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DECOMPOSITION OF THE RANDOM VARIABLE WHOSE DISTRIBUTION IS THE RIESZ-NÁGY-TAKÁCS DISTRIBUTION

In Soo Baek*

Abstract. We give a series of discrete random variables which converges to a random variable whose distribution function is the Riesz-Nágy-Takács (RNT) distribution. We show this using the correspondence theorem that if the moments coincide then their corresponding distribution functions also coincide.

1. Introduction

Usually the random variables are classified into the discrete random variables and the absolutely continuous random variables. The discrete random variable gives a jump function as its distribution function while the absolutely continuous random variable gives an absolutely continuous function as its distribution. The absolutely continuous function has its derivative almost everywhere in the Lebesgue measure sense. More precisely,

$$P(X \in B) = \sum_{x \in B} p(x),$$

where $p(x) = P(X = x)$ if X is a discrete random variable while

$$P(X \in B) = \int_{x \in B} f(x)dx,$$

where $f(x) = F'(x)$ and $F(x) = P(X \leq x)$ if X is an absolutely continuous random variable. We note that the singular function is a continuous strictly increasing function but its derivative is zero almost everywhere

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in the Lebesgue measure sense. If a random variable has the singular function as its distribution function, it is neither discrete nor absolutely continuous. More precisely, if $F(x) = P(X \leq x)$ is the singular function, then

$$P(a < X \leq b) = \sum_{a<x\leq b} p(x) = 0$$

if we assume that X is discrete, and

$$P(a < X \leq b) = \int_{x\in(a,b]} f(x)dx = 0$$

if we assume that X is absolutely continuous, whereas

$$P(a < X \leq b) = F(b) - F(a) > 0.$$

The Riesz-Nágy-Takács (RNT) function is such a singular function. It is well-known that the distribution function is decomposed into 3 parts, namely discrete and absolutely continuous and singular parts. More precisely,

$$F(x) = F_d(x) + F_a(x) + F_s(x),$$

where F_d is a jump function and F_a is an absolutely continuous function and F_s is a singular function. They also give the corresponding measure $\mu_F, \mu_{F_d}, \mu_{F_a}, \mu_{F_s}$ defined by $\mu_F((a,b]) = F(b) - F(a)$ etc. The Lebesgue decomposition theorem argues that $\mu_F = \mu_0 + \mu_1$, where $\mu_0 \ll \lambda$ and $\mu_1 \perp \lambda$ for Lebesgue measure λ. In fact, $\mu_0 = \mu_{F_a}$ and $\mu_1 = \mu_{F_d} + \mu_{F_s}$.

In this paper, we show that the random variable having the RNT function as its distribution is the limit of a series whose terms are discrete random variables. More precisely, the random variable Y whose distribution is the RNT function F, that is $F(x) = P(Y \leq x)$ satisfies $Y = \lim_{n\to\infty} Y_n$, where $Y_n = \sum_{k=1}^{n} X_k$ for discrete random variables X_k. We construct X_k according to the moment generating function $M_n(z) \equiv M_{Y_n}(z) = E(e^{zY_n})$.

2. Preliminaries

Consider $a \in (0,1)$ and $t \in (0,1)$. The Riesz-Nágy-Takács (RNT) distribution $F = F_{a,t}$ is defined on $[0,1]$ by $F(0) = 0$ and

$$F(x) = \sum_{j\geq 1} t^{m_j} \left(\frac{1-t}{t}\right)^{j-1}, \quad x \in (0,1] \tag{2.1}$$

where we represent $x \in (0,1]$ in the form

$$x = \sum_{j\geq 1} a^{m_j} \left(\frac{1-a}{a}\right)^{j-1} \tag{2.2}$$

for some integers m_i with $1 \leq m_1 < m_2 < \cdots < m_n < \cdots$. We note that if $a \neq t$, then $F_{a,t}$ is a singular function, whereas it reduces to the identity function for $a = t$.

PROPOSITION 2.1. ([2]) The n-th moment c_n of the RNT distribution $F = F_{a,t}$ satisfies the recurrence relation

$$[1 - ta^n - (1-t)(1-a)^n]c_n = (1-t)\sum_{j=0}^{n-1} \binom{n}{j} a^{n-j}(1-a)^j c_j \tag{2.3}$$

for $n \geq 1$, with $c_0 = 1$.

3. Main results

Generating functions of one type or another are a standard device in tackling recurrence relations. Thus, on introducing the moment exponential generating function $C(z)$ given by

$$C(z) = \sum_{n\geq 0} c_n z^n/n! \tag{3.1}$$

we can convert (2.3) to a functional equation for $C(z)$:

$$C(z) - tC(az) = (1-t)C((1-a)z)e^{az}. \tag{3.2}$$

While this too may be of limited value in terms of deriving an explicit solution, it is sometimes possible to extract information on asymptotic behaviour from functional equations.

THEOREM 3.1.

$$C(z) - tC(az) = (1-t)C((1-a)z)e^{az},$$

with

$$C(z) = \sum_{n\geq 0} c_n z^n/n!,$$

where c_n *satisfy* (2.3) *for* $n \geq 1$, *with* $c_0 = 1$.

Proof. From the above Theorem, we have

$$c_n = \frac{1-t}{1-ta^n-(1-t)(1-a)^n}\sum_{j=0}^{n-1}\binom{n}{j}a^{n-j}(1-a)^j c_j. \tag{3.3}$$

Noting $C((1-a)z) = \sum_{n\geq 0} c_n(1-a)^n z^n/n!$ and $e^{az} = \sum_{n\geq 0} a^n z^n/n!$, we have $e^{az}C((1-a)z) = \sum_{n\geq 0} b_n z^n$, where

$$b_n = \sum_{j=0}^{n} \frac{c_j(1-a)^j a^{n-j}}{j!(n-j)!}.$$

Noting $C(az) = \sum_{n\geq 0} c_n a^n z^n/n!$ from (3.1), and from (3.3)

$$\begin{aligned} b_n &= \frac{c_n(1-a)^n}{n!} + \sum_{j=0}^{n-1}\frac{c_j(1-a)^j a^{n-j}}{j!(n-j)!}\times\frac{n!}{n!} \\ &= \frac{c_n(1-a)^n}{n!} + \frac{c_n}{n!}\times\frac{1-ta^n-(1-t)(1-a)^n}{1-t} \\ &= \frac{c_n}{n!}\times\frac{1-ta^n}{1-t}, \end{aligned}$$

we have

$$e^{az}C((1-a)z) = \frac{1}{1-t}C(z) - \frac{t}{1-t}C(az).$$

Hence it follows. □

Corollary 3.2. *For $a = \frac{1}{2}$,*

$$C(z) = \prod_{n=1}^{\infty}(t+(1-t)e^{z/2^n}). \tag{3.4}$$

Proof. When $a = \frac{1}{2}$, we have

$$C(z) = tC(az) + (1-t)C((1-a)z)e^{az} = (t+(1-t)e^{z/2})C(\frac{z}{2}).$$

By induction with $C(0) = 1$, it follows. □

From now on, we [1] consider the self-similar measure $\gamma = \gamma_p$ on the self-similar attractor $[0, 1]$ having the n-th cylinder $c_n(\pi(\omega))$ for $\pi(\omega) \in [0, 1] = \pi(\Omega)$, where $\Omega = \{0, 1\}^{\mathbb{N}}$.

Lemma 3.3. *Let $\Omega = \{0, 1\}^{\mathbb{N}}$. Let $l_n(\omega)$ be the left end point of $c_n(\pi(\omega))$. Let $Y_n(\omega) = X_1(\omega) + ... + X_n(\omega)$ with $Y_n(\omega) = l_n(\omega)$. Then $X_n(\omega) = l_n(\omega) - l_{n-1}(\omega)$ for the integer $n \geq 2$ with $X_1(\omega) = l_1(\omega)$.*

Proof. It follows from the construction of $Y_n(\omega)$. □

Theorem 3.4. *For $a \neq b(=1-a)$, we have*

$$X_n(\Omega) = \{0, a^n, a^{n-1}b, a^{n-2}b^2, ..., ab^{n-1}\},$$

with $P(X_n = 0) = p$, $P(X_n = a^{n-k}b^k) = \binom{n-1}{k}p^{n-k-1}q^{k+1}$ for $0 \leq k \leq n-1$ with $q = 1-p$. Further for $d_k = l_k(\omega) - l_{k-1}(\omega)$, where $k \in \{1, ..., n\}$

$$P(X_1 = x_1, ..., X_n = x_n) = \begin{cases} \gamma(c_n(\pi(\omega))), & \text{if} \quad x_k = d_k \\ 0 & \text{otherwise} \end{cases}. \tag{3.5}$$

Proof. It follows from the definition of the self-similar measure $\gamma = \gamma_p$ on the self-similar attractor $[0,1] = \pi(\Omega)$. □

Theorem 3.5.

$$M_n(z) = \sum P(X_1 = x_1, ..., X_n = x_n)e^{z(x_1 + ... + x_n)}. \tag{3.6}$$

Further

$$M_n(z) \to C(z).$$

Proof. For $p = t$, (3.2) gives

$$\begin{aligned} C(z) &= tC(az) + (1-t)e^{az}C((1-a)z) \\ &= t(tC(a^2z) + (1-t)e^{a^2z}C((1-a)az) \\ &\quad +(1-t)e^{az}[tC(a(1-a)z) + (1-t)e^{a(1-a)z}C((1-a)^2z)] \\ &= ... \end{aligned}$$

with $\lim_{n\to\infty} C(a^kb^{n-k}z) = C(0) = 1$ for all $z \in [0,1]$. It follows from (3.5) and the above fact. □

Theorem 3.6.

$$Y_n \to Y,$$

where Y has its distribution as $F = F_{a,p}$ which is the RNT function.

Proof. It follows from the correspondence theorem with (3.6). □

Theorem 3.7. *For $a = \frac{1}{2}$, consider independent $X_1, ..., X_n$ satisfying $P(X_k = 0) = q$ and $P(X_k = 1/2^k) = p$, where $1 \leq k \leq n$ for every positive integer n . In this case, $Y = \lim_{n\to\infty} Y_n$, where $Y_n(\omega) = X_1(\omega) + ... + X_n(\omega)$ has its distribution as $F_{1/2,q}$ which is the RNT function.*

Proof. It follows from the monotone convergence theorem and the correspondence theorem from (3.4). □

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Department of Mathematics
Busan University of Foreign Studies
Busan 608-738, Republic of Korea
E-mail: isbaek@bufs.ac.kr