THE ONE-SIDED QUADRANGULAR FUZZY SETS

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ABSTRACT. We define one-sided quadrangular fuzzy sets, a left quadrangular fuzzy set and a right quadrangular fuzzy set. And then we generalize the results of addition, subtraction, multiplication, and division based on the Zadeh's extension principle for two one-sided quadrangular fuzzy sets. In addtion, we find the condition that the result of addition or subtraction for two one-sided quadrangular fuzzy sets becomes a triangular fuzzy number.

1. Introduction

We can already find so many generalized results of four operations based on Zadeh's extension principle for two triangular fuzzy numbers, two generalized triangular fuzzy sets, and two generalized trapezoidal fuzzy sets([1], [2]). Nevertheless, there are not the results of quadrangular fuzzy sets.

In this paper, we consider the results of operations for two quadrangular fuzzy sets, in particular for two one-sided quadrangular fuzzy sets. Thus we define one-sided quadrangular fuzzy sets and generalize the resutls of four operations for them. The four operations for two one-sided quadrangular fuzzy sets are based on the Zadeh's extension principle just like our other papers([1], [2]). The definition of the operations for two fuzzy sets, (A, μ_A) and (B, μ_B), is given in definition 2.4 once again.

Meanwhile, we define a pentagonal fuzzy number in definition 2.3 to generalize the results of four operations for two one-sided quadrangular fuzzy sets. In addtion, we look for the conditions that the result of addition or subtraction for two one-sided quadrangular fuzzy sets becomes a triangular fuzzy number, and provide some examples for them.

2. Preliminaries

We use the following definition 2.1. which is already defined([2]) to generalize the results of four operations for two fuzzy sets.

DEFINITION 2.1. The set $A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\}$ is said to be the α-*cut* of a fuzzy set A.

The following definition of a triangular fuzzy number is already used in a few of papers([1], [2]).

DEFINITION 2.2. A *triangular fuzzy number* is a fuzzy set $A = (a_1, a_2, a_3)$ having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_3 \leq x \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x < a_3. \end{cases}$$

We define a pentagonal fuzzy number to generalize the results of this paper as follows.

DEFINITION 2.3. A *pentagonal fuzzy number* is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_5 \leq x \\ \frac{x-a_1}{2(a_2-a_1)}, & a_1 \leq x < a_2 \\ \frac{x-a_3}{2(a_3-a_2)} + 1, & a_2 \leq x < a_3 \\ \frac{a_3-x}{2(a_4-a_3)} + 1, & a_3 \leq x < a_4 \\ \frac{a_5-x}{2(a_5-a_4)}, & a_4 \leq x < a_5 \end{cases}$$

where $a_i \in \mathbb{R}, i = 1, 2, 3, 4, 5$, $a_1 + a_3 \neq 2a_2$, and $a_3 + a_5 \neq 2a_4$. This pentagonal fuzzy number is denoted by $A = (a_1, a_2, a_3, a_4, a_5)$.

The following definition of four operations for two fuzzy sets is based on the Zadeh's extention principle([3], [4], [5]).

DEFINITION 2.4. The addition, subtraction, multiplication, and division of two fuzzy sets, A and B, are defined as
  1. $A(+)B : \mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B$
  2. $A(-)B : \mu_{A(-)B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B$
  3. $A(\cdot)B : \mu_{A(\cdot)B}(z) = \sup_{z=xy} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B$
  4. $A(/)B : \mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$

We define a one-sided quadrangular fuzzy set and generalize about four operations for two left(right) quadrangular fuzzy sets in the following section 3. In particular, we investigate, given a left quadrangular fuzzy set A, there is a left quadrangular fuzzy set B such that $A(+)B$ becomes a triangular fuzzy number. Similarly, we show, given a right quadrangular fuzzy set A, there is a left quadrangular fuzzy set B such that $A(-)B$ becomes a triangular fuzzy number.

3. The one-sided quadrangular fuzzy sets

DEFINITION 3.1. A left quadrangular fuzzy set is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_4 \leq x \\ \frac{x-a_1}{2(a_2-a_1)}, & a_1 \leq x < a_2 \\ \frac{x-a_3}{2(a_3-a_2)} + 1, & a_2 \leq x < a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 \leq x < a_4 \end{cases}$$

where $a_i \in \mathbb{R}, i = 1, 2, 3, 4$ and $a_1 + a_3 \neq 2a_2$. This left quadrangular fuzzy set is denoted by $A = (a_1, a_2^*, a_3, a_4)$.

REMARK 3.2. In particular, in order for a left quadrangular A to be a triangular fuzzy number, it is necessary that $2a_2 = a_1 + a_3$ should be met.

Proof. In order for A to be a triangular fuzzy number, the slope of $\mu_A(x)$ in $[a_1, a_2]$ should be same to the slope in $[a_2, a_3]$. It means that $\frac{1}{2(a_2-a_1)} = \frac{1}{2(a_3-a_2)}$. Therefore, it is $2a_2 = a_1 + a_3$. □

DEFINITION 3.3. A right quadrangular fuzzy set is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_4 \leq x \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2 \\ \frac{a_2-x}{2(a_3-a_2)} + 1, & a_2 \leq x < a_3 \\ \frac{a_4-x}{2(a_4-a_3)}, & a_3 \leq x < a_4 \end{cases}$$

where $a_i \in \mathbb{R}, i = 1, 2, 3, 4$ and $a_2 + a_4 \neq 2a_3$. This right quadrangular fuzzy set is denoted by $A = (a_1, a_2, a_3^*, a_4)$.

REMARK 3.4. In particular, in order for a right quadrangular A to be a triangular fuzzy number, it is necessary that $2a_3 = a_2 + a_4$ should be met.

Firstly, we generalize the results of addition, subtraction, multiplication, and division for two left quadrangular fuzzy sets in the following thm 3.5.

THEOREM 3.5. *For two left quadrangular fuzzy sets, $A = (a_1, a_2^*, a_3, a_4)$ and $B = (b_1, b_2^*, b_3, b_4)$, the results of four operations are as follows;*

1. *$A(+)B$ is a left quadrangular fuzzy set or a triangular fuzzy number.*
2. *$A(-)B$ is a pentagonal fuzzy number.*
3. *$A(\cdot)B$ is a fuzzy set on $(a_1 b_1, a_4 b_4)$, but doesn't need to be a left quadrangular fuzzy set or a pentagonal fuzzy number.*
4. *$A(/)B$ is a fuzzy set on $\left(\frac{a_1}{b_4}, \frac{a_4}{b_1}\right)$, but doesn't need to be a left quadrangular fuzzy set or a pentagonal fuzzy number.*

Proof. Note that

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_4 \le x \\ \frac{x - a_1}{2(a_2 - a_1)}, & a_1 \le x < a_2 \\ \frac{x - a_3}{2(a_3 - a_2)} + 1, & a_2 \le x < a_3 \\ \frac{a_4 - x}{a_4 - a_3}, & a_3 \le x < a_4 \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < b_1, b_4 \le x \\ \frac{x - b_1}{2(b_2 - b_1)}, & b_1 \le x < b_2 \\ \frac{x - b_3}{2(b_3 - b_2)} + 1, & b_2 \le x < b_3 \\ \frac{b_4 - x}{b_4 - b_3}, & b_3 \le x < b_4 \end{cases}$$

Four operations are calculated by using α-cuts. Let $A_\alpha = \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right]$ and $B_\alpha = \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right]$ are the α-cuts of A and B. We have to consider α-cuts for two cases, $0 < \alpha \le \frac{1}{2}$ and $\frac{1}{2} < \alpha \le 1$. First, $\alpha = \dfrac{a_1^{(\alpha)} - a_1}{2(a_2 - a_1)}$ and $\alpha = \dfrac{a_4 - a_2^{(\alpha)}}{2(a_4 - a_3)}$ for $0 < \alpha \le \frac{1}{2}$. Thus,

$$A_\alpha = \left[2\alpha(a_2 - a_1) + a_1, \alpha(a_3 - a_4) + a_4\right]$$

and

$$B_\alpha = \left[2\alpha(b_2 - b_1) + b_1, \alpha(b_3 - b_4) + b_4\right].$$

Secondly, $\alpha = \dfrac{a_2 - a_1^{(\alpha)}}{2(a_3 - a_2)} + 1$ and $\alpha = \dfrac{a_4 - a_2^{(\alpha)}}{2(a_4 - a_3)}$ for $\frac{1}{2} < \alpha \le 1$. Thus,

$$A_\alpha = \left[2(\alpha - 1)(a_3 - a_2) + a_3, \alpha(a_3 - a_4) + a_4 \right]$$

and

$$B_\alpha = \left[2(\alpha - 1)(b_3 - b_2) + b_3, \alpha(b_3 - b_4) + b_4 \right].$$

1. Addition : For $0 < \alpha \leq \frac{1}{2}$,

$$
\begin{aligned}
A_\alpha(+)B_\alpha &= \left[a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)} \right] \\
&= \left[2\alpha(a_2 - a_1) + a_1 + 2\alpha(b_2 - b_1) + b_1, \right. \\
&\qquad \left. 2(\alpha - 1)(a_3 - a_2) + a_3 + 2(\alpha - 1)(b_3 - b_2) + b_3 \right]
\end{aligned}
$$

and for $\frac{1}{2} < \alpha \leq 1$,

$$
\begin{aligned}
A_\alpha(+)B_\alpha &= \left[2(\alpha - 1)(a_3 - a_2) + a_3 + 2(\alpha - 1)(b_3 - b_2) + b_3, \right. \\
&\qquad \left. \alpha(a_3 - a_4) + a_4 + \alpha(b_3 - b_4) + b_4 \right]
\end{aligned}
$$

If $x \in \left[a_1 + b_1, a_2 + b_2 \right]$, then $x = 2\alpha(a_2 - a_1) + a_1 + 2\alpha(b_2 - b_1) + b_1$. Thus $\alpha = \frac{x - a_1 - b_1}{2(a_2 + b_2 - a_1 - b_1)}$. Similarly, we obtain $\alpha = \frac{x - a_3 - b_3}{2(a_3 + b_3 - a_2 - b_2)} + 1$ for $x \in \left[a_2 + b_2, a_3 + b_3 \right]$ and $\alpha = \frac{a_4 + b_4 - x}{a_4 + b_4 - a_3 - b_3}$ for $x \in \left[a_3 + b_3, a_4 + b_4 \right]$. Thus

$$
\mu_{A(+)B}(x) = \begin{cases}
0, & x < a_1 + b_1, a_4 + b_4 \leq x \\
\frac{x - a_1 - b_1}{2(a_2 + b_2 - a_1 - b_1)}, & a_1 + b_1 \leq x < a_2 + b_2 \\
\frac{x - a_3 - b_3}{2(a_3 + b_3 - a_2 - b_2)} + 1, & a_2 + b_2 \leq x < a_3 + b_3 \\
\frac{a_4 + b_4 - x}{a_4 + b_4 - a_3 - b_3}, & a_3 + b_3 \leq x < a_4 + b_4.
\end{cases}
$$

It generally satisfies $a_1 + b_1 + a_3 + b_3 \neq 2(a_2 + b_2)$ because of $a_1 + a_3 \neq 2a_2$ and $b_1 + b_3 \neq 2b_2$, but it can satisfy $a_1 + b_1 + a_3 + b_3 = 2(a_2 + b_2)$. If $a_1 + b_1 + a_3 + b_3 \neq 2(a_2 + b_2)$ then $A(+)B = \left(a_1 + b_1, (a_2 + b_2)^*, a_3 + b_3, a_4 + b_4 \right)$ and if $a_1 + b_1 + a_3 + b_3 = 2(a_2 + b_2)$ then $A(+)B = (a_1 + b_1, a_3 + b_3, a_4 + b_4)$. Therefore, $A(+)B$ is a left quadrangular fuzzy set or a triangular fuzzy number.

2. Subtraction : For $0 < \alpha \leq \frac{1}{2}$,

$$
\begin{aligned}
A_\alpha(-)B_\alpha &= \left[a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)} \right] \\
&= \left[2\alpha(a_2 - a_1) + a_1 - \alpha(b_3 - b_4) - b_4, \right. \\
&\qquad \left. \alpha(a_3 - a_4) + a_4 - 2\alpha(b_2 - b_1) - b_1 \right]
\end{aligned}
$$

and for $\frac{1}{2} < \alpha \le 1$,

$$A_\alpha(-)B_\alpha = \big[2(\alpha - 1)(a_3 - a_2) + a_3 - \alpha(b_3 - b_4) - b_4,$$
$$\alpha(a_3 - a_4) + a_4 - 2(\alpha - 1)(b_3 - b_2) - b_3\big].$$

If $x \in [a_1 - b_4, a_2 - \frac{b_3 + b_4}{2}]$, then $x = 2\alpha(a_2 - a_1) + a_1 - \alpha(b_3 - b_4) - b_4$ and $\alpha = \dfrac{x - a_1 + b_4}{2a_2 - 2a_1 + b_4 - b_3}$. Similarly, we have $\alpha = \dfrac{x - a_3 + b_3}{2a_3 - 2a_2 + b_4 - b_3} + 1$ for $x \in [a_2 - \frac{b_3 + b_4}{2}, a_3 - b_3]$, $\alpha = \dfrac{a_3 - b_3 - x}{a_4 - a_3 + 2b_3 - 2b_2} + 1$ for $x \in [a_3 - b_3, \frac{a_3 + a_4}{2} - b_2]$, and $\alpha = \dfrac{a_4 - b_1 - x}{a_4 - a_3 + 2b_2 - 2b_1}$ for $x \in [\frac{a_3 + a_4}{2} - b_2, a_4 - b_1]$.
Thus

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < a_1 - b_4, a_4 - b_1 \le x \\ \frac{x - a_1 + b_4}{2a_2 - 2a_1 + b_4 - b_3}, & a_1 - b_4 \le x < a_2 - \frac{b_3 + b_4}{2} \\ \frac{x - a_3 + b_3}{2a_3 - 2a_2 - b_3 + b_4} + 1, & a_2 - \frac{b_3 + b_4}{2} \le x < a_3 - b_3 \\ \frac{a_3 - b_3 - x}{a_4 - a_3 + 2b_3 - 2b_2} + 1, & a_3 - b_3 \le x < \frac{a_3 + a_4}{2} - b_2 \\ \frac{a_4 - b_1 - x}{a_4 - a_3 + 2b_2 - 2b_1}, & \frac{a_3 + a_4}{2} - b_2 \le x < a_4 - b_1. \end{cases}$$

Also, it satisfies that $2(a_2 - \frac{b_3 + b_4}{2}) \ne a_1 - b_4 + a_3 - b_3$ and $2(\frac{a_3 + a_4}{2} - b_2) \ne a_3 - b_3 + a_4 - b_1$ because of $2a_2 \ne a_1 + a_3$ and $2b_2 \ne b_1 + b_3$. Therefore, $A(-)B$ is a pentagonal fuzzy number and it is denoted by $A(-)B = (a_1 - b_4, a_2 - \frac{b_3 + b_4}{2}, a_3 - b_3, \frac{a_3 + a_4}{2} - b_2, a_4 - b_1)$.

3. Multiplication : For $0 < \alpha \le \frac{1}{2}$,

$$A_\alpha(\cdot)B_\alpha = \big[a_1^{(\alpha)} b_1^{(\alpha)}, a_2^{(\alpha)} b_2^{(\alpha)}\big]$$
$$= \big[\{2\alpha(a_2 - a_1) + a_1\}\{2\alpha(b_2 - b_1) + b_1\},$$
$$\{\alpha(a_3 - a_4) + a_4\}\{\alpha(b_3 - b_4) + b_4\}\big]$$

and for $\frac{1}{2} < \alpha \le 1$,

$$A_\alpha(\cdot)B_\alpha = \big[\{2(\alpha - 1)(a_3 - a_2) + a_3\}\{2(\alpha - 1)(b_3 - b_2) + b_3\},$$
$$\{\alpha(a_3 - a_4) + a_4\}\{\alpha(b_3 - b_4) + b_4\}\big].$$

If $x \in [a_1 b_1, a_2 b_2]$, then $x = \{2\alpha(a_2 - a_1) + a_1\}\{2\alpha(b_2 - b_1) + b_1\}$. Thus $\alpha = \frac{1}{4pq}\big(-b_1 p - a_1 q + \sqrt{(b_1 p + a_1 q)^2 - 4pq(a_1 b_1 - x)}\big)$ where $p = a_2 - a_1$ and $q = b_2 - b_1$. Similarly, we obtain $\alpha = \frac{1}{4rs}\big(4rs - b_3 r - a_3 s + \sqrt{(4rs - b_3 r - a_3 s)^2 - 4rs(4rs - 2b_3 r - 2a_3 s + a_3 b_3 - x)}\big)$ where $r = a_3 - a_2$ and $s = b_3 - b_2$ for $x \in [a_2 b_2, a_3 b_3]$ and $\alpha = \frac{1}{4tu}\big(-b_4 t - $

$a_4u - \sqrt{(b_4t + a_4u)^2 - 4tu(a_4b_4 - x)}\big)$ where $t = a_3 - a_4$ and $s = b_3 - b_4$ for $x \in \big[a_3b_3, a_4b_4\big]$. Therefore, the membership function $\mu_{A(\cdot)B}(x)$ of $A(\cdot)B$ is

$$
\begin{cases}
0, & x < a_1b_1, \ \ a_4b_4 \leq x \\
\frac{1}{4pq}\big(-b_1p - a_1q\big) + \sqrt{(b_1p + a_1q)^2 - 4pq(a_1b_1 - x)}\big), & a_1b_1 \leq x < a_2b_2 \\
\frac{1}{4rs}\big(\sqrt{(4rs - b_3r - a_3s)^2 - 4rs(4rs - 2b_3r - 2a_3s + a_3b_3 - x)} \\
\qquad +4rs - b_3r - a_3s\big), & a_2b_3 \leq x < a_3b_3 \\
\frac{1}{4tu}\big(-b_4t - a_4u - \sqrt{(b_4t + a_4u)^2 - 4tu(a_4b_4 - x)}\big), & a_3b_3 \leq x < a_4b_4
\end{cases}
$$

where $p = a_2 - a_1$, $q = b_2 - b_1$, $r = a_3 - a_2$, $s = b_3 - b_2$, $t = a_3 - a_4$, and $s = b_3 - b_4$. Hence $A(\cdot)B$ is a fuzzy set on (a_1b_1, a_4b_4), but doesn't need to be a left quadrangular fuzzy set or a pentagonal fuzzy number.

4. Division : For $0 < \alpha \leq \frac{1}{2}$,

$$
\begin{aligned}
A_\alpha(/)B_\alpha &= \big[a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}\big] \\
&= \left[\frac{2\alpha(a_2 - a_1) + a_1}{\alpha(b_3 - b_4) + b_4}, \frac{\alpha(a_3 - a_4) + a_4}{2\alpha(b_2 - b_1) + b_1}\right]
\end{aligned}
$$

and for $\frac{1}{2} < \alpha \leq 1$,

$$
A_\alpha(/)B_\alpha = \left[\frac{2(\alpha - 1)(a_3 - a_2) + a_3}{\alpha(b_3 - b_4) + b_4}, \frac{\alpha(a_3 - a_4) + a_4}{2(\alpha - 1)(b_3 - b_2) + b_3}\right].
$$

If $x \in [a_1/b_4, a_2/\frac{b_3+b_4}{2}]$, then $x = \dfrac{2\alpha(a_2 - a_1) + a_1}{\alpha(b_3 - b_4) + b_4}$. Thus

$$
\alpha = \frac{b_4x + a_1}{(b_3 - b_4)x - 2(a_2 - a_1)}.
$$

We also have

$$
\alpha = \frac{b_4x + 2a_2 - a_3}{(b_3 - b_4)x - 2(a_3 - a_2)} \quad for \quad x \in [a_2/\frac{b_3 + b_4}{2}, a_3/b_3],
$$

$$
\alpha = \frac{(3b_3 - b_2)x + a_4}{2(b_3 - b_2)x + a_4 - a_3} \quad for \quad x \in [a_3/b_3, \frac{a_3 + a_4}{2}/b_2],
$$

and

$$
\alpha = \frac{b_1x + a_4}{2(b_2 - b_1)x + a_4 - a_3} \quad for \quad x \in [\frac{a_3 + a_4}{2}/b_2, a_4/b_1].
$$

Therefore

$$
\mu_{A(/)B}(x) = \begin{cases} 0, & x < a_1/b_4, \, a_4/b_1 \leq x \\ \frac{b_4 x + a_1}{(b_3 - b_4)x - 2(a_2 - a_1)}, & a_1/b_4 \leq x < a_2/\frac{b_3 + b_4}{2} \\ \frac{b_4 x + 2a_2 - a_3}{(b_3 - b_4)x - 2(a_3 - a_2)}, & a_2/\frac{b_3 + b_4}{2} \leq x < a_3/b_3 \\ \frac{(3b_3 - b_2)x + a_4}{2(b_3 - b_2)x + a_4 - a_3}, & a_3/b_3 \leq x < \frac{a_3 + a_4}{2}/b_2 \\ \frac{b_1 x + a_4}{2(b_2 - b_1)x + a_4 - a_3}, & \frac{a_3 + a_4}{2}/b_2 \leq x < a_4/b_1. \end{cases}
$$

Hence $A(/)B$ is a fuzzy set on $\left(\frac{a_1}{b_4}, \frac{a_4}{b_1}\right)$, but doesn't need to be a left quadrangular fuzzy set or a pentagonal fuzzy number. $\square$

REMARK 3.6. Given a left quadrangular fuzzy set A, there is a left quadrangular fuzzy set B such that $A(+)B$ becomes a triangular fuzzy number.

Proof. Let $A = (a_1, a_2^*, a_3, a_4)$. We seek a function based on the left quadrangular fuzzy set $B = (b_1, b_2^*, b_3, b_4)$. Since b_1 and b_4 are arbitrary we can randomly take them. Then b_2 and b_3 should satisfy the following relation

$$
\frac{1}{2(a_2 + b_2 - a_1 - b_1)} = \frac{1}{2(a_3 + b_3 - a_2 - b_2)}
$$

Therefore, we have $b_3 = 2a_2 + 2b_2 - a_1 - a_3 - b_1$. $\square$

EXAMPLE 3.7. For $A = (1, 2^*, 4, 5)$, we have $B = (3, 6^*, 8, 9)$ that $A(+)B$ becomes a triangular fuzzy number. Since $b_3 = -a_1 - b_1 + 2a_2 + 2b_2 - a_3$, $A(+)B = (4, 12, 14)$ and

$$
\mu_{A(+)B}(x) = \begin{cases} 0, & x < 4, \, 14 \leq x \\ \frac{x-4}{8}, & 4 \leq x < 12 \\ \frac{14-x}{2}, & 12 \leq x < 14. \end{cases}
$$

Subsequently, we generalize the results of four operations for two right quadrangular fuzzy sets in the following theorem 3.8. For that, let $A = (a_1, a_2, a_3^*, a_4)$ and $B = (b_1, b_2, b_3^*, b_4)$, where $a_i, b_i \in \mathbb{R}, i = 1, 2, 3, 4$, $a_2 + a_4 \neq 2a_3$, and $b_2 + b_4 \neq 2b_3$.

THEOREM 3.8. For two right quadrangular fuzzy sets, $A = (a_1, a_2, a_3^*, a_4)$ and $B = (b_1, b_2, b_3^*, b_4)$, the results of four operations are as follows;

1. $A(+)B$ is a right quadrangular fuzzy set or a triangular fuzzy number.
2. $A(-)B$ is a pentagonal fuzzy number.
3. $A(\cdot)B$ is a fuzzy set on $(a_1 b_1, a_4 b_4)$, but doesn't need to be a right quadrangular fuzzy set or a pentagonal fuzzy number.

4. $A(/)B$ is a fuzzy set on $\left(\frac{a_1}{b_4}, \frac{a_4}{b_1}\right)$, but doesn't need to be a right quadrangular fuzzy set or a pentagonal fuzzy number.

Proof. The proof is similar with Theorem 3.5. □

REMARK 3.9. Given a right quadrangular fuzzy set A, there is a right quadrangular fuzzy set B such that $A(+)B$ becomes a triangular fuzzy number.

Proof. The proof is similar with Remark 3.6. □

We generalize about addition and subtraction for a left quadrangular fuzzy set and a right quadrangular fuzzy set. For that, let $A = (a_1, a_2^*, a_3, a_4)$ and $B = (b_1, b_2, b_3^*, b_4)$, where $a_i, b_i \in \mathbb{R}, i = 1, 2, 3, 4$, $a_1 + a_3 \neq 2a_2$, and $b_2 + b_4 \neq 2b_3$. Meanwhile it's better to consider $B(-)A$ because $A(-)B \neq B(-)A$.

THEOREM 3.10. *For a left quadrangular fuzzy set A and a right quadrangular fuzzy set B, the results of $A(+)B$, $B(+)A$, $A(-)B$, and $B(-)A$ are as follows;*

1. *$A(+)B$ and $B(+)A$ are the pentagonal fuzzy numbers.*
2. *$A(-)B$ is a left quadrangular fuzzy set or a triangular fuzzy number.*
3. *$B(-)A$ is a right quadrangular fuzzy set or a triangular fuzzy number.*

Proof. We calculate addition and subtration using α-cuts as in the preceding. Let $A_\alpha = \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right]$ and $B_\alpha = \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right]$ are the α-cuts of A and B. We use A_α and B_α in Theorem 3.5 for two cases. First, $\alpha = \dfrac{b_1^{(\alpha)} - b_1}{b_2 - b_1}$ and $\alpha = \dfrac{b_4 - b_2^{(\alpha)}}{2(b_4 - b_3)}$ for $0 < \alpha \leq \frac{1}{2}$. Secondly, $\alpha = \dfrac{b_1^{(\alpha)-b_1}}{b_2 - b_1}$ and $\alpha = \dfrac{b_2 - b_2^{(\alpha)}}{2(b_3 - b_2)} + 1$ for $\frac{1}{2} < \alpha \leq 1$. Thus, for $0 < \alpha \leq \frac{1}{2}$,

$$A_\alpha = \left[2\alpha(a_2 - a_1) + a_1, \alpha(a_3 - a_4) + a_4\right],$$

$$B_\alpha = \left[\alpha(b_2 - b_1) + b_1, 2\alpha(b_4 - b_3) + b_4\right]$$

and for $\frac{1}{2} < \alpha \leq 1$,

$$A_\alpha = \left[2(\alpha - 1)(a_3 - a_2) + a_3, \alpha(a_3 - a_4) + a_4\right],$$

$$B_\alpha = \left[\alpha(b_2 - b_1) + b_1, 2(\alpha - 1)(b_3 - b_2) + b_2\right].$$

1. $A(+)B$ and $B(+)A$: For $0 < \alpha \le \frac{1}{2}$,

$$A_\alpha(+)B_\alpha = \big[2\alpha(a_2 - a_1) + a_1 + \alpha(b_2 - b_1) + b_1,$$
$$\alpha(a_3 - a_4) + a_4 - 2\alpha(b_4 - b_3) + b_4\big]$$

and for $\frac{1}{2} < \alpha \le 1$,

$$A_\alpha(+)B_\alpha = \big[2(\alpha - 1)(a_3 - a_2) + a_3 + \alpha(b_2 - b_1) + b_1,$$
$$\alpha(a_3 - a_4) + a_4 - 2(\alpha - 1)(b_3 - b_2) + b_2\big].$$

If $x \in [a_1 + b_1, a_2 + \frac{b_1 + b_2}{2}]$, then $x = 2\alpha(a_2 - a_1) + a_1 + \alpha(b_2 - b_1) + b_1$. Thus $\alpha = \dfrac{x - a_1 - b_1}{2a_2 - 2a_1 + b_2 - b_1}$. Similarly, we have $\alpha = \dfrac{x - a_3 - b_2}{2a_3 - 2a_2 + b_2 - b_1} + 1$ for $x \in [a_2 + \frac{b_1 + b_2}{2}, a_3 + b_2]$, $\alpha = \dfrac{a_3 + b_2 - x}{a_4 - a_3 + 2b_3 - 2b_2} + 1$ for $x \in [a_3 + b_2, \frac{a_3 + a_4}{2} + b_3]$, and $\alpha = \dfrac{a_4 + b_4 - x}{a_4 - a_3 + 2b_4 - 2b_3}$ for $x \in [\frac{a_3 + a_4}{2} + b_3, a_4 + b_4]$. Thus

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < a_1 + b_1, \ a_4 + b_4 \le x \\ \frac{x - a_1 - b_1}{2a_2 - 2a_1 + b_2 - b_1}, & a_1 + b_1 \le x < a_2 + \frac{b_1 + b_2}{2} \\ \frac{x - a_3 - b_2}{2a_3 - 2a_2 + b_2 - b_1} + 1, & a_2 + \frac{b_1 + b_2}{2} \le x < a_3 + b_2 \\ \frac{a_3 + b_2 - x}{a_4 - a_3 + 2b_3 - 2b_2} + 1, & a_3 + b_2 \le x < \frac{a_3 + a_4}{2} + b_3 \\ \frac{a_4 + b_4 - x}{a_4 - a_3 + 2b_4 - 2b_3}, & \frac{a_3 + a_4}{2} + b_3 \le x < a_4 + b_4. \end{cases}$$

Also, it satisfies that $2(a_2 + \frac{b_1 + b_2}{2}) \ne a_1 + b_1 + a_3 + b_2$ and $2(\frac{a_3 + a_4}{2} + b_3) \ne a_3 + b_2 + a_4 + b_4$ because of $2a_2 \ne a_1 + a_3$ and $2b_3 \ne b_2 + b_4$. Therefore, $A(+)B$ is a pentagonal fuzzy number and $A(+)B = (a_1 + b_1, a_2 + \frac{b_1 + b_2}{2}, a_3 + b_2, \frac{a_3 + a_4}{2} + b_3, a_4 + b_4)$. Similiarly, $B(+)A$ is a pentagonal fuzzy number and it is denoted by $B(+)A = (b_1 + a_1, b_2 + \frac{a_1 + a_2}{2}, b_3 + a_2, \frac{b_3 + b_4}{2} + a_3, b_4 + a_4)$.

2. $A(-)B$: For $0 < \alpha \le \frac{1}{2}$,

$$A_\alpha(-)B_\alpha = \big[2\alpha(a_2 - a_1) + a_1 - 2\alpha(b_4 - b_3) - b_4,$$
$$\alpha(a_3 - a_4) + a_4 - \alpha(b_2 - b_1) - b_1\big]$$

and for $\frac{1}{2} < \alpha \le 1$,

$$A_\alpha(-)B_\alpha = \big[2(\alpha - 1)(a_3 - a_2) + a_3 - 2(\alpha - 1)(b_3 - b_2) - b_2,$$
$$\alpha(a_3 - a_4) + a_4 - \alpha(b_2 - b_1) - b_1\big].$$

We have $\alpha = \dfrac{x - a_1 + b_4}{2(a_2 - a_1 + b_4 - b_3)}$ for $x \in \left[a_1 - b_4, a_2 - b_3\right]$,

$\alpha = \dfrac{x - a_3 + b_2}{2(a_3 - a_2 + b_3 - b_2)} + 1$ for $x \in \left[a_2 - b_3, a_3 - b_2\right]$, and

$\alpha = \dfrac{a_4 - b_1 - x}{a_4 - a_3 + b_2 - b_1}$ for $x \in \left[a_3 - b_2, a_4 - b_1\right]$. Thus

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < a_1 - b_4, a_4 - b_1 \leq x \\ \frac{x - a_1 + b_4}{2(a_2 - a_1 + b_4 - b_3)}, & a_1 - b_4 \leq x < a_2 - b_3 \\ \frac{x - a_3 + b_2}{2(a_3 - a_2 + b_3 - b_2)} + 1, & a_2 - b_3 \leq x < a_3 - b_2 \\ \frac{a_4 - b_1 - x}{a_4 - a_3 + b_2 - b_1}, & a_3 - b_2 \leq x < a_4 - b_1. \end{cases}$$

It generally satisfies $a_1 - b_4 + a_3 - b_2 \neq 2(a_2 - b_3)$ because of $a_1 + a_3 \neq 2a_2$ and $b_2 + b_4 \neq 2b_3$, but it can satisfy $a_1 - b_4 + a_3 - b_2 = 2(a_2 - b_3)$. Hence, if $a_1 - b_4 + a_3 - b_2 \neq 2(a_2 - b_3)$ then $A(-)B = \left(a_1 - b_4, (a_2 - b_3)^*, a_3 - b_2, a_4 - b_1\right)$ and if $a_1 - b_4 + a_3 - b_2 = 2(a_2 - b_3)$ then $A(-)B = (a_1 - b_4, a_3 - b_2, a_4 - b_1)$. Therefore, $A(-)B$ is a left quadrangular fuzzy set or a triangular fuzzy number.

3. $B(-)A$: The proof is similar with the proof of $A(-)B$.   □

REMARK 3.11. Given a left quadrangular fuzzy set A, there is a right quadrangular fuzzy set B such that $A(-)B$ becomes a triangular fuzzy number.

Proof. Let $A = (a_1, a_2^*, a_3, a_4)$. We seek a function based on the set $B = (b_1, b_2, b_3^*, b_4)$. Since b_1 and b_4 are arbitrary we can randomly take them. Then b_2 and b_3 should satisfy the following relation

$$\frac{1}{2(a_2 - a_1 + b_4 - b_3)} = \frac{1}{2(a_3 - a_2 + b_3 - b_2)}.$$

Therefore, we have $b_2 = 2b_3 - 2a_2 - b_4 + a_1 + a_3$.   □

EXAMPLE 3.12. For $A = (1, 2^*, 4, 5)$, we have $B = (3, 6, 7^*, 9)$ that $A(-)B$ becomes a triangular fuzzy number. Since $b_2 = 2b_3 - 2a_2 - b_4 + a_1 + a_3$, $A(-)B = (-8, -2, 2)$.

REMARK 3.13. Given a right quadrangular fuzzy number B, there is a left quadrangular fuzzy number A such that $B(-)A$ becomes a triangular fuzzy number.

Proof. The proof is similar with Remark 3.11.   □

EXAMPLE 3.14. For $B = (2, 4, 5^*, 8)$, we have $A = (2, 3^*, 6, 8)$ that $B(-)A = (-6, -2, 6)$ becomes a triangular fuzzy number.

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