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SKEW-SYMMETRIC SOLVENT FOR SOLVING A POLYNOMIAL EIGENVALUE PROBLEM

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Abstract. In this paper a nonlinear matrix equation is considered which has the form

$$P(X) = A_0X^m + A_1X^{m-1} + \cdots + A_{m-1}X + A_m = 0,$$

where X is an $n \times n$ unknown real matrix and A_m, A_{m-1}, $\ldots$, A_0 are $n \times n$ matrices with real elements. Newtons method is applied to find the skew-symmetric solvent of the matrix polynomial $P(X)$. We also suggest an algorithm which converges the skew-symmetric solvent even if the Fréchet derivative of $P(X)$ is singular.

1. Introduction

For solving an m-th order ordinary differential equation which has a form

$$A_0\frac{d^m}{dt^m}x(t) + A_1\frac{d^{m-1}}{dt^{m-1}}x(t) + \cdots + A_{m-1}\frac{d}{dt}x(t) + A_mx(t) = 0,$$

where A_m, A_{m-1}, $\ldots$, A_0 are $n \times n$ real matrices, we need to consider the polynomial eigenvalue problem

$$P(\lambda)v = (\lambda^m A_0 + \lambda^{m-1}A_1 + \cdots + \lambda A_{m-1} + A_m)v = 0. \tag{1.1}$$

For solving the problem (1.1) we may consider the matrix equation

$$P(X) = A_0X^m + A_1X^{m-1} + \cdots + A_{m-1}X + A_m = 0. \tag{1.2}$$

If $m = 2$ the matrix equation (1.1) can be rewritten by

$$Q(\lambda)v = (\lambda^2 A_0 + \lambda A_1 + A_2)v = 0, \tag{1.3}$$

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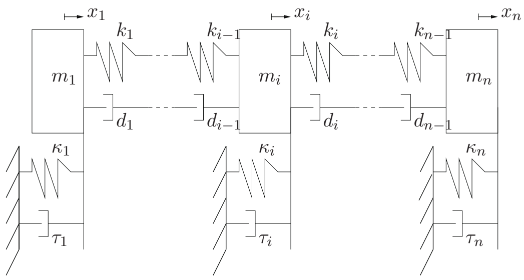

FIGURE 1. An n degree of freedom damped mass-spring system. [9]

which arise from a freedom damped mass-spring system [2]. Figure 1 shows a connected damped mass-spring system. The i-th mass of weight m_i is connected to the $(i+1)$-th mass by a spring with constant k_i and damper with constant d_i, and ground by a spring with constant κ_i and damper constant τ_i.

Mehrmann and Watkins [6] showed that When $A_0 = A_0^T$, $A_1 = -A_1^T$, $A_2 = A_2^T$ in the quadratic eigenvalue problem (1.3), it has a Hamiltonian eigenstructure. An application of finding skew-symmetric solvent of matrix polynomial comes from the polynomial eigenvalue problem (1.1), since any skew-symmetric matrix has a pair of purely imaginary eigenvalues [4], [7]. In this paper we suggest an algorithm for solving skew-symmetric solvent of matrix polynomial.

2. Newton's methods for nonlinear matrix equation

From the Fréchet derivative in Newton's method of the matrix polynomial (1.2), it is necessary to find the solution $H \in \mathbb{C}^{n\times n}$ of the equation

$$P_X(H) = \sum_{i=1}^{m}\left[\left(\sum_{\mu=0}^{m-i} A_\mu X^{m-(\mu+i)}\right) H X^{i-1}\right] = -P(X). \tag{2.1}$$

REMARK 2.1. Recall that P_X is regular if and only if

$$\inf_{\|H\|=1} \|P_X(H)\| > 0.$$

Kratz and Stickel [5] used the Schur algorithm to solve (2.1). For a given $X \in \mathbb{C}^{n\times n}$, compute the Schur decomposition of X

$$Q^* X Q = U \tag{2.2}$$

where Q is unitary and U is upper triangular. Then, substituting (2.2) into (2.1), the system is transformed to

$$\sum_{i=1}^{m} \left(\sum_{\mu=0}^{m-i} A_\mu X^{m-(\mu+i)} \right) H' U^{i-1} = F \tag{2.3}$$

where $H' = HQ$ and $F = -P(X)Q$. Taking the vec operator both sides of (2.3) makes a linear system such that

$$\widetilde{F}\mathrm{vec}(H') = \mathrm{vec}(F) \tag{2.4}$$

where the matrix $\widetilde{F} \in \mathbb{C}^{n\times n}$ is

$$\widetilde{F} = \sum_{i=1}^{m} \left[\left(U^{i-1}\right)^T \otimes \left(\sum_{\mu=0}^{m-i} A_\mu X^{m-(\mu+i)} \right) \right]. \tag{2.5}$$

Seo and Kim [8] defined $\widetilde{F_{ij}} = \sum_{i=1}^{m} \left[U^{i-1}\right]_{ji} \left(\sum_{\mu=1}^{m-i} A_\mu X^{m-(\mu+i)}\right)$ to reduce the system size of the equation (2.4) to $n \times n$, then $\widetilde{F}$ in (2.5) is represented by

$$\widetilde{F} = \begin{bmatrix} \widetilde{F_{11}} & & & \\ \widetilde{F_{21}} & \widetilde{F_{22}} & & 0 \\ \vdots & \vdots & \ddots & \\ \widetilde{F_{n1}} & \widetilde{F_{n2}} & \cdots & \widetilde{F_{nn}} \end{bmatrix}. \tag{2.6}$$

If we suppose that the matrices $\widetilde{F_{ii}}$ are nonsingular, then using the block forward substitution, the equation (2.4) can be changed to n linear systems with size $n \times n$ such that

$$\begin{aligned} h_1' &= \widetilde{F_{11}}^{-1} f_1 \\ h_2' &= \widetilde{F_{22}}^{-1} (f_2 - \widetilde{F_{21}} h_1') \\ &\vdots \\ h_n' &= \widetilde{F_{nn}}^{-1} (f_n - \widetilde{F_{n1}} h_1' - \cdots - \widetilde{F_{n,n-1}} h_{n-1}'), \end{aligned}$$

where h_i' and f_i are the ith columns of H' and F, respectively.

3. Skew-symmetric solvents of the matrix polynomial $P(X)$

Here, we consider an algorithm to compute skew-symmetric solutions of the q-th Newton iteration (2.1).

ALGORITHM 3.1.

1. *Input $n \times n$ real matrices $A_0, A_1, \cdots, A_m$ and skew-symmetric matrix $X_q \in \mathbb{R}^{n\times n}$.*
2. *Choose a skew-symmetric starting matrix $H_{q_0} \in \mathbb{R}^{n\times n}$.*
3. $k = 0$;

$$R_0 = -P(X_q) - \left(\sum_{i=1}^{m}\sum_{\mu=0}^{m-i} A_\mu X_q^{m-(\mu+i)} H_{q_0} X_q^{i-1}\right)$$

$$Z_0 = \sum_{i=1}^{m}\left(\sum_{\mu=0}^{m-i} A_\mu X_q^{m-(\mu+i)}\right)^T R_0 \left(X_q^{i-1}\right)^T$$

$$P_0 = \frac{1}{2}(Z_0 - Z_0^T)$$

4. **while** $R_k \neq 0$

$$H_{q_{k+1}} = H_{q_k} + \frac{\|R_k\|^2}{\|P_k\|^2} P_k$$

$$R_{k+1} = -P(X_q) - \left(\sum_{i=1}^{m}\sum_{\mu=0}^{m-i} A_\mu X_q^{m-(\mu+i)} H_{q_{k+1}} X_q^{i-1}\right)$$

$$Z_{k+1} = \sum_{i=1}^{m}\left(\sum_{\mu=0}^{m-i} A_\mu X_q^{m-(\mu+i)}\right)^T R_{k+1} \left(X_q^{i-1}\right)^T$$

$$P_{k+1} = \frac{1}{2}(Z_{k+1} - Z_{k+1}^T) + \frac{\operatorname{tr}(Z_{k+1}P_k)}{\|P_k\|^2} P_k.$$

end

REMARK 3.2. The matrices P_k and H_{q_k} are skew-symmetric in Algorithm 3.1.

By Algorithm 3.1, we can obtain some properties which are useful for the proof of our convergence theory.

LEMMA 3.3. *Let H_q be a skew-symmetric solution of the q-th Newton iteration* (2.1)*, then*

$$\operatorname{tr}\left[P_k^T (H_q - H_{q_k})\right] = \|R_k\|^2, \quad \textit{for} \quad k = 0, 1, \cdots. \tag{3.1}$$

Proof. When $k=0$, we obtain

$$\begin{aligned}
&\operatorname{tr}\left[P_0^T\left(H_q-H_{q0}\right)\right]\\
=&\ \operatorname{tr}\left[\frac{1}{2}\left(Z_0-Z_0^T\right)^T\left(H_q-H_{q0}\right)\right]\\
=&\ \operatorname{tr}\left[Z_0^T\left(H_q-H_{q0}\right)\right]\\
=&\ \operatorname{tr}\left\{\left[\sum_{i=1}^{m}\left(\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}\right)^T R_0\left(X_q^{i-1}\right)^T\right]^T\left(H_q-H_{q0}\right)\right\}\\
=&\ \operatorname{tr}\left\{R_0^T\left[\sum_{i=1}^{m}\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}\left(H_q-H_{q0}\right)X_q^{i-1}\right]\right\}\\
=&\ \operatorname{tr}\left\{R_0^T\left[-P(X_q)-\left(\sum_{i=1}^{m}\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}H_{q0}X_q^{i-1}\right)\right]\right\}\\
=&\ \|R_0\|^2,
\end{aligned}$$

by Algorithm 3.1.
We assume that (3.1) holds for $k=l$, then

$$\begin{aligned}
&\operatorname{tr}\left[P_{l+1}^T\left(H_q-H_{q_{l+1}}\right)\right]\\
=&\ \operatorname{tr}\left\{\left[\frac{1}{2}\left(Z_{l+1}-Z_{l+1}^T\right)+\frac{\operatorname{tr}\left(Z_{l+1}P_l\right)}{\|P_l\|^2}P_l\right]^T\left(H_q-H_{q_{l+1}}\right)\right\}\\
=&\ \operatorname{tr}\left[Z_{l+1}^T\left(H_q-H_{q_{l+1}}\right)\right]+\frac{\operatorname{tr}\left(Z_{l+1}P_l\right)}{\|P_l\|^2}\operatorname{tr}\left[P_l^T\left(H_q-H_{q_{l+1}}\right)\right]\\
=&\ \operatorname{tr}\left\{\left[\sum_{i=1}^{m}\left(\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}\right)^T R_{l+1}\left(X_q^{i-1}\right)^T\right]^T\left(H_q-H_{q_{l+1}}\right)\right\}\\
=&\ \operatorname{tr}\left\{R_{l+1}^T\left[\sum_{i=1}^{m}\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}\left(H_q-H_{q_{l+1}}\right)X_q^{i-1}\right]\right\}\\
=&\ \operatorname{tr}\left\{R_{l+1}^T\left[-P(X_q)-\left(\sum_{i=1}^{m}\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}H_{q_{l+1}}X_q^{i-1}\right)\right]\right\}\\
=&\ \operatorname{tr}\left(R_{l+1}^T R_{l+1}\right)=\ \|R_{l+1}\|^2,
\end{aligned}$$

from Algorithm 3.1 and from the following result

$$\begin{aligned}
\operatorname{tr}\left[P_l^T\left(H_q-H_{q_{l+1}}\right)\right] &= \operatorname{tr}\left[P_l^T\left(H_q-H_{q_l}-\frac{\|R_l\|^2}{\|P_l\|^2}P_l\right)\right] \\
&= \operatorname{tr}\left[P_l^T\left(H_q-H_{q_l}\right)\right]-\frac{\|R_l\|^2}{\|P_l\|^2}\operatorname{tr}\left(P_l^TP_l\right) \\
&= \|R_l\|^2-\|R_l\|^2 \\
&= 0.
\end{aligned}$$

□

LEMMA 3.4. *Suppose that the q-th Newton iteration* (2.1) *is consistent and there exists a integer number l such that $R_k \neq 0$ for all $k=0,1,\cdots,l$. Then by Lemma 3.3 $P_k \neq 0$ and we have*

(3.2) $\operatorname{tr}\left(R_k^TR_j\right)=0$ *and* $\operatorname{tr}\left(P_k^TP_j\right)=0$ *for $k>j=0,1,\cdots,l,\ l\geq 1$.*

Proof. We prove the conclusion (3.2) using the principle induction. i) We firstly prove $\operatorname{tr}\left(R_k^TR_{k-1}\right)=0$ and $\operatorname{tr}\left(P_k^TP_{k-1}\right)=0$ for $k=0,1,\cdots,l$. When $l=1$, from Algorithm 3.1

$$\begin{aligned}
&\operatorname{tr}\left(R_1^TR_0\right) \\
&= \operatorname{tr}\left\{\left[R_0-\frac{\|R_0\|^2}{\|P_0\|^2}\left(\sum_{i=1}^{m}\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}P_0X_q^{i-1}\right)\right]^TR_0\right\} \\
&= \operatorname{tr}\left(R_0^TR_0\right)-\frac{\|R_0\|^2}{\|P_0\|^2}\operatorname{tr}\left[\left(\sum_{i=1}^{m}\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}P_0X_q^{i-1}\right)^TR_0\right] \\
&= \|R_0\|^2 \\
&\quad -\frac{\|R_0\|^2}{\|P_0\|^2}\operatorname{tr}\left\{P_0^T\left[\sum_{i=1}^{m}\left(\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}\right)^TR_0\left(X_q^{i-1}\right)^T\right]\right\} \\
&= \|R_0\|^2-\frac{\|R_0\|^2}{\|P_0\|^2}\operatorname{tr}\left(P_0^TZ_0\right) \\
&= \|R_0\|^2-\frac{\|R_0\|^2}{\|P_0\|^2}\operatorname{tr}\left[P_0^T\frac{1}{2}\left(Z_0-Z_0^T\right)\right] \\
&= \|R_0\|^2-\frac{\|R_0\|^2}{\|P_0\|^2}\operatorname{tr}\left(P_0^TP_0\right) \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{tr}\left(P_1^T P_0\right) &= \operatorname{tr}\left\{\left[\frac{1}{2}\left(Z_1 - Z_1^T\right) + \frac{\operatorname{tr}\left(Z_1 P_0\right)}{\|P_0\|^2} P_0\right]^T P_0\right\} \\
&= \operatorname{tr}\left(Z_1^T P_0\right) + \frac{\operatorname{tr}\left(Z_1 P_0\right)}{\|P_0\|^2} \operatorname{tr}\left(P_0^T P_0\right) \\
&= \operatorname{tr}\left(P_0^T Z_1\right) + \operatorname{tr}\left(Z_1 P_0\right) \\
&= -\operatorname{tr}\left(Z_1 P_0\right) + \operatorname{tr}\left(Z_1 P_0\right) \\
&= 0.
\end{aligned}$$

If we assume that $\operatorname{tr}\left(R_s^T R_{s-1}\right) = 0$ and $\operatorname{tr}\left(P_s^T P_{s-1}\right) = 0$ hold for $l = s$, then we obtain

$$\begin{aligned}
&\operatorname{tr}\left(R_{s+1}^T R_s\right) \\
&= \operatorname{tr}\left\{\left[R_s - \frac{\|R_s\|^2}{\|P_s\|^2}\left(\sum_{i=1}^{m}\sum_{\mu=0}^{m-i} A_\mu X_q^{m-(\mu+i)} P_s X_q^{i-1}\right)\right]^T R_s\right\} \\
&= \operatorname{tr}\left(R_s^T R_s\right) - \frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left[\left(\sum_{i=1}^{m}\sum_{\mu=0}^{m-i} A_\mu X_q^{m-(\mu+i)} P_s X_q^{i-1}\right)^T R_s\right] \\
&= \|R_s\|^2 \\
&\quad -\frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left\{P_s^T\left[\sum_{i=1}^{m}\left(\sum_{\mu=0}^{m-i} A_\mu X_q^{m-(\mu+i)}\right)^T R_s\left(X_q^{i-1}\right)^T\right]\right\} \\
&= \|R_s\|^2 - \frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left(P_s^T Z_s\right) \\
&= \|R_s\|^2 - \frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left[P_s^T \frac{1}{2}\left(Z_s - Z_s^T\right)\right] \\
&= \|R_s\|^2 - \frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left[P_s^T\left(P_s - \frac{\operatorname{tr}\left(Z_s P_{s-1}\right)}{\|P_{s-1}\|^2} P_{s-1}\right)\right] \\
&= \|R_s\|^2 - \|R_s\|^2 + \frac{\|R_s\|^2 \operatorname{tr}\left(Z_s P_{s-1}\right)}{\|P_s\|^2\|P_{s-1}\|^2}\operatorname{tr}\left(P_s^T P_{s-1}\right) \\
&= 0,
\end{aligned}$$

and

$$
\begin{aligned}
&\operatorname{tr}\left(P_{s+1}^T P_s\right)\\
&=\operatorname{tr}\left\{\left[\frac{1}{2}\left(Z_{s+1}-Z_{s+1}^T\right)+\frac{\operatorname{tr}\left(Z_{s+1}P_s\right)}{\|P_s\|^2}P_s\right]^T P_s\right\}\\
&=\operatorname{tr}\left(Z_{s+1}^T P_s\right)+\frac{\operatorname{tr}\left(Z_{s+1}P_s\right)}{\|P_s\|^2}\operatorname{tr}\left(P_s^T P_s\right)\\
&=\operatorname{tr}\left(P_s^T Z_{s+1}\right)+\operatorname{tr}\left(Z_{s+1}P_s\right)\\
&=-\operatorname{tr}\left(Z_{s+1}P_s\right)+\operatorname{tr}\left(Z_{s+1}P_s\right)\\
&=0.
\end{aligned}
$$

ii) Suppose that $\operatorname{tr}\left(R_s^T R_j\right)=0$ and $\operatorname{tr}\left(P_s^T P_j\right)=0$ hold for all $j=0,1,\cdots,s-1$. Then, from Algorithm 3.1 and i) we get

$$
\begin{aligned}
&\operatorname{tr}\left(R_{s+1}^T R_j\right)\\
&=\operatorname{tr}\left\{\left[R_s-\frac{\|R_s\|^2}{\|P_s\|^2}\left(\sum_{i=1}^{m}\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}P_s X_q^{i-1}\right)\right]^T R_j\right\}\\
&=\operatorname{tr}\left(R_s^T R_j\right)-\frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left[\left(\sum_{i=1}^{m}\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}P_s X_q^{i-1}\right)^T R_j\right]\\
&=-\frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left\{P_s^T\left[\sum_{i=1}^{m}\left(\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}\right)^T R_j\left(X_q^{i-1}\right)^T\right]\right\}\\
&=-\frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left(P_s^T Z_j\right)\\
&=-\frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left[P_s^T\frac{1}{2}\left(Z_j-Z_j^T\right)\right]\\
&=-\frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left[P_s^T\left(P_j-\frac{\operatorname{tr}\left(Z_jP_{j-1}\right)}{\|P_{j-1}\|^2}P_{j-1}\right)\right]\\
&=-\frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left(P_s^T P_j\right)+\frac{\|R_s\|^2\operatorname{tr}\left(Z_jP_{j-1}\right)}{\|P_s\|^2\|P_{j-1}\|^2}\operatorname{tr}\left(P_s^T P_{j-1}\right)\\
&=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{tr}\left(P_{s+1}^T P_j\right)\\
&= \operatorname{tr}\left\{\left[\frac{1}{2}\left(Z_{s+1}-Z_{s+1}^T\right)+\frac{\operatorname{tr}\left(Z_{s+1}P_s\right)}{\|P_s\|^2}P_s\right]^T P_j\right\}\\
&= \operatorname{tr}\left(Z_{s+1}^T P_j\right)+\frac{\operatorname{tr}\left(Z_{s+1}P_s\right)}{\|P_s\|^2}\operatorname{tr}\left(P_s^T P_j\right)\\
&= \operatorname{tr}\left\{\left[\sum_{i=1}^{m}\left(\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}\right)^T R_{s+1}\left(X_q^{i-1}\right)^T\right]^T P_j\right\}\\
&= \operatorname{tr}\left[R_{s+1}^T\left(\sum_{i=1}^{m}\sum_{\mu=0}^{m-i}A_\mu X_q^{m-(\mu+i)}P_j X_q^{i-1}\right)\right]\\
&= \operatorname{tr}\left[R_{s+1}^T\frac{\|P_j\|^2}{\|R_j\|^2}\left(R_j-R_{j+1}\right)\right]\\
&= \frac{\|P_j\|^2}{\|R_j\|^2}\operatorname{tr}\left(R_{s+1}^T R_j\right)-\frac{\|P_j\|^2}{\|R_j\|^2}\operatorname{tr}\left(R_{s+1}^T R_{j+1}\right)\\
&= 0,
\end{aligned}
$$

for all $j = 0, 1, \cdots, s-1$. Hence, we complete the proof by i) and ii). □

From Lemma 3.4 we know that, if there is a positive number l such that $R_k \neq 0$ for all $k = 0, 1, \cdots, l$, then, the matrices R_k and R_j are orthogonal for $k \neq j$.

THEOREM 3.5. *Let the q-th Newton iteration* (2.1) *has a skew-symmetric solution H_q. Then for a given skew-symmetric starting matrix, the solution H_q can be found, at most, in n^2 steps.*

This theorem can be proved by the similar way of Theorem 3.3 in [1].

Proof. From Lemma 3.4, the set $\{R_0, R_1, \cdots, R_{n^2-1}\}$ is an orthogonal basis of $\mathbb{R}^{n\times n}$. Since the q-th Newton iteration (2.1) has a skew-symmetric solution, and using Lemma 3.3, $P_k \neq 0$ for k. By Algorithm 3.1 and Lemma 3.4 we obtain $H_{q_{n^2}}$ and R_{n^2} , and $\operatorname{tr}\left(R_{n^2}^T R_k\right) = 0$ for $k = 0, 1, \cdots, n^2-1$. However, $\operatorname{tr}\left(R_{n^2}^T R_k\right) = 0$ holds only when $R_{n^2} = 0$, which implies that $H_{q_{n^2}}$ is a solution of the q-th Newton iteration. Thus $H_{q_{n^2}}$ is a skew-symmetric matrix. □

From Newton's method and the above theorem, we have the following result.

THEOREM 3.6. *Suppose that the matrix polynomial has a skew-symmetric solvent and each Newton iteration is consistent for a skew-symmetric starting matrix X_0. The sequence $\{X_k\}$ is generated by Newton's method with X_0 such that*

$$\lim_{k\to\infty} X_k = S,$$

and the matrix S satisfies $P(S) = 0$, then S is a skew-symmetric solvent.

The proof of the theorem is also similar to Theorem 3.4 in [1].

Proof. If H_k is skew-symmetric solution of kth Newton iteration then $(k+1)$th approximation matrix is

$$X_{k+1} = X_0 + H_0 + \cdots + H_k.$$

By the properties of skew-symmetric matrix X_{k+1} is also skew-symmetric. Since, the Newton sequence $\{X_k\}$ converges to a solvent S, it is a skew-symmetric solvent. □

In this paper, we consider an iterative method for finding a skew-symmetric solution of matrix equation in (2.1). Then we incorporated the iterative method into Newtons method to compute the skew-symmetric solvent of matrix polynomial $P(X)$ in (1.2).

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