CERTAIN RESULTS ON THE q-GENOCCHI NUMBERS AND POLYNOMIALS

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ABSTRACT. In this work, we deal with q-Genocchi numbers and polynomials. We derive not only new but also interesting properties of the q-Genocchi numbers and polynomials. Also, we give Cauchy-type integral formula of the q-Genocchi polynomials and derive distribution formula for the q-Genocchi polynomials. In the final part, we introduce a definition of q-Zeta-type function which is interpolation function of the q-Genocchi polynomials at negative integers which we express in the present paper.

1. Preliminaries

Let p be a fixed odd prime number. Now, we need the definitions of the some notations for this work such that let $\mathbb{Q}_p$ be the field p-adic rational numbers and let $\mathbb{C}_p$ be the completion of algebraic closure of $\mathbb{Q}_p$. That is,

$$\mathbb{Q}_p = \left\{ x = \sum_{n=-k}^{\infty} a_n p^n : 0 \le a_n \le p-1 \right\}.$$

Then $\mathbb{Z}_p$ is integral domain which is defined by

$$\mathbb{Z}_p = \left\{ x = \sum_{n=0}^{\infty} a_n p^n : 0 \le a_n \le p-1 \right\}$$

or

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \le 1 \right\}.$$

We assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$ as an indeterminate. The p-adic absolute value $|.|_p$, is normally given by

$$|x|_p = p^{-r}$$

where $x = p^r \frac{s}{t}$ with $(p, s) = (p, t) = (s, t) = 1$ and $r \in \mathbb{Q}$.

The q-extension of x with the display notation of $[x]_q$ is introduced by

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

We note that $\lim_{q \to 1} [x]_q = x$ (see[1-22]).

Also, we use notation $\mathbb{N}^*$ which means the combinations of zero and Natural numbers.

We consider that η is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient

$$\Phi_\eta (x, y) = \frac{\eta (x) - \eta (y)}{x - y},$$

have a limit $\acute{\eta}(a)$ as $(x, y) \to (a, a)$ and denote this by $\eta \in UD(\mathbb{Z}_p)$. Then, for $\eta \in UD(\mathbb{Z}_p)$, we can discuss the following

$$\frac{1}{[p^n]_q} \sum_{0 \leq \xi < p^n} \eta (\xi) q^\xi = \sum_{0 \leq \xi < p^n} \eta (\xi) \mu_q (\xi + p^n \mathbb{Z}_p),$$

which represents as a p-adic q-analogue of Riemann sums for η. The integral of η on $\mathbb{Z}_p$ will be defined as the limit $(n \to \infty)$ of these sums, when it exists. The p-adic q-integral of function $\eta \in UD(\mathbb{Z}_p)$ is defined by T. Kim in [7], [11], [16] by

$$(1.1) \qquad I_q (\eta) = \int_{\mathbb{Z}_p} \eta (\xi) \, d\mu_q (\xi) = \lim_{n \to \infty} \frac{1}{[p^n]_q} \sum_{\xi=0}^{p^n-1} \eta (\xi) q^\xi.$$

The bosonic integral is considered as a bosonic limit $q \to 1$, $I_1 (\eta) = \lim_{q \to 1} I_q (\eta)$. Similarly, the fermionic p-adic integral on $\mathbb{Z}_p$ is introduced by T. Kim as follows:

$$(1.2) \qquad\qquad I_{-q} (\eta) = \int_{\mathbb{Z}_p} \eta (\xi) \, d\mu_{-q} (\xi)$$

(for more details, see [16-18]).

From (1.2), it is well-known equality

$$(1.3) \qquad\qquad q I_{-q} (\eta_1) + I_{-q} (\eta) = [2]_q \eta (0),$$

where $\eta_1 (x) = \eta (x + 1)$ (for details, see[2-4, 11-14, 16-22]).

The q-Genocchi polynomials with wegiht 0 are given by Araci et al., as follows:

(1.4)
$$\frac{\widetilde{G}_{n+1,q}(x)}{n+1} = \int_{\mathbb{Z}_p} (x+\xi)^n \, d\mu_{-q}(\xi).$$

From (1.4), we have

$$\widetilde{G}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^l \widetilde{G}_{n-l,q}$$

where $\widetilde{G}_{n,q}(0) := \widetilde{G}_{n,q}$ are called q-Genocchi numbers with weight 0. Then, q-Genocchi numbers are defined as

$$\widetilde{G}_{0,q} = 0 \text{ and } q\left(\widetilde{G}_q + 1\right)^n + \widetilde{G}_{n,q} = \begin{cases} [2]_q, & \text{if } n = 1 \\ 0, & \text{if } n \neq 1, \end{cases}$$

with the usual convention about replacing $\left(\widetilde{G}_q\right)^n$ by $\widetilde{G}_{n,q}$ is used (for details, see [4]).

In [8], Kim and Lee derived some properties for the q-Euler numbers and polynomials. In [4], Araci also studied not only new but also interesting properties of the q-Genocchi numbers and polynomials with weight 0. Actually, Genocchi numbers and polynomials, Euler numbers and polynomials and their q-extensions have been studied in several different ways for a long time (for details, see [1-21]). Our aim in this paper is to present some new interesting properties of the q-Genocchi numbers and polynomials. Our applications for the q-Genocchi polynomials will seem to be useful in mathematics for engineerings.

2. Some poperties of q-Genocchi numbers and polynomials with weight 0

Let $\eta(x) = e^{t(x+\xi)}$. Then, by using (1.3), we see that

$$t \int_{\mathbb{Z}_p} e^{t(x+\xi)} d\mu_{-q}(\xi) = \frac{[2]_q t}{qe^t + 1} e^{xt}.$$

From the last equality and (1.4), we get Araci, Acikgoz and Qi's q-Genocchi polynomials with weight 0 in [3] as follows:

$$(2.1) \quad t \int_{\mathbb{Z}_p} e^{t(x+\xi)} d\mu_{-q}(\xi) = \sum_{n=0}^{\infty} \widetilde{G}_{n,q}(x) \frac{t^n}{n!}$$

$$= \frac{[2]_q t}{qe^t + 1} e^{xt}, \ |\log q + t| < \pi.$$

Substituting $x \to x + y$ into (2.1), then we can write

$$\sum_{n=0}^{\infty} \widetilde{G}_{n,q}(x+y) \frac{t^n}{n!} = \frac{[2]_q t}{qe^t + 1} e^{(x+y)t}$$

$$= \left(\sum_{n=0}^{\infty} \widetilde{G}_{n,q}(x) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} y^n \frac{t^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \widetilde{G}_{k,q}(x) y^{n-k} \right) \frac{t^n}{n!}$$

From the above applications, we can easily express the following theorem:

THEOREM 2.1. *The following holds:*

$$(2.2) \quad \widetilde{G}_{n,q}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{G}_{k,q}(x) y^{n-k}.$$

By (2.2), we consider the following

$$\widetilde{G}_{n,q}(x+y) = ny^{n-1} + \sum_{k=2}^{n} \binom{n}{k} \widetilde{G}_{k,q}(x) y^{n-k}.$$

From this, it follows that

$$\widetilde{G}_{n,q}(x+y) - ny^{n-1} = \sum_{k=2}^{n} \binom{n}{k} \widetilde{G}_{k,q}(x) y^{n-k}$$

can be derived and so we reach the following theorem:

THEOREM 2.2. *For $n \in \mathbb{N}^*$, one has*

$$(2.3) \quad \sum_{k=0}^{n} \frac{\binom{n}{k}}{(k+2)(k+1)} \widetilde{G}_{k+2,q}(x) y^{n-k}$$

$$= \frac{\widetilde{G}_{n+2,q}(x+y) - (n+2) y^{n+1}}{(n+2)(n+1)}.$$

Replacing y by $-y$ into (2.3), then we get

$$(2.4) \qquad \frac{\widetilde{G}_{n+2,q}\left(x-y\right)-\left(n+2\right)\left(-1\right)^{n+1}y^{n+1}}{\left(n+2\right)\left(n+1\right)}$$

$$= \sum_{k=0}^{n} \frac{\binom{n}{k}\left(-1\right)^{n-k}}{\left(k+2\right)\left(k+1\right)}\widetilde{G}_{k+2,q}\left(x\right)y^{n-k}.$$

By (2.4), it follows that

$$(2.5) \qquad \sum_{k=0}^{n} \frac{\binom{n}{k}\left(-1\right)^{k}}{\left(k+2\right)\left(k+1\right)}\widetilde{G}_{k+2,q}\left(x\right)y^{n-k}$$

$$= \frac{\left(-1\right)^{n}\widetilde{G}_{n+2,q}\left(x-y\right)+\left(n+2\right)y^{n+1}}{\left(n+1\right)\left(n+2\right)}.$$

Therefore, from the expressions of (2.3) and (2.5), we procure the following theorem:

THEOREM 2.3. *The following holds true:*

$$(2.6) \qquad \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\binom{n}{2k}}{\left(k+1\right)\left(2k+1\right)}\widetilde{G}_{2k+2,q}\left(x\right)y^{n-2k}$$

$$= \frac{\left(-1\right)^{n}\widetilde{G}_{n+2,q}\left(x-y\right)+\widetilde{G}_{n+2,q}\left(x+y\right)}{\left(n+1\right)\left(n+2\right)}$$

where $\left[.\right]$ is Gauss' symbol.

By (2.4), we have the following identity

$$(2.7) \qquad \sum_{k=2}^{n} \frac{\binom{n}{k}\left(-1\right)^{k}}{k\left(k-1\right)}\widetilde{G}_{k,q}\left(x\right)y^{n-k}$$

$$= \frac{\left(-1\right)^{n}\widetilde{G}_{n,q}\left(x-y\right)+ny^{n-1}}{n\left(n-1\right)}.$$

By (2.4), (2.5) and (2.7), then we have the following theorem:

THEOREM 2.4. *For $n \in \mathbb{N}^{*}$, we get*

$$(2.8) \qquad \frac{\left(-1\right)^{n}\widetilde{G}_{n+2,q}\left(x-y\right)+\widetilde{G}_{n+2,q}\left(x+y\right)}{\left(n+2\right)\left(n+1\right)}$$

$$= \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \frac{\binom{n}{2k}}{\left(k+1\right)\left(2k+1\right)}\widetilde{G}_{2k+2,q}\left(x\right)y^{n-2k}.$$

Taking $y = 1$ into (2.3), it follows that

$$(2.9) \qquad q \sum_{k=0}^{n} \frac{\binom{n}{k}}{(k+2)(k+1)} \widetilde{G}_{k+2,q}(x) = \frac{q\widetilde{G}_{n+2,q}(x+1)}{(n+1)(n+2)} - \frac{q}{n+1}.$$

We need the following for sequel of this paper:

$$\sum_{n=0}^{\infty} \left(q\widetilde{G}_{n,q}(x+1) + \widetilde{G}_{n,q}(x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left([2]_q x^n \right) \frac{t^{n+1}}{n!},$$

from the above, we easily develop the following:

$$(2.10) \qquad q\widetilde{G}_{n+1,q}(x+1) + \widetilde{G}_{n+1,q}(x) = (n+1)[2]_q x^n.$$

By (2.9) and (2.10), we state the following theorem:

THEOREM 2.5. *The following holds:*

$$(2.11) \qquad \sum_{k=0}^{n} \frac{\binom{n}{k}}{(k+2)(k+1)} \widetilde{G}_{k+2,q}(x)$$

$$= \frac{[2]_q x^{n+1}}{qn+q} - \frac{\widetilde{G}_{n+2,q}(x)}{(qn+q)(n+2)} - \frac{1}{qn+q}.$$

On account of the equality $\lim_{q \to 1} \widetilde{G}_{n,q}(x) = \widetilde{G}_{n,1}(x) := G_n(x)$, where $G_n(x)$ are known as Genocchi polynomials which are defined by means of the following exponential generating function:

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t + 1} e^{xt} \quad (|t| < \pi),$$

(see [1-4, 12, 13, 21]). As an application, as $q \to 1$ in (2.11), we discover the following corollary:

COROLLARY 2.6. *The following*

$$\sum_{k=0}^{n} \frac{\binom{n}{k}}{(k+2)(k+1)} G_{k+2}(x)$$

$$= \frac{2x^{n+1}}{n+1} - \frac{G_{n+2}(x)}{(n+1)(n+2)} - \frac{1}{n+1}$$

is true.

Let us take $y = 1$ and $n \to 2n$ into (2.6), becomes

(2.12)

$$\sum_{k=0}^{n} \frac{\binom{2n}{2k}}{(k+1)(2k+1)} \widetilde{G}_{2k+2,q}(x)$$

$$= \frac{\widetilde{G}_{2n+2,q}(x-1) + \widetilde{G}_{2n+2,q}(x+1)}{(2n+1)(2n+2)}$$

$$= \frac{\frac{1}{q}\left(q\widetilde{G}_{2n+2,q}(x+1) + \widetilde{G}_{2n+2,q}(x)\right) + q\widetilde{G}_{2n+2,q}(x) + \widetilde{G}_{2n+2,q}(x-1)}{(2n+1)(2n+2)}$$

$$- \frac{\widetilde{G}_{2n+2,q}(x)}{q(2n+1)(2n+2)} - \frac{q\widetilde{G}_{2n+2,q}(x)}{(2n+1)(2n+2)}$$

$$= \frac{(n+2)[2]_q x^{n+1}}{(2n+1)(2n+2)} + \frac{(n+2)[2]_q (x-1)^{n+1}}{(2n+1)(2n+2)} - \frac{\widetilde{G}_{2n+2,q}(x)}{q(2n+1)(2n+2)}$$

$$- \frac{q\widetilde{G}_{2n+2,q}(x)}{(2n+1)(2n+2)}$$

After these applications, we conclude with the following theorem:

THEOREM 2.7. *The following identity*

(2.13)
$$\sum_{k=0}^{n} \frac{\binom{2n}{2k}}{(k+1)(2k+1)} \widetilde{G}_{2k+2,q}(x)$$

$$= \frac{(n+2)[2]_q x^{n+1}}{(2n+1)(2n+2)} + \frac{(n+2)[2]_q (x-1)^{n+1}}{(2n+1)(2n+2)}$$

$$- \frac{\widetilde{G}_{2n+2,q}(x)}{q(2n+1)(2n+2)} - \frac{q\widetilde{G}_{2n+2,q}(x)}{(2n+1)(2n+2)}$$

is true.

Now, we analyse as $q \to 1$ for the equation (2.13) and so we readily state the following corollary which seems to be interesting property for the Genocchi polynomials.

COROLLARY 2.8. *The following equality holds:*

$$(2.14) \qquad \sum_{k=0}^{n} \frac{\binom{2n}{2k}}{(k+1)(2k+1)} G_{2k+2}(x)$$

$$= \frac{2(n+2)x^{n+1}}{(2n+1)(2n+2)} + \frac{2(n+2)(x-1)^{n+1}}{(2n+1)(2n+2)}$$

$$- \frac{G_{2n+2}(x)}{(2n+1)(2n+2)} - \frac{G_{2n+2}(x)}{(2n+1)(2n+2)}.$$

Substituting $n \to 2n+1$ and $y = 1$ into (2.8), we compute

$$\sum_{k=0}^{n} \frac{\binom{2n+1}{2k}}{(k+1)(2k+1)} \widetilde{G}_{2k+2,q}(x)$$

$$= \frac{\widetilde{G}_{2n+3,q}(x+1) - \widetilde{G}_{2n+3,q}(x-1)}{(2n+3)(2n+2)}$$

$$= \frac{\frac{1}{q}\left(q\widetilde{G}_{2n+3,q}(x+1) + \widetilde{G}_{2n+3,q}(x)\right) - \left(\widetilde{G}_{2n+3,q}(x) + \widetilde{G}_{2n+3,q}(x-1)\right)}{(2n+3)(2n+2)}$$

$$+ \left(\frac{q-1}{q}\right) \frac{\widetilde{G}_{2n+3,q}(x)}{(2n+3)(2n+2)}$$

$$= \frac{[2]_q \, x^{2n+2}}{q(2n+2)} - \frac{[2]_q (x-1)^{2n+2}}{(2n+2)} + \left(\frac{q-1}{q}\right) \frac{\widetilde{G}_{2n+3,q}(x)}{(2n+3)(2n+2)}.$$

Therefore, we obtain the following theorem:

THEOREM 2.9. *The following equality holds:*

$$\sum_{k=0}^{n} \frac{\binom{2n+1}{2k}}{(k+1)(2k+1)} \widetilde{G}_{2k+2,q}(x)$$

$$= \frac{[2]_q \, x^{2n+2}}{q(2n+2)} - \frac{[2]_q (x-1)^{2n+2}}{(2n+2)} + \left(\frac{q-1}{q}\right) \frac{\widetilde{G}_{2n+3,q}(x)}{(2n+3)(2n+2)}.$$

As $q \to 1$ in the above theorem, then we easily derive the following corollary:

COROLLARY 2.10. *For $n \in \mathbb{N}^*$, then we have*

$$\sum_{k=0}^{n} \frac{\binom{2n+1}{2k}}{(k+1)(2k+1)} G_{2k+2,q}(x) = \frac{2x^{2n+2}}{(2n+2)} - \frac{2(x-1)^{2n+2}}{(2n+2)}.$$

3. Further remarks

In this final part, we remember the definition of the generating function of the q-Genocchi polynomials, as follows:

$$(3.1) \qquad \mathcal{F}_q(x,t) = \frac{[2]_q \, t}{q e^t + 1} e^{xt} = \sum_{n=0}^{\infty} \widetilde{G}_{n,q}(x) \frac{t^n}{n!}.$$

Applying k-th derivative to (3.1), then we attain the following

$$(3.2) \qquad \frac{d^k}{dt^k}\left(\frac{[2]_q \, t}{q e^t + 1} e^{xt} \right) = \frac{d^k}{dt^k}\left(\sum_{n=0}^{\infty} \widetilde{G}_{n,q}(x) \frac{t^n}{n!} \right).$$

Taking $\lim_{t \to 0}$ on the both sides in (3.2), then we arrive at the following theorem:

THEOREM 3.1. *The following equality*

$$(3.3) \qquad \widetilde{G}_{k,q}(x) = \lim_{t \to 0}\left[\frac{d^k}{dt^k}\left(\frac{[2]_q \, t}{q e^t + 1} e^{xt} \right) \right]$$

is true.

We now consider Cauchy-type integral formula of the q-Genocchi polynomials which is a vital and important in complex analysis, is an important statement about line integrals for holomorphic functions in the complex plane. So, by using equation of (3.3), we can develop the following theorem:

THEOREM 3.2. *The following Cauchy-type integral holds true:*

$$\widetilde{G}_{n,q}(x) = \frac{n!}{2\pi i} \int_C \mathcal{F}_q(x,t) \frac{dt}{t^{n+1}}$$

where C is a loop which starts at $-\infty$, encircles the origin once in the positive direction, and the returns $-\infty$.

Distribution formula for the special polynomials is important to study regarding p-adic Measure theory. That is,

$$
\int_{\mathbb{Z}_p} (x+y)^n \, d\mu_{-q}(y) = \lim_{n\to\infty} \frac{1}{[dp^n]_{-q}} \sum_{\xi=0}^{dp^n-1} (-1)^\xi (x+\xi)^n q^\xi
$$

$$
= \frac{d^n}{[d]_{-q}} \sum_{a=0}^{d-1} (-1)^a q^a \left(\lim_{n\to\infty} \frac{1}{[p^n]_{-q^d}} \sum_{\xi=0}^{p^n-1} (-1)^\xi \left(\frac{x+a}{d}+\xi \right)^n q^{d\xi} \right)
$$

$$
= \frac{d^n}{[d]_{-q}} \sum_{a=0}^{d-1} (-1)^a q^a \frac{\widetilde{G}_{n+1,q}\left(\frac{x+a}{d}\right)}{n+1}.
$$

As a result, we obtain the following theorem.

THEOREM 3.3. *For $n \in \mathbb{N}^*$, then we have*

$$
[d]_{-q}\, \widetilde{G}_{n,q}(dx) = d^{n-1} \sum_{a=0}^{d-1} (-1)^a q^a \widetilde{G}_{n,q}\left(x+\frac{a}{d}\right).
$$

By using definition of the geometric series in (3.1), we easily see that

$$
\sum_{m=0}^{\infty} \widetilde{G}_{m,q}(x) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left([2]_q (m+1) \sum_{n=0}^{\infty} (-1)^n q^n n^m \right) \frac{t^{m+1}}{(m+1)!},
$$

by comparing the coefficients on the both sides, then we have, for $m \in \mathbb{N}$

(3.4)
$$
\frac{\widetilde{G}_{m+1,q}(x)}{m+1} = [2]_q \sum_{n=1}^{\infty} (-1)^n q^n n^m.
$$

By (3.4), we define q-Zeta-type function as follows: For $s \in \mathbb{C}$ and $\Re e(s) > 1$,

(3.5)
$$
\widetilde{\zeta}(s,x:q) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{n^s}.
$$

As $q \to 1$ in (3.5), it leads to the following

$$
\lim_{q\to 1} \widetilde{\zeta}(s,x:q) = \widetilde{\zeta}(s,x:1) := \zeta(s,x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}
$$

which is well-known Euler-Zeta function (see [7]). By (3.4) and (3.5), we get

$$
\widetilde{\zeta}(-n,x:q) = \frac{\widetilde{G}_{n+1,q}(x)}{n+1}.
$$

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