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THE HESTOCK AND HENSTOCK DELTA INTEGRALS

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Abstract. In this paper, we study the Henstock delta integral, which generalizes the Henstock integral. In particular, we study the relation between the Henstock and Henstock delta integrals.

1. Introduction and preliminaries

The Henstock delta integral on time scales was introduced by Allan Peterson and Bevan Thompson [2].

In this paper, we investigate the relation between the Henstock and Henstock delta integrals.

First, we introduce some concepts related to the notion of time scales. A time scale $\mathbb{T}$ is any closed nonempty subset of $\mathbb{R}$, with the topology inherited from the standard topology on the real numbers $\mathbb{R}$. For each $t \in \mathbb{T}$, we define the forward jump operator $\sigma(t)$ by

$$\sigma(t) = \inf\{z > t : z \in \mathbb{T}\}$$

and the backward jump operator $\rho(t)$ by

$$\rho(t) = \sup\{z < t : z \in \mathbb{T}\}$$

where $\inf \phi = \sup \mathbb{T}$ and $\sup \phi = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say the t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense,while if $\rho(t) = t$, we say that t is left-dense. The forward graininess function $\mu(t)$ is defined by $\mu(t) = \sigma(t) - t$, and the backward graininess function $\nu(t)$ is defined by $\nu(t) = t - \rho(t)$.

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For $a, b \in \mathbb{T}$, we define the time scale interval in $\mathbb{T}$ by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

2. The Henstock and Henstock delta integrals

Definition 2.1. ([2]) $\delta = (\delta_L, \delta_R)$ is a $\triangle$-gauge on $[a, b]_{\mathbb{T}}$ by $\delta_L(t) > 0$ on $(a, b]_{\mathbb{T}}$, $\delta_R(t) > 0$ on $[a, b)_{\mathbb{T}}$, $\delta_L(a) \geq 0, \delta_R(b) \geq 0$, and $\delta_R(t) \geq \mu(t)$ for each $t \in [a, b)_{\mathbb{T}}$.

Definition 2.2. ([2]) A collection $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^n$ of tagged intervals is a Henstock partition of $[a, b]_{\mathbb{T}}$ if $\bigcup_{i=1}^n [t_{i-1}, t_i]_{\mathbb{T}} = [a, b]_{\mathbb{T}}$, $[t_{i-1}, t_i]_{\mathbb{T}} \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)]$ and $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}}$ for each $i = 1, 2, \cdots, n$.

For Henstock partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$, we write

$$S(f, \mathcal{P}) = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

whenever $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$.

Definition 2.3. ([2]). A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is Henstock delta integrable (or $H_\triangle$-integral) on $[a, b]_{\mathbb{T}}$ if there exists a number A such that for each $\epsilon > 0$ there exists a $\triangle$-gauge δ on $[a, b]_{\mathbb{T}}$ such that

$$\left| S(f, \mathcal{D}) - A \right| < \epsilon$$

for every δ-fine Henstock partition $\mathcal{D}$ of $[a, b]_{\mathbb{T}}$. A number A is called the $H_\triangle$-integral of f on $[a, b]_{\mathbb{T}}$, and we write $A = (H_\triangle) \int_a^b f \triangle t$.

Recall that $f : [a, b] \to \mathbb{R}$ is Henstock integrable (or H-integrable) on $[a, b]$ if there exists a number A such that for each $\varepsilon > 0$ there exists a gauge $\delta : [a, b] \to \mathbb{R}^+$ on $[a, b]$ such that

$$\left| S(f, \mathcal{P}) - A \right| < \epsilon$$

for every δ-fine Henstock partition $\mathcal{P}$ of $[a, b]$.

Theorem 2.4. *A function $f : [a, b] \to \mathbb{R}$ is H-integrable on $[a, b]$ if and only if f is $H_\triangle$-integrable on $[a, b]$.*

Proof. Let f be H-integrable on $[a,b]$ and let $\epsilon > 0$. Then there exists a gauge $\delta : [a,b] \to \mathbb{R}^+$ such that

$$\Big|S(f,\mathcal{P}) - (H)\int_a^b f\Big| < \epsilon$$

for every δ-fine Henstock partition $\mathcal{P}$ of $[a,b]$.

Define $\delta^* = (\delta_L, \delta_R)$ by $\delta_L(t) = \delta_R(t) = \frac{\delta}{2}$ for each $t \in [a,b]$. Assume that $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ is a δ^*-fine partition of $[a,b]$. Then $\mathcal{P}$ is δ-fine and

$$\big|S(f,\mathcal{P}) - (H)\int_a^b f\big| < \epsilon.$$

Hence f is $H_\triangle$-integrable on $[a,b]$ and $(H_\triangle)\int_a^b f\triangle t = (H)\int_a^b f$. Conversely, assume that f is $H_\triangle$-integrable on $[a,b]$ and let $\epsilon > 0$. Then there exists a $\triangle$-gauge $\delta^* = (\delta_L, \delta_R)$ such that

$$\Big|S(f,\mathcal{P}) - (H_\triangle)\int_a^b f\triangle t\Big| < \epsilon$$

for every δ^*-fine partition $\mathcal{P}$ of $[a,b]$.

Define

$$\delta(t) = \begin{cases} \min\{\delta_L(t), \delta_R(t)\} & \text{if} \quad \mathrm{t} \in (a,b) \\ \delta_R(t) \quad \text{if} \quad t = a \\ \delta_L(t) \quad \text{if} \quad t = b. \end{cases}$$

Assume that $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ is a δ-fine partition of $[a,b]$.

Then $\mathcal{P}$ is δ^*-fine and

$$\big|S(f,\mathcal{P}) - (H_\triangle)\int_a^b f\triangle t\big| < \epsilon.$$

Hence f is H-integrable on $[a,b]$ and $(H)\int_a^b f = (H_\triangle)\int_a^b f\triangle t$. □

Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a function on $[a,b]_{\mathbb{T}}$, and let $\{(a_k, b_k)\}_{k=1}^\infty$ be the sequence of intervals contiguous to $[a,b]_{\mathbb{T}}$ in $[a,b]$.

Define a function $f^* : [a,b] \to \mathbb{R}$ on $[a,b]$ by

$$f^*(t) = \begin{cases} f(a_k) & \text{if} \quad t \in (a_k, b_k) \quad \text{for some} \quad k \\ f(t) & \text{if} \quad t \in [a,b]_{\mathbb{T}}. \end{cases}$$

Then we have the following theorem.

Theorem 2.5. *If $f^* : [a,b] \to \mathbb{R}$ is H-integrable on $[a,b]$, then $f : [a,b]_T \to \mathbb{R}$ is $H_\triangle$-integrable on $[a,b]_T$ and $(H_\triangle)\int_a^b f\triangle t = (H)\int_a^b f^*$.*

Proof. Let $f^* : [a,b] \to \mathbb{R}$ be H-integrable on $[a,b]$ and let $\epsilon > 0$. By theorem 2.4, there exists a $\triangle$-gauge $\delta = (\delta_L, \delta_R)$ on $[a,b]$ such that

$$\Big|S(f^*, \mathcal{P}) - (H)\int_a^b f^*\Big| < \frac{\epsilon}{2}$$

for every δ-fine Henstock partition $\mathcal{P}$ of $[a,b]$.

Define a $\triangle$-gauge $\delta^* = (\delta^*_L, \delta^*_R)$ on $[a,b]_{\mathbb{T}}$ by

$$\delta^*_L(t) = \delta_L(t)$$

$$\delta^*_R(t) = \begin{cases} \delta_R(t) \text{ if } t \text{ is a right-dense point of } [a,b]_{\mathbb{T}} \\ \sigma(t) - t \text{ if } t \text{ is a right-scattered point of } [a,b]_{\mathbb{T}}. \end{cases}$$

Let $\mathcal{D} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ be a δ^*-fine partition of $[a,b]_{\mathbb{T}}$. Define $A = \{i : \xi_i \text{ is a right-scattered point of } [a,b]_T \text{ and}$

$[\xi_i, \sigma(\xi_i)] \subset [t_{i-1}, t_i]\}$,

$A_1 = \{i \in A : t_{i-1} = \xi_i\}, A_2 = \{i \in A : t_{i-1} < \xi_i\}$, and

$B = \{1, 2, 3, \cdots, n\} - A$.

Let $D_0 = \{(\xi_i, [t_{i-1}, t_i]) : i \in B\}$. Then D_0 is a δ-fine partial partition of $[a,b]$. For each $i \in A$, there is a δ-fine partition D'_i of $[\xi_i, \sigma(\xi_i)]$ such that

$$\Big|S(f^*, D'_i) - (H)\int_{\xi_i}^{\sigma(\xi_i)} f^*\Big| < \frac{\epsilon}{2n}.$$

For each $i \in A$, let $D_i = \begin{cases} D'_i \text{ if } i \in A \\ D'_i \bigcup \{(\xi_i, [t_{i-1}, \xi_i])\} \text{ if } i \in A_2. \end{cases}$

Then $\mathcal{P} = D_0 \cup [\bigcup_{i \in A} D_i]$ is a δ-fine partition of $[a,b]$ and we have

$$\begin{aligned} &\Big|S(f, D) - (H)\int_a^b f^*\Big| \\ &\le \Big|S(f, D) - S(f^*, \mathcal{P})\Big| + \Big|S(f^*, \mathcal{P}) - (H)\int_a^b f^*\Big| \\ &\le \sum_{i \in A_1} \big|f(\xi_i)(t_i - t_{i-1}) - S(f^*, D_i)\big| \\ &\quad + \sum_{i \in A_2} \big|f(\xi_i)(t_i - t_{i-1}) - S(f^*, D_i)\big| + \frac{\epsilon}{2} \\ &= \sum_{i \in A_1} \big|f(\xi_i)(\sigma(\xi_i) - \xi_i) - S(f^*, D'_i)\big| \\ &\quad + \sum_{i \in A_2} \big|f(\xi_i)(\sigma(\xi_i) - \xi_i) - S(f^*, D'_i)\big| + \frac{\epsilon}{2} \end{aligned}$$

$$= \sum_{i\in A} \left|(H)\int_{\xi_i}^{\sigma(\xi_i)} f^* - S(f^*, D_i')\right| + \frac{\epsilon}{2}$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$$

Hence, f is $H_\triangle$-integral on $[a,b]_\mathbb{T}$ and $(H_\triangle)\int_a^b f\triangle t = (H)\int_a^b f^*$. □

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