, $L^{\infty}\left(\mathbb{D}, d A_{\alpha}\right)$ is a proper subset of $C G$. Since $f(z)=0$ for all $|z|>\frac{1}{2}, \lim _{z \rightarrow \partial \mathbb{D}} \widetilde{T_{f}^{\alpha}}(z)=0=\lim _{z \rightarrow \partial \mathbb{D}}\left\|\left(T_{f}^{\alpha}\right)_{z} 1\right\|_{p, \alpha}$. Since $T_{f}^{\alpha}\left(z^{n}\right) \neq 0$ for all $n \in \mathbb{N}$, $T_{f}^{\alpha}$ has an infinite-dimensional range and hence it is not compact, that is, the vanishing property does not imply the compactness of Toeplitz operators.

## 3. Some operators

This section contains the boundedness of some operators. We begin by starting well-known lemma (see Lemma 3.10 in [5]) which is some integral estimates.

Lemma 3.1. Suppose $a-1<\alpha$. If $a+b<2+\alpha$ then
$\int_{\mathbb{D}} \frac{d A_{\alpha}(w)}{\left(1-|w|^{2}\right)^{a}|1-\bar{z} w|^{b}}$ is bounded on $\mathbb{D}$.
Note that $\left(T_{u}^{\alpha}\right)^{*}=T_{\bar{u}}^{\alpha}$. Thus for $z \in \mathbb{D}$,

$$
\left(T_{u}^{\alpha}\right)^{*} K_{w}^{\alpha}(z)=<\left(T_{u}^{\alpha}\right)^{*} K_{w}^{\alpha}, K_{z}^{\alpha}>=<K_{w}^{\alpha}, T_{u}^{\alpha} K_{z}^{\alpha}>=\overline{T_{u}^{\alpha} K_{z}^{\alpha}(w)} .
$$

Moreover, $\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{t, \alpha}$ in the right side of the next lemma may not be finite but it will be infinite, making the corresponding inequality true.

Lemma 3.2. Suppose $u \in L^{1}\left(\mathbb{D}, d A_{\alpha}\right)$ and $0<a<1$. If $2<\frac{2+\alpha}{a}<t$ then there is a constant $c$ such that

$$
\int_{\mathbb{D}} \frac{\left|\left(T_{u}^{\alpha} K_{z}^{\alpha}\right)(w)\right|}{\left(1-|w|^{2}\right)^{a}} d A_{\alpha}(w) \leq \frac{c| |\left(T_{u}^{\alpha}\right)_{z} 1| |_{t, \alpha}}{\left(1-|z|^{2}\right)^{a}}
$$

for all $z \in \mathbb{D}$ and

$$
\int_{\mathbb{D}} \frac{\left|\left(T_{u}^{\alpha} K_{z}^{\alpha}\right)(w)\right|}{\left(1-|z|^{2}\right)^{a}} d A_{\alpha}(z) \leq \frac{\left.c| |\left(T_{u}^{\alpha}\right)_{w} 1\right|_{t, \alpha}}{\left(1-|w|^{2}\right)^{a}}
$$

for all $w \in \mathbb{D}$.
Proof. Take any $z$ in $\mathbb{D}$. Since $U_{z}^{\alpha} 1=k_{z}^{\alpha}, T_{u}^{\alpha} K_{z}^{\alpha}=\frac{\left(T_{u}^{\alpha}\right)_{z} 1 \circ \varphi_{z}\left(\varphi_{z}^{\prime}\right)^{1+\frac{\alpha}{2}}}{\left(1-|z|^{2}\right)^{1+\frac{\alpha}{2}}}$ and hence put $w=\varphi_{z}(\lambda)$ to obtain the following :

$$
\begin{aligned}
& \int_{\mathbb{D}} \frac{\left|T_{u}^{\alpha} K_{z}^{\alpha}(w)\right|}{\left(1-|w|^{2}\right)^{a}} d A_{\alpha}(w) \\
& =\int_{\mathbb{D}} \frac{\left|\left(T_{u}^{\alpha}\right)_{z} 1(\lambda)\right|\left|\varphi_{z}^{\prime}\left(\varphi_{z}(\lambda)\right)\right|^{1+\frac{\alpha}{2}}}{\left(1-|z|^{2}\right)^{1+\frac{\alpha}{2}}\left(1-\left|\varphi_{z}(\lambda)\right|^{2}\right)^{a}}\left|\varphi_{z}^{\prime}(\lambda)\right|^{2}\left(1-\left|\varphi_{z}(\lambda)\right|^{2}\right)^{\alpha} d A(\lambda) \\
& =\frac{1}{\left(1-|z|^{2}\right)^{a}} \int_{\mathbb{D}} \frac{\left|\left(T_{u}^{\alpha}\right)_{z} 1(\lambda)\right|}{|1-\bar{z} \lambda|^{2-2 a+\alpha}\left(1-|\lambda|^{2}\right)^{a-\alpha}} d A(\lambda) \\
& \leq \frac{\|\left.\left(T_{u}^{\alpha}\right)_{z} 1\right|_{t, \alpha}}{\left(1-|z|^{2}\right)^{a}}\left(\int_{\mathbb{D}} \frac{d A_{\alpha}(\lambda)}{\left(1-|\lambda|^{2}\right)^{a t^{\prime}}|1-\bar{z} \lambda|^{(2-2 a+\alpha) t^{\prime}}}\right)^{\frac{1}{t^{\prime}}}
\end{aligned}
$$

Here, the inequality comes from Hölder's inequality. If $(2-a+\alpha) t^{\prime}-\alpha<2$ then the final integral is finite. Since $\frac{2+\alpha}{a}<t$, $t^{\prime}<\frac{2+\alpha}{2-a+\alpha}$. This makes the corresponding inequality true. The second inequality follows from the above observation.

Corollary 3.3. Suppose $0<a<1$ and $\|u\|_{G}$ is finite with respect to $\|\cdot\|_{p, \alpha}$ for some $p \in(2, \infty)$, that is, $u \in C G$. If $2<\frac{2+\alpha}{a}<p$ then there is a constant $c$ such that

$$
\int_{\mathbb{D}} \frac{\left|\left(T_{u}^{\alpha} K_{z}^{\alpha}\right)(w)\right|}{\left(1-|w|^{2}\right)^{a}} d A_{\alpha}(w) \leq \frac{c\left\|\left(T_{u}^{\alpha}\right)_{z}\right\|_{p, \alpha}}{\left(1-|z|^{2}\right)^{a}} \preceq \frac{\|u\|_{G}}{\left(1-|z|^{2}\right)^{a}}
$$

for all $z \in \mathbb{D}$ and

$$
\int_{\mathbb{D}} \frac{\left|\left(T_{u}^{\alpha} K_{z}^{\alpha}\right)(w)\right|}{\left(1-|z|^{2}\right)^{a}} d A_{\alpha}(z) \leq \frac{c\left\|\left(T_{u}^{\alpha}\right)_{w}\right\|_{p, \alpha}}{\left(1-|w|^{2}\right)^{a}} \preceq \frac{\|u\|_{G}}{\left(1-|w|^{2}\right)^{a}}
$$

for all $w \in \mathbb{D}$.
Proof. If follows immediately from the definition of $\|u\|_{G}$ and Lemma 3.2.

Proposition 3.4. If $u \in C G$ and $\|u\|_{G}$ is finite with respect to $\|\cdot\|_{t, \alpha}$ then $\left|T_{\bar{u}}^{\alpha}(h)(w)\right| \leq \frac{1}{\left(1-|w|^{2}\right)^{1+\frac{\alpha}{2}}}\|h\|_{2, \alpha}\|u\|_{t, \alpha}$ for every $h \in A_{\alpha}^{2}$ and every $w \in \mathbb{D}$.

Proof. Suppose $h \in A_{\alpha}^{2}$ and $w \in \mathbb{D}$. Then

$$
\left(T_{\bar{u}}^{\alpha} h\right)(w)=<T_{\bar{u}}^{\alpha} h, K_{w}^{\alpha}>=\frac{1}{\left(1-|w|^{2}\right)^{1+\frac{\alpha}{2}}} \times<h, \bar{u} k_{w}^{\alpha}>
$$

By Hölder's inequality, we get $\left|<h, \bar{u} k_{w}^{\alpha}>\right| \leq\|h\|_{t^{\prime}, \alpha}\left\|\bar{u} k_{w}^{\alpha}\right\|_{t, \alpha}$. Since $1<t^{\prime}<2$ and $A_{\alpha}(\mathbb{D})=1$, $\|h\|_{t^{\prime}, \alpha} \leq\|h\|_{t, \alpha}$ and hence one has the result.

Suppose $f \in A_{\alpha}^{2}$ and $z \in \mathbb{D}$. Then

$$
\begin{aligned}
\left(T_{u}^{\alpha} f\right)(z) & =<T_{u}^{\alpha} f, K_{z}^{\alpha}> \\
& =\int_{\mathbb{D}} f(w) \overline{\left(\left(T_{u}^{\alpha}\right)^{*} K_{z}^{\alpha}\right)(w)} d A_{\alpha}(w) \\
& =\int_{\mathbb{D}} f(w) T_{u}^{\alpha} K_{w}^{\alpha}(z) d A_{\alpha}(w)
\end{aligned}
$$

Thus $T_{u}^{\alpha}$ is the integral operator with kernel $T_{u}^{\alpha} K_{w}^{\alpha}(z)$ and hence we find some upper bound of $\left\|T_{u}^{\alpha}\right\|_{p}$ to use the Schur test (see page 126 of [3]), where $\left\|T_{u}^{\alpha}\right\|_{p}$ is the operator norm on $A_{\alpha}^{p}$.

Theorem 3.5. Suppose $u \in C G$ and $\|u\|_{G}$ is finite with respect to $\|\cdot\|_{p, \alpha}$. If $p p^{\prime}(2+\alpha)<t$ then $T_{u}^{\alpha}$ is a bounded linear operator on $A_{\alpha}^{p}$ and $A_{\alpha}^{p^{\prime}}$ and $\left\|T_{u}^{\alpha}\right\|_{p} \preceq\|u\|_{G}$.

Proof. Since $0<\frac{1}{p p^{\prime}}<1$, let $h(\lambda)=\frac{1}{\left(1-|\lambda|^{2}\right)^{p p^{\prime}}}$. Then $h$ is a positive measurable function. Since $\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{t, \alpha}$ and $\left\|\left(T_{\bar{u}}^{\alpha}\right)_{z} 1\right\|_{t, \alpha}$ are less than or equal to $\|u\|_{G}$, the results follow from Lemma 3.2 and the Schur test.

Using the concept of a Carleson measure, we get the boundness and compactness of Toeplitz operators.

Proposition 3.6. Suppose $u \in C G$ and $\|u\|_{G}$ is finite with respect to $\|\cdot\|_{t, \alpha}$.
(1) Then $|u| d A_{\alpha}$ is a Carleson measure on $A_{\alpha}^{p}$ and hence $T_{u}^{\alpha}$ is a bounded linear operator.
(2) If $\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{t, \alpha} \rightarrow 0$ as $z \rightarrow \partial \mathbb{D}$ then $T_{u}^{\alpha}$ is compact.

Proof. (1) For $z \in \mathbb{D},|\widetilde{u}(z)|=\left|<T_{u}^{\alpha} k_{z}^{\alpha}, k_{z}^{\alpha}>\right|$

$$
\begin{aligned}
& =\left(1-|z|^{2}\right)^{1+\frac{\tilde{\alpha}}{2}}\left|<T_{u}^{\alpha} K_{z}^{\alpha}, k_{z}^{\alpha}>\right| \\
& \leq\left(1-|z|^{2}\right)^{1+\frac{\alpha}{2}}\left\|T_{u}^{\alpha} K_{z}^{\alpha}\right\|_{2, \alpha} \\
& =\left(1-|z|^{2}\right)^{1+\frac{\alpha}{2}}\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{2, \alpha} \\
& \leq\left(1-|z|^{2}\right)^{1+\frac{\alpha}{2}}\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{t, \alpha},
\end{aligned}
$$

where the last inequality follows from $A_{\alpha}(\mathbb{D})=1$.
Since $\widetilde{u}$ is bounded, $|u| d A_{\alpha}$ is a Carleson measure on $A_{\alpha}^{p}$.
(2) In the proof of $(1)$, for $z \in \mathbb{D},|\widetilde{u}(z)| \leq\left(1-|z|^{2}\right)^{1+\frac{\alpha}{2}}\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{t, \alpha}$ and hence $|u| d A_{\alpha}$ is a vanishing Carleson measure. Thus $T_{u}^{\alpha}$ is a compact linear operator.

Corollary 3.7. Suppose $u \in C G$ and $\|u\|_{G}$ is finite with respect to $\|\cdot\|_{p, \alpha}$. If $\|u\|_{G}$ vanishes on $\partial \mathbb{D}$ then $T_{u}^{\alpha}$ and $T_{\bar{u}}^{\alpha}$ are compact opeators.

Proof. It follows immediately from the fact that $\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{t, \alpha}$ and $\left\|\left(T_{\bar{u}}^{\alpha}\right)_{z} 1\right\|_{t, \alpha}$ are less than or equal to $\|u\|_{G}$.

Proposition 3.8. Suppose $u \in C G$ and $\|u\|_{G}$ is finite with respect to $\|\cdot\|_{t, \alpha}$. If $T_{u}^{\alpha}$ is a compact operator then $\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{2, \alpha} \rightarrow 0$ as $z \rightarrow \partial \mathbb{D}$ and hence $\widetilde{u}$ has the vanishing property on $\partial \mathbb{D}$. Moreover, $\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{t, \alpha} \rightarrow$ 0 as $z \rightarrow \partial \mathbb{D}$.

Proof. We note that $H^{\infty}$ is dense in $A_{\alpha}^{2}$. Take any $f$ in $A_{\alpha}^{2}$. Then $<$ $f, k_{z}^{\alpha}>=\left(1-|z|^{2}\right)^{1+\frac{\alpha}{2}} f(z)$ and hence $k_{z}^{\alpha} \rightarrow 0$ weakly in $A_{\alpha}^{2}$ as $z \rightarrow \partial \mathbb{D}$. Since $\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{2, \alpha}=\left\|T_{u}^{\alpha} k_{z}^{\alpha}\right\|_{2, \alpha}$ and $T_{u}^{\alpha}$ is compact, $\left\|\left(T_{u}^{\alpha}\right)_{z} 1\right\|_{2, \alpha} \rightarrow 0$ as $z \rightarrow \partial \mathbb{D}$.

For $u \in L^{1}\left(\mathbb{D}, d A_{\alpha}\right)$, we define an operator $H_{u}^{\alpha}: A_{\alpha}^{2} \rightarrow\left(A_{\alpha}^{2}\right)^{\perp}$ by $H_{u}^{\alpha}(g)=\left(I-P_{\alpha}\right)(u g), g \in A_{\alpha}^{2}$. Then $H_{u}^{\alpha}$ is called the Hankel operator on the weighted Bergman space with symbol $u$. Since $L^{\infty}\left(\mathbb{D}, d A_{\alpha}\right)$ is dense in $L^{1}\left(\mathbb{D}, d A_{\alpha}\right), H_{u}^{\alpha}$ is densely defined and if $u \in$ $L^{\infty}\left(\mathbb{D}, d A_{\alpha}\right)$ then $\left\|H_{u}^{\alpha}\right\| \leq\|u\|_{\infty}$ and hence $H_{u}^{\alpha}$ is bounded. By Lemma 2.1, $\left(T_{u}^{\alpha}\right)_{z}=T_{u o \varphi_{z}}^{\alpha}$ and hence $\left\|\left(H_{u}^{\alpha}\right)_{z} 1\right\|_{2, \alpha}=\left\|H_{u}^{\alpha} k_{z}^{\alpha}\right\|_{2, \alpha} \leq\left\|H_{u}^{\alpha}\right\|$ and $\left(H_{u}^{\alpha}\right)_{z}=\left(I-T_{u}^{\alpha}\right)_{z}=I-T_{u \circ \varphi_{z}}^{\alpha}=H_{u \circ \varphi_{z}}^{\alpha}$. Thus one has the following properties :

Proposition 3.9. Suppose $u_{1}, u_{2} \in L^{1}\left(\mathbb{D}, d A_{\alpha}\right)$ and $u_{1}=u_{2} \circ \varphi_{z}$ for some $z \in \mathbb{D}$. Then the following pairs are unitary equivalent :
(1) $T_{u_{1}}^{\alpha}$ and $T_{u_{2}}^{\alpha}$
(2) $H_{u_{1}}^{\alpha}$ and $H_{u_{2}}^{\alpha}$.

Proposition 3.10. Suppose $H_{u}^{\alpha}$ is bounded, where $u \in L^{1}\left(\mathbb{D}, d A_{\alpha}\right)$. Then $\left(H_{u}^{\alpha}\right)_{z} 1$ and $H_{u}^{\alpha} k_{z}^{\alpha}$ are in $L^{2}\left(\mathbb{D}, d A_{\alpha}\right)$ and $H_{u \circ \varphi_{z}}^{\alpha}$ is bounded.

Proof. By the above observation, $\left(H_{u}^{\alpha}\right)_{z} 1$ and $H_{u}^{\alpha} k_{z}^{\alpha}$ are in $L^{2}\left(\mathbb{D}, d A_{\alpha}\right)$. Take any $f$ in $A_{\alpha}^{2}$. Since $\left(H_{u}^{\alpha}\right)_{z}=H_{u \circ \varphi_{z}}^{\alpha},\left\|H_{u \varphi_{z}}^{\alpha}(f)\right\|_{2, \alpha}=\left\|\left(H_{u}^{\alpha}\right)_{z} f\right\|_{2, \alpha}$ $=\left\|H_{u}^{\alpha} U_{z}^{\alpha}(f)\right\|_{2, \alpha} \leq\left\|H_{u}^{\alpha}\left|\|| | f\|_{2, \alpha}\right.\right.$. This completes the proof.

Proposition 3.11. If $u^{2} \in C G$ then $H_{u}^{\alpha}$ is bounded and hence we get the results of Proposition 3.10.

Proof. Take any $f$ in $A_{\alpha}^{2}$. By Proposition 3.6, $|u|^{2} d A_{\alpha}$ is a Carleson measure on $A_{\alpha}^{2}$ and hence there is a constant $c$ such that

$$
\int_{\mathbb{D}}|f(z)|^{2}|u(z)|^{2} d A_{\alpha}(z) \leq c\|f\|_{2, \alpha}^{2} .
$$

Then $\left\|H_{u}^{\alpha}(f)\right\|_{2, \alpha}^{2}=\left\|\left(I-P_{\alpha}\right)(u f)\right\|_{2, \alpha}^{2} \leq\|u f\|_{2, \alpha}^{2} \leq c\|f\|_{2, \alpha}^{2}$. Thus $H_{u}^{\alpha}$ is bounded.

Consider some products of Toeplitz operators and Hankel operators. Suppos $u, v \in L^{1}\left(\mathbb{D}, d A_{\alpha}\right)$ and $f, g \in A_{\alpha}^{2}$. Since $<v f, P_{\alpha}(u g)>$ $=<P_{\alpha}(v f), P_{\alpha}(u g)>$ and $<\bar{u} T_{v}^{\alpha}(f), g>=<T_{\bar{u}}^{\alpha} T_{v}^{\alpha}(f), g>,<$ $\left(H_{u}^{\alpha}\right)^{*} H_{v}^{\alpha}(f), g>=<\bar{u} v f, g>-<T_{v}^{\alpha}(f), u g>-<v f, P_{\alpha}(u g)>$ $+<T_{\bar{u}}^{\alpha} T_{v}^{\alpha}(f), g>=<\left(T_{\bar{u} v}^{\alpha}-T_{\bar{u}}^{\alpha} T_{v}^{\alpha}\right)(f), g>$ and hence $\left(H_{u}^{\alpha}\right)^{*} H_{v}^{\alpha}=$ $T_{\bar{u} v}^{\alpha}-T_{\bar{u}}^{\alpha} T_{v}^{\alpha}$. In particular, if $u=v$ then $\left(H_{u}^{\alpha}\right)^{*} H_{u}^{\alpha}=T_{|u|^{2}}^{\alpha}-T_{\bar{u}}^{\alpha} T_{u}^{\alpha}$. If $H_{u}^{\alpha}$ is compact then $\left(H_{u}^{\alpha}\right)^{*} H_{u}^{\alpha}$ is compact. Proposition 3.8 implies that $\left(\left(H_{u}^{\alpha}\right)^{*} H_{u}^{\alpha}\right)^{\sim}(z) \rightarrow 0$ as $z \rightarrow \partial \mathbb{D}$ and hence $\left\|H_{u} k_{z}^{\alpha}\right\|_{2, \alpha} \rightarrow 0$ as $z \rightarrow \partial \mathbb{D}$ because $\left\|H_{u}^{\alpha} k_{z}^{\alpha}\right\|_{2, \alpha}^{2}=<H_{u}^{\alpha} k_{z}^{\alpha}, H_{u}^{\alpha} k_{z}^{\alpha}>=\left(\left(H_{u}^{\alpha}\right)^{*} H_{u}^{\alpha}\right)^{\sim}(z)$.
Suppose $u, v, u^{2}, v^{2}$ are in $C G$ and $T_{u}^{\alpha}$ and $H_{u}^{\alpha}$ are compact. Then $\left(T_{u}^{\alpha}\right)^{*}$ and $\left(H_{u}^{\alpha}\right)^{*}$ are also compact. Since $U_{z}^{\alpha}$ is a bounded linear operator and $\left(T_{u}^{\alpha}\right)_{z}=U_{z}^{\alpha} T_{u}^{\alpha} U_{z}^{\alpha}$, the above equality implies that the following are compact :
(1) $T_{u}^{\alpha} T_{v}^{\alpha}$
(2) $T_{u}^{\alpha} T_{\bar{v}}^{\alpha}$
(3) $T_{\bar{u}}^{\alpha} T_{u}^{\alpha}$
(4) $\left(H_{u}^{\alpha}\right)^{*} H_{v}^{\alpha}$
(5) $H_{u}^{\alpha}\left(H_{v}^{\alpha}\right)^{*}$
(6) $T_{\bar{u} v}^{\alpha}$
(7) $T_{u \bar{v}}^{\alpha}$
(8) $T_{|u|^{2}}^{\alpha}$
(9) $H_{u}^{\alpha} T_{u}^{\alpha}$
(10) $H_{u}^{\alpha} T_{\bar{u}}^{\alpha}$
(11) $H_{v}^{\alpha} T_{u}^{\alpha}$
(12) $T_{u \circ \varphi_{z}}^{\alpha} T_{v \circ \varphi_{z}}^{\alpha}$
(13) $T_{u \circ \varphi_{z}}^{\alpha} T_{v \circ \varphi_{z}}^{\alpha}$
(14) $H_{u \circ \varphi_{z}}^{\alpha}\left(H_{v}^{\alpha}\right)^{*}$
(15) $\left(H_{u}^{\alpha}\right)^{*} H_{v \circ \varphi_{z}}^{\alpha}$.

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