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VAGUE p-IDEALS AND VAGUE a-IDEALS IN BCI-ALGEBRAS

Yun Sun Hwang* and Sun Shin Ahn**

Abstract. The notion of vague p-ideals and vague a-ideals of BCI-algebras is introduced, and several properties of them are investigated. We show that a vague set of a BCI-algebra is a vague a-ideal if and only if it is both a vague q-ideal and a vague p-ideal.

1. Introduction

Several authors from time to time have made a number of generalizations of Zadeh's fuzzy set theory [11]. Of these, the notion of vague set theory introduced by Gau and Buehrer [3] is of interest to us. Using the vague set in the sense of Gau and Buehrer, Biswas [2] studied vague groups. Jun and Park [6,10] studied vague ideals and vague deductive systems in subtraction algebras. In [8], the concept of vague BCK/BCI-algebras is discussed. S. S. Ahn, Y. U. Cho and C. H. Park [1] studied vague quick ideals of BCK/BCI-algebras. Y. B. Jun and K. J. Lee ([7]) introduced the notion of positive implicative vague ideals in BCK-algebras. They established relations between a vague ideal and a positive implicative ideals. In [5], the notion of vague q-ideal of BCI-algebras was introduced and several properties of them were investigated.

In this paper, we also use the notion of vague set in the sense of Gau and Buehrer to discuss the vague theory in BCI-algebras. We introduce the notion of vague p-ideal and q-ideal of BCI-algebras and investigate several properties of them. We show that a vague set of a BCI-algebra is a vague a-ideal if and only if it is both a vague q-ideal and a vague p-ideal.

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Correspondence should be addressed to Sun Shin Ahn, sunshine@dongguk.edu.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(a1) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(a2) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(a3) $(\forall x \in X)\ (x * x = 0)$,
(a4) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \Rightarrow x = y)$.

In any BCI-algebra X we can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X has the following properties:

(b1) $(\forall x \in X)\ (x * 0 = x)$.
(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.
(b3) $(\forall x, y \in X)\ (0 * (x * y) = (0 * x) * (0 * y))$.
(b4) $(\forall x, y \in X)\ (x * (x * (x * y)) = x * y)$.
(b5) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$.
(b6) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$.
(b7) $(\forall x, y, z \in X)\ (0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))$.
(b8) $(\forall x, y \in X)\ (0 * (0 * (x * y)) = (0 * y) * (0 * x))$.

A non-empty subset S of a BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset A of a BCI-algebra X is called an *ideal* of X if it satisfies:

(c1) $0 \in A$,
(c2) $(\forall x \in A)\ (\forall y \in X)\ (y * x \in A \Rightarrow y \in A)$.

Note that every ideal A of a BCI-algebra X satisfies:

$$(\forall x \in A)\,(\forall y \in X)\,(y \leq x \Rightarrow y \in A).$$

A non-empty subset A of a BCI-algebra X is called a *q-ideal* of X if it satisfies (c1) and

(c3) $(\forall x, y, z \in X)(x * (y * z) \in A, y \in A \Rightarrow x * z \in A)$.

A non-empty subset A of a BCI-algebra X is called a *p-ideal* of X if it satisfies (c1) and

(c4) $(\forall x, y, z \in X)((x * z) * (y * z) \in A, y \in A \Rightarrow x \in A)$.

A non-empty subset A of a BCI-algebra X is called a *a-ideal* of X if it satisfies (c1) and

(c5) $(\forall x, y, z \in X)((x * z) * (0 * y) \in A, z \in A \Rightarrow y * x \in A)$.

Note that any q-ideal(p-ideal, a-ideal) is an ideal, but the converse is not true in general.

We refer the reader to the book [4] for further information regarding BCI-algebras.

DEFINITION 2.1. [2] A *vague set* A in the universe of discourse U is characterized by two membership functions given by:

1. A true membership function
$$t_A : U \to [0,1],$$
and
2. A false membership function
$$f_A : U \to [0,1],$$

where $t_A(u)$ is a lower bound on the grade of membership of u derived from the "evidence for u", $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and
$$t_A(u) + f_A(u) \leq 1.$$
Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0,1]$. This indicates that if the actual grade of membership of u is $\mu(u)$, then
$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$
The vague set A is written as
$$A = \{\langle u, [t_A(u), f_A(u)]\rangle \mid u \in U\},$$
where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of u in A, denoted by $V_A(u)$.

For $\alpha, \beta \in [0,1]$ we now define (α,β)-cut and α-cut of a vague set. Recall that if $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are two subintervals of $[0,1]$, we can define a relation by $I_1 \succeq I_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$([2]).

DEFINITION 2.2. [2] Let A be a vague set of a universe X with the true-membership function t_A and the false-membership function f_A. The (α,β)-*cut* of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by
$$A_{(\alpha,\beta)} = \{x \in X \mid V_A(x) \succeq [\alpha, \beta]\}.$$

Clearly $A_{(0,0)} = X$. The (α,β)-cuts of the vague set A are also called *vague-cuts* of A.

DEFINITION 2.3. [2] The α-*cut* of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = X$, and if $\alpha \geq \beta$ then $A_\alpha \subseteq A_\beta$ and $A_{(\alpha,\beta)} = A_\alpha$.
Equivalently, we can define the α-cut as

$$A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}.$$

3. Vague p-ideals

For our discussion, we shall use the following notations on interval arithmetic:

Let $I[0,1]$ denote the family of all closed subintervals of $[0,1]$. We define the term "imax" to mean the maximum of two intervals as

$$\text{imax}(I_1, I_2) := [\max(a_1, a_2), \max(b_1, b_2)],$$

where $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2] \in I[0,1]$. Similarly we define "imin". The concepts of "imax" and "imin" could be extended to define "isup" and "iinf" of infinite number of elements of $I[0,1]$.

It is obvious that $L = \{I[0,1], \text{isup}, \text{iinf}, \succeq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$ (see [2]).

In what follows let X denote a BCI-algebra unless specified otherwise.

Definition 3.1. [8] A vague set A of a BCI-algebra X is called a *vague BCI-algebra* of X if the following condition is true:

(d0) $(\forall x \in X)(V_A(x * y) \succeq \text{imin}\{V_A(x), V_A(y)\})$.

that is,

$$t_A(x * y) \geq \min\{t_A(x), t_A(y)\},$$
$$1 - f_A(x * y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}$$

for all $x, y \in X$.

Definition 3.2. [8] A vague set A of X is called a *vague ideal* of a BCI-algebra X if the following conditions are true:

(d1) $(\forall x \in X)(V_A(0) \succeq V_A(x))$,
(d2) $(\forall x, y \in X)(V_A(x) \succeq \text{imin}\{V_A(x * y), V_A(y)\})$.

that is,

$$t_A(0) \geq t_A(x), 1 - f_A(0) \geq 1 - f_A(x),$$
$$\text{and } t_A(x) \geq \min\{t_A(x * y), t_A(y)\}$$
$$1 - f_A(x) \geq \min\{1 - f_A(x * y), 1 - f_A(y)\}$$

for all $x, y \in X$.

Proposition 3.3. [8] *Every vague ideal of a BCI-algebra X satisfies the following properties:*

(i) $(\forall x, y \in X)(x \leq y \Rightarrow V_A(x) \succeq V_A(y))$,

(ii) $(\forall x, y, z \in X)(V_A(x * z) \succeq imin\{V_A((x * y) * z), V_A(y)\})$.

Definition 3.4. [5] A vague set A of X is called a *vague q-ideal* of X if it satisfies (d1) and

(d3) $(\forall x, y, z \in X)(V_A(x * z) \succeq \text{imin}\{V_A(x * (y * z)), V_A(y)\})$.

that is,

$$t_A(x * z) \geq \min\{t_A(x * (y * z)), t_A(y)\},$$
$$1 - f_A(x * z) \geq \min\{1 - f_A(x * (y * z)), 1 - f_A(y)\}$$

for all $x, y, z \in X$.

Definition 3.5. A vague set A of X is called a *vague p-ideal* of a BCI-algebra X if it satisfies (d1) and

(d4) $(\forall x, y, z \in X)(V_A(x) \succeq \text{imin}\{V_A((x * z) * (y * z)), V_A(y)\})$.

that is,

$$t_A(x) \geq \min\{t_A((x * z) * (y * z)), t_A(y)\},$$
$$1 - f_A(x) \geq \min\{1 - f_A((x * z) * (y * z)), 1 - f_A(y)\}$$

for all $x, y, z \in X$.

Example 3.6. *Let $X := \{0, a, b, c\}$ be a BCI-algebra([9]) in which the $*$-operation is given by the following table:*

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let A be the vague set in X defined as follows:

$$A = \{\langle 0, [0.7, 0.2]\rangle, \langle a, [0.7, 0.2]\rangle, \langle b, [0.5, 0.4]\rangle, \langle c, [0.5, 0.4]\rangle\}.$$

It is routine to verify that A is a vague p-ideal of X.

Theorem 3.7. *Every vague p-ideal of a BCI-algebra X is a vague ideal of X.*

Proof. Let A be a vague p-ideal of X. Putting $z := 0$ in (d4), for any $x, y \in X$ we have

$$\begin{aligned} V_A(x) &\succeq \text{imin}\{V_A((x * 0) * (y * 0)), V_A(y)\} \\ &= \text{imin}\{V_A(x * y), V_A(y)\}. \end{aligned}$$

Hence (d2) holds. Thus A is a vague ideal of a BCI-algebra X. $\square$

The converse of Theorem 3.7 is not true in general as seen the following example.

EXAMPLE 3.8. *Let $X := \{0, a, 1, 2, 3\}$ be a BCI-algebra([9]) in which the $*$-operation is given by the following table:*

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Let B be the vague set in X defined as follows:

$B = \{\langle 0, [0.8, 0.2]\rangle, \langle a, [0.7, 0.3]\rangle, \langle 1, [0.5, 0.4]\rangle, \langle 2, [0.5, 0.4]\rangle.\langle 3, [0.5, 0.4]\rangle\}$.

It is routine to verify that B is a vague ideal of X. But it is not a vague p-ideal of X, since $V_B(a) \not\succeq \text{imin}\{V_B((a * 1) * (0 * 1)), V_B(0)\}$.

LEMMA 3.9. *Let A be a vague ideal of X. Then $V_A(0*(0*x)) \succeq V_A(x)$ for all $x \in X$.*

Proof. Let A be a vague ideal of X. For any $x \in X$, we have

$$\begin{aligned} V_A(0 * (0 * x)) &\succeq \text{imin}\{V_A((0 * (0 * x)) * x), V_A(x)\} \\ &= \text{imin}\{V_A((0 * x) * (0 * x)), V_A(x)\} \\ &= \text{imin}\{V_A(0), V_A(x)\} = V_A(x). \end{aligned}$$

This completes the proof. $\square$

PROPOSITION 3.10. *Let A be a vague ideal of a BCI-algebra X. If A satisfies $V_A(x * y) \succeq V_A((x * z) * (y * z))$ for all $x, y, z \in X$, then A is a vague p-ideal of a BCI-algebra X.*

Proof. For any $x, y, z \in X$, we have

$$\begin{aligned} V_A(x) &\succeq \text{imin}\{V_A(x * y), V_A(y)\} \\ &\succeq \text{imin}\{V_A((x * z) * (y * z)), V_A(y)\}. \end{aligned}$$

Hence (d4) holds. Thus A is a vague p-ideal of X. $\square$

THEOREM 3.11. *A vague ideal A of a BCI-algebra X is a vague p-ideal of X if and only if*

$(*)$ $(\forall x \in X)(V_A(x) \succeq V_A(0 * (0 * x))$.

Proof. Assume that a vague ideal A of X is a vague p-ideal. Putting $z := x$ and $y := 0$ in (d4), for any $x \in X$ we have

$$\begin{aligned} V_A(x) &\succeq \text{imin}\{V_A((x * x) * (0 * x)), V_A(0)\} \\ &= \text{imin}\{V_A(0 * (0 * x)), V_A(0)\} \\ &= V_A(0 * (0 * x)). \end{aligned}$$

Conversely, suppose that a vague ideal A of a BCI-algebra X satisfies $(*)$. Using Lemma 3.9, (b7), (b8) and $(*)$, for any $x, y, z \in X$ we have

$$\begin{aligned} V_A((x * z) * (y * z)) &\preceq V_A(0 * (0 * ((x * z) * (y * z)))) \\ &= V_A((0 * y) * (0 * x)) \\ &= V_A(0 * (0 * (x * y))) \\ &\preceq V_A(x * y). \end{aligned}$$

By Proposition 3.10, A is a vague p-ideal of X. $\square$

4. Vague a-ideals

Definition 4.1. A vague set A of X is called a *vague a-ideal* of a BCI-algebra X if it satisfies (d1) and

(d5) $(\forall x, y, z \in X)(V_A(y * x) \succeq \text{imin}\{V_A((x * z) * (0 * y)), V_A(z)\})$.

that is,

$$\begin{aligned} t_A(y * x) &\geq \min\{t_A((x * z) * (0 * y)), t_A(z)\}, \\ 1 - f_A(y * x) &\geq \min\{1 - f_A((x * z) * (0 * y)), 1 - f_A(z)\} \end{aligned}$$

for all $x, y, z \in X$.

Example 4.2. *Consider $X = \{0, a, b, c\}$ as in Example 3.6. Let C be the vague set in X defined as follows:*

$$C = \{\langle 0, [0.7, 0.1]\rangle, \langle a, [0.7, 0.1]\rangle, \langle b, [0.5, 0.3]\rangle, \langle c, [0.5, 0.3]\rangle\}.$$

It is routine to verify that C is a vague a-ideal of X.

Theorem 4.3. *Every vague a-ideal of a BCI-algebra X is both a vague ideal of X and a vague BCI-algebra of X.*

Proof. Let A be any vague a-ideal of a BCI-algebra X. Putting $z = y = 0$ in (d5), for any $x \in X$ we have

$$\begin{aligned} V_A(0 * x) &\succeq \text{imin}\{V_A((x * 0) * (0 * 0)), V_A(0)\} \\ &= \text{imin}\{V_A(x), V_A(0)\} \\ &= V_A(x). \end{aligned} \tag{4.1}$$

Taking $x = z = 0$ in (d5), for any $y \in X$ we have

$$\begin{aligned} V_A(y) = V_A(y * 0) &\succeq \text{imin}\{V_A((0*0)*(0*y)), V_A(0)\} \\ &= \text{imin}\{V_A(0*(0*y)), V_A(0)\} \\ &= V_A(0*(0*y)). \qquad (4.2) \end{aligned}$$

Putting $y = 0$ in (d5), for any $x, z \in X$ we have

$$\begin{aligned} V_A(0*x) &\succeq \text{imin}\{V_A((x*z)*(0*0)), V_A(z)\} \\ &= \text{imin}\{V_A(x*z), V_A(z)\}. \qquad (4.3) \end{aligned}$$

Using (4.2) and (4.1), we have

$$V_A(x) \succeq V_A(0*(0*x)) \succeq V_A(0*x).$$

Hence $V_A(x) \succeq \text{imin}\{V_A(x*z), V_A(z)\}$ and so (d2) holds. Thus A is a vague ideal of X.

Using (d2), we have

$(**)$ $(\forall x, y \in X)(V_A(x*y) \succeq \text{imin}\{V_A((x*y)*z), V_A(z)\})$.

Putting $z = x$ in $(**)$ and use (4.1), for any $x, y \in X$ we have

$$\begin{aligned} V_A(x*y) &\succeq \text{imin}\{V_A((x*y)*x), V_A(x)\} \\ &= \text{imin}\{V_A((x*x)*y), V_A(x)\} \\ &= \text{imin}\{V_A(0*y), V_A(x)\} \\ &\succeq \text{imin}\{V_A(y), V_A(x)\}. \end{aligned}$$

Thus A is a vague BCI-algebra of X. □

The converse of Theorem 4.3 is not true in general as seen the following example.

EXAMPLE 4.4. *Let $X := \{0, a, b\}$ be a BCI-algebra([9]) in which the $*$-operation is given by the following table:*

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Let D be the vague set in X defined as follows:

$$D = \{\langle 0, [0.8, 0.1]\rangle, \langle a, [0.5, 0.3]\rangle, \langle b, [0.5, 0.3]\rangle\}.$$

It is routine to verify that D is both a vague ideal of X and a vague BCI-algebra of X. But it is not a vague a-ideal of X since

$$V_D(a*0) \not\succeq \text{imin}\{V_D((0*0)*(0*a)), V_D(0)\}.$$

LEMMA 4.5. [8] *Every vague BCI-algebra X of a BCI-algebra X satisfies:*

$$(\forall x \in X)(V_A(0) \succeq V_A(x)).$$

PROPOSITION 4.6. *Let A be a vague set of a BCI-algebra X. If A is a vague ideal of X, then it satisfies: for any $x, y, z \in X$,*

$$x * y \leq z \Rightarrow V_A(x) \succeq imin\{V_A(y), V_A(z)\}.$$

Proof. Assume that A is a vague ideal of X and let $x, y, z \in X$ be such that $x * y \leq z$. It follows from Proposition 3.3(i) that $V_A(z) \preceq V_A(x * y)$. Using (d2), we have

$$V_A(x) \succeq \mathrm{imin}\{V_A(x * y), V_A(y)\} \succeq \mathrm{imin}\{V_A(y), V_A(z)\}.$$

This completes the proof. □

Next we give the characterizations of vague a-ideals.

THEOREM 4.7. *Let A be a vague ideal of a BCI-algebra X. Then the following are equivalent:*

(1) *A is a vague a-ideal of X.*
(2) $(\forall x, y, z \in X)(V_A(y * (x * z)) \succeq V_A((x * z) * (0 * y)))$.
(3) $(\forall x, y \in X)(V_A(y * x) \succeq V_A(x * (0 * y))$.

Proof. (1)⇒(2) Let $s := (x * z) * (0 * y)$ for any $x, y, z \in X$. Then $((x * z) * s) * (0 * y) = ((x * z) * (0 * y)) * s = 0$. Using (d5), for any $x, y, z \in X$ we have

$$\begin{aligned} V_A(y * (x * z)) &\succeq \mathrm{imin}\{V_A(((x * z) * s) * (0 * y)), V_A(s)\} \\ &= \mathrm{imin}\{V_A(0), V_A(s)\} \\ &= V_A(s) \\ &= V_A((x * z) * (0 * y)). \end{aligned}$$

Hence (2) holds.
(2)⇒(3) Let $z := 0$ in (2). We obtain (3).
(3)⇒(1) Let $x, y, z \in X$. Using (b6) and (a2), we have $(x * (0 * y)) * ((x * z) * (0 * y)) \leq x * (x * z) \leq z$ and so $(x * (0 * y)) * ((x * z) * (0 * y)) \leq z$. It follows from Proposition 4.6 that $V_A((x * (0 * y)) \succeq \mathrm{imin}\{V_A((x * z) * (0 * y)), V_A(z)\}$. Using (3), we have

$$V_A(y * x) \succeq V_A(x * (0 * y)) \succeq \mathrm{imin}\{V_A((x * z) * (0 * y)), V_A(z)\}.$$

Hence (d5) holds. Thus A is a vague a-ideal of X. □

Now, we discuss the relations among vague a-ideals, vague p-ideals, and vague q-ideals of a BCI-algebra X and give another characterization of vague a-ideals of a BCI-algebra X.

THEOREM 4.8. [5] *Let A be a vague ideal of a BCI-algebra X. Then the following are equivalent:*

(1) *A is a vague q-ideal of X.*
(2) $(\forall x, y \in X)(V_A(x * y) \succeq V_A(x * (0 * y))$.
(3) $(\forall x, y, z \in X)(V_A((x * y) * z) \succeq V_A(x * (y * z))$.

THEOREM 4.9. [5] *Every vague q-ideal of a BCI-algebra X is both a vague ideal of X and a vague BCI-algebra of X.*

THEOREM 4.10. *Any vague a-ideal of a BCI-algebra X is a vague p-ideal of X.*

Proof. Let A be a vague a-ideal of X. Then it is a vague ideal of X by Theorem 4.3. Setting $x = z = 0$ in Theorem 4.7(2), we have

$$V_A(y * (0 * 0)) \succeq V_A((0 * 0) * (0 * y),$$

i.e., $V_A(y) \succeq V_A(0 * (0 * y))$. By Theorem 3.11, A is a vague p-ideal of X. $\square$

The converse of Theorem 4.10 is not true in general as the following example.

EXAMPLE 4.11. *Let $X := \{0, a, b\}$ be a BCI-algebra([9]) in which the $*$-operation is given by the following table:*

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Let E be the vague set in X defined as follows:

$$E = \{\langle 0, [0.8, 0.1]\rangle, \langle a, [0.5, 0.3]\rangle, \langle b, [0.5, 0.3]\rangle\}.$$

It is routine to verify that E is a vague p-ideal of X. But it is not a vague a-ideal of X since

$$V_E(b * a) \not\succeq imin\{V_E((a * 0) * (0 * b)), V_E(0)\}.$$

THEOREM 4.12. *Any vague a-ideal of a BCI-algebra X is a vague q-ideal X.*

Proof. Let A be any vague a-ideal of X. Then it is a vague ideal of X by Theorem 4.3. In order to prove that A is a vague q-ideal from Theorem 4.8(2), it suffices to show that $V_A(x*y) \succeq V_A(x*(0*y))$ for all $x, y \in X$. Since for any $x, y \in X$

$$\begin{aligned}
(0*(0*(y*(0*x))))&*(x*(0*y)) \\
&=[(0*(0*y))*(0*(0*(0*x)))]*(x*(0*y)) \\
&=((0*(0*y))*(0*x))*(x*(0*y)) \\
&\leq(x*(0*y))*(x*(0*y))=0,
\end{aligned}$$

we have $(0*(0*(y*(0*x))))*(x*(0*y))=0$ and so

$$0*(0*(y*(0*x)))\leq x*(0*y).$$

It follows from Theorem 4.10, Theorem 3.11 and Proposition 3.3(i) that

$$V_A(y*(0*x)) \succeq V_A(0*(0*(y*(0*x)))) \succeq V_A(x*(0*y)).$$

Using Theorem 4.7(3), we have $V_A(x*y) \succeq V_A(y*(0*x))$. Hence $V_A(x*y) \succeq V_A(x*(0*y))$. Thus A is a vague q-ideal of X. □

The converse of Theorem 4.12 is not true in general as seen the following example.

EXAMPLE 4.13. *Consider a BCI-algebra $X=\{0,a,b\}$ as in Example 4.4. Let F be the vague set in X defined as follows:*

$$F=\{\langle 0,[0.8,0.1]\rangle,\langle a,[0.5,0.4]\rangle,\langle b,[0.5,0.4]\rangle\}.$$

It is routine to verify that F is a vague q-ideal of X. But it is not a vague a-ideal of X since

$$V_F(a*0) \not\succeq imin\{V_F((0*0)*(0*a)),V_F(0)\}.$$

LEMMA 4.14. *Let A be a both a vague BCI-algebra X and a vague ideal of a BCI-algebra X. Then $V_A(0*x) \succeq V_A(x)$ for all $x \in X$.*

Proof. Put $x=0$ in (d0). Then for all $y \in X$

$$\begin{aligned}
V_A(0*y) \succeq&\mathrm{imin}\{V_A(0),V_A(y)\} \\
=&V_A(y).
\end{aligned}$$

This completes the proof. □

THEOREM 4.15. *A vague set A of a BCI-algebra X is a a-ideal of X if and only if it is both a vague p-ideal and a vague q-ideal of X.*

Proof. Assume that A is a vague p-ideal and a vague q-ideal of X. By Theorem 4.9, A is both a vague BCI-algebra of X and a vague ideal of X. In order to prove that A is a vague a-ideal from Theorem 4.7(3), it suffices to show that $V_A(y*x) \succeq V_A(x*(0*y))$ for all $x, y \in X$. Since for any $x, y \in X$

$$\begin{aligned}(0*(y*x))*(x*y) =& ((0*y)*(0*x))*(x*y)\\ =& ((0*(x*y))*y)*(0*x)\\ =& (((0*x)*(0*y))*y)*(0*x)\\ =& (0*(0*y))*y\\ =& (0*y)*(0*y) = 0,\end{aligned}$$

we obtain $0*(y*x) \leq x*y$. It follows from Proposition 3.3(i) that $V_A(x*y) \preceq V_A(0*(y*x))$. Using Lemma 4.14 and Theorem 3.11, we have

$$V_A(x*y) \preceq V_A(0*(y*x)) \preceq V_A(0*(0*(y*x))) \preceq V_A(y*x).$$

By Theorem 4.8(2), $V_A(x*(0*y)) \preceq V_A(x*y) \preceq V_A(y*x)$. Thus A is a vague a-ideal of X.

Conversely, if A is a vague a-ideal of X, then A is a vague p-ideal and a vague q-ideal of X by Theorem 4.10 and Theorem 4.12. $\square$

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Department of Mathematics Education
Dongguk University
Seoul 10-715, Republic of Korea
E-mail: `hwangyunsun@nate.com`

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Department of Mathematics Eduaction
Dongguk University
Seoul 100-715, Republic of Korea
E-mail: `sunshine@dongguk.edu`