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MINIMAL QUASI-F COVERS OF SOME EXTENSION

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Abstract. Observing that every Tychonoff space X has an extension kX which is a weakly Lindelöf space and the minimal quasi-F cover $QF(kX)$ of kX is a weakly Lindelöf, we show that Φ_{kX} : $QF(kX) \rightarrow kX$ is a $z^{\#}$-irreducible map and that $QF(\beta X) = \beta QF(kX)$. Using these, we prove that $QF(kX) = kQF(X)$ if and only if $\Phi_X^k : kQF(X) \rightarrow kX$ is an onto map and $\beta QF(X) = QF(\beta X)$.

1. Introduction

All spaces in this paper are assumed to be Tychonoff and βX (vX, resp.) denotes the Stone-Čech compactification (Hewitt realcompactification, resp.) of X.

Iliadis constructed the absolute of a Hausdorff space X, which is the minimal exteamally disconnected cover $(E(X), \pi_X)$ of X and they turn out to be the perfect onto projective covers ([6]). To generalize extremally disconnected spaces, basically disconnected spaces, quasi-F spaces and cloz-spaces have been introduced and their minimal covers have been studied by various aurthors ([1], [4], [5], [8], [9]). In these ramifications, minimal covers of compact spaces can be nisely characterized.

In particular, Henriksen and Gillman intoduced the concept of quasi-F spaces in which every dense cozero-set is C^*-embedded ([2]). Each space X has the minimal quasi-F cover $(QF(X), \Phi_X)$ ([5]). In [5], authors investigated when $\beta QF(X) = QF(\beta X)$ and $QF(X) = \Phi_{\beta X}^{-1}(X)$, where $(QF(\beta X), \Phi_{\beta X})$ is the minimal quasi-F cover of βX.

It is well-known that each space has the minimal extremally disconnected cover $(E(X), k_X)$ and that $\beta E(X) = E(\beta X)$ ([8]). Moreover,

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internal characterizations of a space X that is equivalent to $E(vX) = vE(X)$ is known ([8]). Similar results for the minimal basically disconnected cover $(\Lambda X, \Lambda_X)$ are given by [7].

For any space X, there is an extension (kX, k_X) of X such that

(1) kX is a weakly Lindelöf space, and
(2) for any continuous map $f : X \to Y$, there is a continuous map $f^k : kX \to kY$ such that $f^k|_X = f$([10]).

The purpose to write this paper is to find the relation of the minimal quasi-F cover $QF(kX)$ of kX and $kQF(X)$. For any space X, we show that $QF(kX)$ is a weakly Lindelöf space and $\Phi_{kX} : QF(kX) \to kX$ is a $z^{\#}$ - irreducible map and that $QF(\beta X) = \beta QF(kX)$. Moreover, we show that $kQF(X) = QF(kX)$ if and only if $\Phi_X^k : kQF(X) \to kX$ is an onto map and $QF(\beta X) = \beta QF(X)$.

For the termlnology, we refer to [2] and [9].

2. Quasi-F covers

Let X be a space. It is well-known that the collection $\mathcal{R}$(X) of all regular closed sets in X, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows :
For any $A \in \mathcal{R}$(X) and any $\mathcal{F} \subseteq \mathcal{R}$(X),

$\bigvee \mathcal{F} = cl_X\big(\cup \{F \mid F \in \mathcal{F}\}\big)$,
$\bigwedge \mathcal{F} = cl_X\big(int_X\big(\cap \{F \mid F \in \mathcal{F}\}\big)\big)$, and
$A' = cl_X(X - A)$.

A sublattic of $\mathcal{R}$(X) is a subset of $\mathcal{R}$(X) that contains $\emptyset$, X and is closed under finite joins and finite meets ([8]).

A map $f : Y \to X$ is called a *covering map* if it is an onto continuous, perpect, and irreducible map ([8]).

Lemma 2.1. ([8])

(1) *Let X be a dense subspace of Y. Then the map $\phi : R(Y) \to R(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism.*
(2) *Let $f : Y \to X$ be a covering map. Then the map $\psi : R(Y) \to R(X)$, defined by $\psi(A) = f(A)$, is a Boolean isomorphism.*

In the above lemma, the inverse map $\phi^{-1} : R(X) \to R(Y)$ of ϕ is given by $\phi^{-1}(B) = cl_Y(B)$ $\big(B \in R(X)\big)$ and the inverse map $\psi^{-1} : R(X) \to R(Y)$ of ψ is given by $\psi^{-1}(B) = cl_Y\big(int_Y\big(f^{-1}(B)\big)\big) = cl_Y\big(f^{-1}\big(int_X(B)\big)\big)$ $\big(B \in R(X)\big)$.

Definition 2.2. A space X is called *a quasi-F space* if for any zerosets A, B in X, $cl_X\big(int_X(A\cap B)\big) = cl_X\big(int_X(A)\big) \cap cl_X\big(int_X(B)\big)$, equivalently, every dense cozero-set in X is C^*-embedded in X.

It is well-known that a space X is a quasi-F space if and only if βX (or vX) is a quasi-F space.

Definition 2.3. Let X be a space. Then a pair (Y, f) is called

(1) *a cover of* X if $f : X \to Y$ is a covering map,
(2) *a quasi-F cover of* X if (Y, f) is a cover of X and Y is a quasi-F space, and
(3) *a minimal quasi-F cover of* X if (Y, f) is a quasi-F cover of X and for any quasi-F cover (Z, g) of X, there is a covering map $h : Z \to Y$ such that $f \circ h = g$.

Let X be a space, $Z(X) = \{Z \mid Z \text{ is a zero-set in } X\}$ and $Z(X)^{\#} = \{cl_X\big(int_X(A)\big) \mid A \in Z(X)\}$. Then $Z(X)^{\#}$ is a sublattice of $R(X)$.

Suppose that X is a compact space. Let $QF(X) = \{\alpha \mid \alpha \text{ is a } Z(X)^{\#}\text{−ultrafilter}\}$ and for any $A \in Z(X)^{\#}$, let $\sum_A^{Z(X)^{\#}} = \{\alpha \in QF(X) \mid A \in \alpha\}$. Then the space $QF(X)$, equipped with the topology for which $\{QF(X) - \sum_A^{Z(X)^{\#}} \mid A \in Z(X)^{\#}\}$ is a base, is a quasi-F space. Define the map $\Phi_X : QF(X) \to X$ by $\Phi_X(\alpha) = \cap\{A \mid A \in \alpha\}$. Then $\big(QF(X), \Phi_X\big)$ is the minimal quasi-F cover of X and for any $A \in Z(X)^{\#}$, $\Phi_X\big(\sum_A^{Z(X)^{\#}}\big) = A$([4]).

Let X, Y be spaces and $f : Y \to X$ a map. For any $U \subseteq X$, let $f_U : f^{-1}(U) \to U$ denote the restriction and co-restriction of f with respect to $f^{-1}(U)$ and U, respectively. For any space X, let $\big(QF(\beta X), \Phi_\beta\big)$ denote the minimal quasi-F cover of βX.

We recall that a covering map $f : Y \to X$ is called $z^{\#}-irreducible$ if $f\big(Z(Y)^{\#}\big) = Z(X)^{\#}$. Let $f : Y \to X$ be a covering map and Z a zero-set in X. By Lemma 2.1, $f\big(cl_Y\big(int_Y\big(f^{-1}(Z)\big)\big)\big) = cl_X\big(int_X(Z)\big)$ and $cl_Y\big(int_Y\big(f^{-1}(Z)\big)\big) \in Z(Y)^{\#}$. Hence $Z(X)^{\#} \subseteq f\big(Z(Y)^{\#}\big)$ and so $f : Y \to X$ is $z^{\#}$-irreducible if and only if $f\big(Z(Y)^{\#}\big) \subseteq Z(X)^{\#}$. Using these we have the following :

Proposition 2.4. *Let $f : Y \to X$ and $g : W \to Y$ be covering maps. Then $f \circ g : W \to X$ is $z^{\#}$-irreducible if and only if $f : Y \to X$ and $g : W \to Y$ are $z^{\#}$-irreducible.*

It is well-known that Φ_β is $z^{\#}$-irreducible ([5]).

3. Minimal quasi-F covers of kX

A z-filter $\mathcal{F}$ on a space X is called *real* if $\mathcal{F}$ is closed under the countable intersection.

For any space X, let $kX = vX \cup \{p \in \beta X - vX \mid$ there is a real $z-$filter $\mathcal{F}$ on X such that $\cap\{cl_{vX}(F) \mid F \in \mathcal{F}\} = \emptyset$ and $p \in \cap\{cl_{\beta X}(F) \mid F \in \mathcal{F}\}\}$. Then kX is an extension of a space X such that $vX \subseteq kX \subseteq \beta X$ ([10]).

We recall that a space X is called *a weakly Lindelöf space* if for any open cover $\mathcal{U}$ of X, there is a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\cup\{V \mid V \in \mathcal{V}\}$ is a dense subset of X .

LEMMA 3.1. ([10]) *For any space X, kX is a weakly Lindelöf space.*

It is well known that a space X is weakly Lindelöf if and only if for any $Z(X)^{\#}$-filter $\mathcal{A}$ with the countable meet property, $\cap\{A \mid A \in \mathcal{A}\} \neq \emptyset$.

Let X be a space. For any $A \in Z(\beta X)^{\#}$, let $\sum_A^{Z(\beta X)^{\#}} = \sum_A$ and $\sum_A \cap QF(kX) = \lambda_A$. Then for any $A \in Z(\beta X)^{\#}$, $\Phi_\beta(\sum_A) = A$, and $\Phi_{kX}(\lambda_A) = A \cap kX$, because $QF(kX) = \Phi_\beta^{-1}(kX)$ and $\Phi_{kX} = \Phi_{\beta_{kX}}$ ([7]).

THEOREM 3.2. *Let X be a space. Then we have the following :*

(1) $QF(kX)$ is a weakly Lindelöf space, and

(2) $\Phi_{kX} : QF(kX) \to kX$ is a $z^{\#}$-irreducible map.

Proof. (1) Let $\mathcal{A}$ be a z-filter on $QF(kX)$ with the countable meet property and $\cap\{A \mid A \in \mathcal{A}\} = \emptyset$. Suppose that $\cap\{\Phi_{kX}(A) \mid A \in \mathcal{A}\} \neq \emptyset$. Pick $x \in \cap\{\Phi_{kX}(A) \mid A \in \mathcal{A}\}$. Since $\mathcal{A}$ is a z-filter on $QF(kX)$, $\mathcal{A}$ has the finite intersection property. Hence $\{A \cap \Phi_{kX}^{-1}(x) \mid A \in \mathcal{A}\}$ is a family of closed set in $\Phi_{kX}^{-1}(x)$ with the finite intersection property. Since $\Phi_{kX}^{-1}(x)$ is a compact subset in $QF(kX)$, $\cap\{A \cap \Phi_{kX}^{-1}(x) \mid A \in \mathcal{A}\} \neq \emptyset$ and so $\cap\{A \mid A \in \mathcal{A}\} \neq \emptyset$. This is a contracdiction. Thus $\cap\{\Phi_{kX}(A) \mid A \in \mathcal{A}\} = \emptyset$. Since kX is a weakly Lindelöf space, there is a sequence (A_n) in $\mathcal{A}$ such that $cl_{kX}\big(\cup\{kX - \Phi_{kX}(A_n) \mid n \in N\}\big) = kX$. Let $A \in \mathcal{A}$. Then $\Phi_{kX}\big(QF(kX) - A\big) \supseteq kX - \Phi_{kX}(A)$ and hence $\Phi_{kX}(A') \supseteq \Phi_{kX}\big(QF(kX) - A\big) \supseteq kX - \Phi_{kX}(A)$. Thus $cl_{kX}\big(\cup\{\Phi_{kX}(A'_n) \mid n \in N\}\big) = kX$. Note that

$$\begin{aligned} kX &= cl_{kX}\big(\cup\{\Phi_{kX}(A'_n) \mid n \in N\}\big) \\ &= cl_{kX}\big(\Phi_{kX}\big(\cup\{A'_n \mid n \in N\}\big)\big) \\ &= \Phi_{kX}\big(cl_{kX}\big(\cup\{A'_n \mid n \in N\}\big)\big) \\ &= \Phi_{kX}\big(\vee\{A'_n \mid n \in N\}\big). \end{aligned}$$

Since Φ_{kX} is a covering map, $\vee\{A'_n \mid n \in N\} = QF(kX)$ and so $\big(\vee\{A'_n \mid n \in N\}\big)' = \wedge\{A_n \mid n \in N\} = \emptyset$. Since $\mathcal{A}$ has the countable meet property, it is a contradiction. Hence $\cap\{A \mid A \in \mathcal{A}\} = \emptyset$ and so $QF(kX)$ is a weakly Lindelöf space.

(2) Take any zero-set Z in $QF(kX)$. Since $QF(kX)$ is a weakly Lindelöf space, $QF(kX) - Z$ is an open weakly Lindelöf subspace of $QF(kX)$. Hence there is a sequence (Z_n) in $Z(\beta X)^{\#}$ such that for any $n \in N$, $QF(kX) - \big(\Sigma_{Z_n} \cap\ QF(kX)\big) \subseteq QF(kX) - Z$ and

$$\begin{aligned} &cl_{QF(kX)}\big(\cup\{QF(kX) - \big(\Sigma_{Z_n} \cap\ QF(kX)\big) \mid n \in N\}\big) \cap \big(QF(kX) - Z\big) \\ &= cl_{QF(kX)}\big(\cup\{QF(kX) - \lambda_{Z_n} \mid n \in N\}\big) \cap \big(QF(kX) - Z\big) \\ &= QF(kX) - Z. \end{aligned}$$

Hence $\vee\{\lambda_{Z'_n} \mid n \in N\} \supseteq QF(kX) - Z \supseteq \cup\{\lambda_{Z'_n} \mid n \in N\}$. Thus $\wedge\{\lambda_{Z_n} \mid n \in N\} = cl_{QF(kX)}\big(int_{QF(kX)}(Z)\big)$. Note that for any $A \in Z(\beta X)^{\#}$, $\Phi_{QF(kX)}(\lambda_A) = A \cap kX$. By Lemma 2.1,

$$\begin{aligned} &\Phi_{QF(kX)}\big(cl_{QF(kX)}\big(int_{QF(kX)}(Z)\big)\big) \\ &= \Phi_{QF(kX)}\big(\wedge\{\lambda_{Z_n} \mid n \in N\}\big) \\ &= \wedge\{\Phi_{QF(kX)}(\lambda_{Z_n}) \mid n \in N\} \\ &= \wedge\{Z_n \cap kX \mid n \in N\}. \end{aligned}$$

and hence $\Phi_{QF(kX)}\big(cl_{QF(kX)}\big(int_{QF(kX)}(Z)\big)\big) \in Z(kX)^{\#}$. Thus $\Phi_{QF(kX)}$ is a $z^{\#}$-irreducible map. □

Let X be a space. Then $\beta QF(X) = QF(\beta X)$ if and only if Φ_X is $z^{\#}$-irreducible ([5]). Using this, we have the following :

Corollary 3.3. *For any space, $QF(\beta X) = \beta QF(kX)$.*

Lemma 3.4. ([10]) *For any continuous map $f : X \to Y$, there is a unique continuous map $f^k : kX \to kY$ such that $f^k \circ k_X = k_Y \circ f$.*

Let X be a space. Then there is a covering map $h : \beta QF(X) \to QF(\beta X)$ such that $\Phi_\beta \circ h \circ \beta_{QF(X)} = \beta_X \circ \Phi_X$. By Lemma 3.4, there is a continuous map $\Phi^k_X : kQF(X) \to kX$ such that $\Phi^k_X \circ k_{QF(X)} = k_X \circ \Phi_X$. Since $\Phi_\beta^{-1}(kX) = QF(kX)$, there is a continuous map $t_X : kQF(X) \to QF(kX)$ such that $j \circ t_X = h \circ \beta_{kQF(X)}$ and $\Phi_{QF(kX)} \circ t_X = \Phi^k_X$, where $j : QF(kX) \to QF(\beta X)$ is a dense embedding. If t_X is a homeomorphism, then we write $kQF(X) = QF(kX)$.

Corollary 3.5. *Let X be a space. If $kQF(X) = QF(kX)$, then $\beta QF(X) = QF(\beta X)$.*

Proof. Since $t_X : kQF(X) \to QF(kX)$ is a homeomorphism and $\Phi_{kX} : QF(kX) \to kX$ is $z^{\#}$-irreducible, $\Phi^k_X : kQF(X) \to kX$ is $z^{\#}$-irreducible. Take any zero-set Z in $\beta QF(X)$. Then, by Lemma 2.1, $cl_{\beta QF(X)}\big(int_{\beta QF(X)}(Z)\big) \cap kQF(X) \in Z(kQF(X))^{\#}$ and

$$\begin{aligned}&\Phi^k_X\big(cl_{\beta QF(X)}\big(int_{\beta QF(X)}(Z)\big) \cap k\Lambda X\big)\\ &= \Phi_\beta\big(h\big(cl_{\beta QF(X)}\big(int_{\beta QF(X)}(Z)\big)\big)\big) \cap kX \in Z(kX)^{\#}.\end{aligned}$$

By Lemma 2.1, $\Phi_\beta\big(h\big(cl_{\beta QF(X)}\big(int_{\beta QF(X)}(Z)\big)\big)\big) \in Z(\beta X)^{\#}$ and so $\Phi_\beta \circ h$ is a $z^{\#}$-irreducible map. Proposition 2.4, $h : \beta QF(X) \to QF(\beta X)$ is a $z^{\#}$-irreducible map. Since $\beta QF(X)$ and $QF(\beta X)$ are quasi-F spaces, h is a homeomorphism. □

Let X be a space such that $\beta QF(X) = QF(\beta X)$. By Corollary 3.3, there is a homeomorphism $m_X : \beta QF(X) \to \beta QF(kX)$ such that $\beta_{QF(kX)} \circ t_X = m_X \circ \beta_{kQF(X)}$. Since $m_X \circ \beta_{kQF(X)}$ is an embedding, t_X is an embedding.

A subspace X of a space Y is called *C^*-embedded in Y* if for any real-valued continuous map $f : X \to R$, there is a continuous map $g : Y \to R$ such that $g|_X = f$. For any space X, X is C^*-embedded in βX and if $X \supseteq Y \supseteq W \supseteq \beta X$, then Y is C^*-embedded in W ([2]). Hence we have the following

Corollary 3.6. *Let X be a space such that $\beta QF(X) = QF(\beta X)$. Then $kQF(X)$ is a C^*-embedded subspace of $QF(kX)$.*

Theorem 3.7. *Let X be a space.Then the following are equivalent :*

(1) *$kQF(X) = QF(kX)$,*
(2) *t_X is an onto map and $\beta QF(X) = QF(\beta X)$, and*
(3) *Φ^k_X is an onto map and $\beta QF(X) = QF(\beta X)$.*

Proof. (1) ⇒ (2) By Corollary 3.5, it is trivial.
(2) ⇒ (3) Since Φ_X and t_X are onto maps, Φ^k_X is an onto map.
(3) ⇒ (1) Let $f = \Phi^k_X$. Take any $x \in kX$. Since f is an onto map and Φ_X is a covering map, $f\big(kQF(X) - QF(X)\big) = kX - X$([8]). Since $\beta_{kX} \circ f = \Phi_\beta \circ h \circ \beta_{kQF(X)}$, $f^{-1}(x) = (\Phi_\beta \circ h)^{-1}(x) = \phi^{-1}_\beta(X) \subseteq kQF(X) - QF(X)$. Since $\Phi_\beta \circ h$ is a covering map, $f^{-1}(x)$ is a compact subset of $kQF(X)$ and hence f is a compact map. By Corollary 3.6, $f^{-1}(x) = \Phi^{-1}_\beta(x) \subseteq QF(kX)$.

Let F be a closed set in $kQF(X)$ and $x \in kX - f(F)$. Then $f^{-1}(x) \cap F = \emptyset$. Since $f^{-1}(x)$ is compact, there are $A, B \in Z(\beta X)^{\#}$ such that $f^{-1}(x) \subseteq \Sigma_A$, $F \subseteq \Sigma_B$ and $A \cap B = \emptyset$. Since $\Phi_\beta(\Sigma_B) = B$ and

$\Phi_\beta^{-1}(x) \cap \Sigma_B = f^{-1}(x) \cap \Sigma_B = \emptyset$, $x \notin B$. Since $cl_{kX}(f(F)) \subseteq B$, $x \notin cl_{kX}(f(F))$. Thus f is a closed map and so f is a perfect map.

Since $m_X \circ \Phi_\beta \circ \beta_{kQF(X)} = \beta_{kX} \circ \Phi_X^k$ and $m_X \circ \Phi_\beta$ is a covering map, Φ_X^k is a covering map. Since $kQF(X)$ is a quasi-F space, there is a covering map $l : kQF(X) \to QF(kX)$ such that $\Phi_{QF(kX)} \circ l = \Phi_X^k$. Since $QF(X) = \Phi_\beta^{-1}(X)$ and $QF(kX) = \Phi_\beta^{-1}(kX)$, $l \circ k_{QF(X)} = t_X \circ k_{QF(X)}$. Since $k_{QF(X)}$ is a dense embedding, $l = t_X$ is a homeomorphism. $\square$

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