

## MINIMAL QUASI-F COVERS OF $vX$

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ABSTRACT. We show that if  $X$  is a space such that  $\beta QF(X) = QF(\beta X)$  and each stable  $Z(X)^\#$ -ultrafilter has the countable intersection property, then there is a homeomorphism  $h_X : vQF(X) \rightarrow QF(vX)$  with  $r_X = \Phi_{vX} \circ h_X$ . Moreover, if  $\beta QF(X) = QF(\beta X)$  and  $vE(X) = E(vX)$  or  $v\Lambda(X) = \Lambda(vX)$ , then  $vQF(X) = QF(vX)$ .

### 1. Introduction

All spaces in this paper are Tychonoff spaces and  $\beta X$  ( $vX$ , resp.) denotes the Stone-Ćech compactification (Hewitt realcompactification, resp.) of a space  $X$ .

Gleason ([3]) and Iliadis ([6]) characterized minimal extremally disconnected covers of spaces and the projective objects in the category of compact spaces and continuous maps and proved that each compact space has the projective cover which is the extremally disconnected cover of the space. Over the next decade numerous authors extended this characterization to other categories of topological spaces ([9]).

A space is called a *quasi- $F$  space* if its dense cozero-set is  $C^*$ -embedded. A quasi- $F$  space, introduced by Henriksen and Gillman, is a generalization of  $F$ -spaces ([2]), in which every cozero-set is  $C^*$ -embedded. Each space has the minimal quasi- $F$  cover  $(QF(X), \Phi_X)$  ([1], [4], [5]). In [5] and [8], authors investigated when  $\beta QF(X) = QF(\beta X)$  and  $QF(X) = \Phi_\beta^{-1}(X)$ , where  $(QF(\beta X), \Phi_\beta)$  is the minimal quasi- $F$  cover of  $\beta X$ .

It is well-known that each space has the minimal extremally disconnected cover  $(E(X), k_X)$  and that  $\beta E(X) = E(\beta X)$  ([9]). Moreover, internal characterizations of a space  $X$  that is equivalent to  $E(vX) =$

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Received December 13, 2012; Accepted January 11, 2013.

2010 Mathematics Subject Classification: Primary 54G10, 54C10; Secondary 54D60.

Key words and phrases: quasi- $F$  space, covering map, realcompactification.

$vE(X)$  is known([9]). Similar results for the minimal basically disconnected cover  $(\Lambda X, \Lambda_X)$  are given by [8].

In this paper, for any space  $X$  such that  $\beta QF(X) = QF(\beta X)$  and each stable  $Z(X)^\#$ -ultrafilter has the countable intersection property, we first show that there is a map  $r_X : vQF(X) \rightarrow vX$  such that  $(vQF(X), r_X)$  is a quasi- $F$  cover of  $vX$  and show that  $vQF(X) = QF(vX)$ , that is, there is a homeomorphism  $h_X : vQF(X) \rightarrow QF(vX)$  with  $r_X = \Phi_{vX} \circ h_X$ . Moreover, if  $\beta QF(X) = QF(\beta X)$  and  $vE(X) = E(vX)$  or  $v\Lambda(X) = \Lambda(vX)$ , then  $vQF(X) = QF(vX)$ .

For the terminology, we refer to [2, 9].

### 2. Quasi-F covers

Let  $X$  be a space. It is well-known that the collection  $\mathcal{R}(X)$  of all regular closed sets in  $X$ , when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows :

For any  $A \in \mathcal{R}(X)$  and any  $\mathcal{F} \subseteq \mathcal{R}(X)$ ,  
 $\bigvee \mathcal{F} = cl_X(\cup\{F \mid F \in \mathcal{F}\})$ ,  
 $\bigwedge \mathcal{F} = cl_X(int_X(\cap\{F \mid F \in \mathcal{F}\}))$ , and  
 $A' = cl_X(X - A)$ .

A sublattice of  $\mathcal{R}(X)$  is a subset of  $\mathcal{R}(X)$  that contains  $\emptyset, X$  and is closed under finite joins and finite meets([9]).

A map  $f : Y \rightarrow X$  is called a *covering map* if it is an onto continuous, perfect, and irreducible map([9]).

LEMMA 2.1. ([8])

- (1) Let  $X$  be a dense subspace of  $Y$ . Then the map  $\phi : R(Y) \rightarrow R(X)$ , defined by  $\phi(A) = A \cap X$ , is a Boolean isomorphism.
- (2) Let  $f : Y \rightarrow X$  be a covering map. Then the map  $\psi : R(Y) \rightarrow R(X)$ , defined by  $\psi(A) = f(A)$ , is a Boolean isomorphism.

In the above lemma, the inverse map  $\phi^{-1} : R(X) \rightarrow R(Y)$  of  $\phi$  is given by  $\phi^{-1}(B) = cl_Y(B)$  ( $B \in R(X)$ ) and the inverse map  $\psi^{-1} : R(X) \rightarrow R(Y)$  of  $\psi$  is given by  $\psi^{-1}(B) = cl_Y(int_Y(f^{-1}(B))) = cl_Y(f^{-1}(int_X(B)))$  ( $B \in R(X)$ ).

LEMMA 2.2. Let  $f : Y \rightarrow X$  be a covering map and  $(A_n)$  a decreasing sequence in  $R(Y)$ . Then  $f(\cap\{A_n \mid n \in N\}) = \cap\{f(A_n) \mid n \in N\}$ .

*Proof.* Let  $(A_n)$  be a decreasing sequence in  $R(Y)$ . Then clearly,  $f(\cap\{A_n \mid n \in N\}) \subseteq \cap\{f(A_n) \mid n \in N\}$ .

Let  $x \in \cap\{f(A_n) \mid n \in N\}$  and  $n \in N$ . Then  $x \in f(A_n)$  and  $A_n \cap f^{-1}(x) \neq \emptyset$  for all  $n \in N$ . Since  $(A_n)$  is a decreasing sequence in  $R(Y)$ ,  $(A_n)$  has the finite intersection property and  $\{A_n \cap f^{-1}(x) \mid n \in N\}$  is a family of closed sets in  $f^{-1}(x)$  with the finite intersection property. Since  $f$  is a compact map,  $f^{-1}(x)$  is a compact subset of  $Y$ . Hence  $\cap\{A_n \cap f^{-1}(x) \mid n \in N\} \neq \emptyset$ . Pick  $y \in \cap\{A_n \cap f^{-1}(x) \mid n \in N\}$ . Then  $y \in \cap\{A_n \mid n \in N\}$  and  $f(y) = x$  and  $\cap\{f(A_n) \mid n \in N\} \subseteq f(\cap\{A_n \mid n \in N\})$ .  $\square$

DEFINITION 2.3. A space  $X$  is called a *quasi-F space* if for any zero-sets  $A, B$  in  $X$ ,  $cl_X(int_X(A \cap B)) = cl_X(int_X(A)) \cap cl_X(int_X(B))$ , equivalently, every dense cozero-set in  $X$  is  $C^*$ -embedded in  $X$ .

It is well-known that a space  $X$  is a quasi-F space if and only if  $\beta X$  (or  $\nu X$ ) is a quasi-F space.

DEFINITION 2.4. Let  $X$  be a space. Then a pair  $(Y, f)$  is called

- (1) a *cover of  $X$*  if  $f : X \rightarrow Y$  is a covering map,
- (2) a *quasi-F cover of  $X$*  if  $(Y, f)$  is a cover of  $X$  and  $Y$  is a quasi-F space, and
- (3) a *minimal quasi-F cover of  $X$*  if  $(Y, f)$  is a quasi-F cover of  $X$  and for any quasi-F cover  $(Z, g)$  of  $X$ , there is a covering map  $h : Z \rightarrow Y$  such that  $f \circ h = g$ .

Let  $X$  be a space,  $Z(X) = \{Z \mid Z \text{ is a zero-set in } X\}$  and  $Z(X)^\# = \{cl_X(int_X(A)) \mid A \in Z(X)\}$ . Then  $Z(X)^\#$  is a sublattice of  $R(X)$ .

Suppose that  $X$  is a compact space. Let  $QF(X) = \{\alpha \mid \alpha \text{ is a } Z(X)^\# \text{-ultrafilter}\}$  and for any  $A \in Z(X)^\#$ , let  $\Sigma_A = \{\alpha \in QF(X) \mid A \in \alpha\}$ . Then the space  $QF(X)$ , equipped with the topology for which  $\{QF(X) - \Sigma_A \mid A \in Z(X)^\#\}$  is a base, is a quasi-F space. Define the map  $\Phi_X : QF(X) \rightarrow X$  by  $\Phi_X(\alpha) = \cap\{A \mid A \in \alpha\}$ . Then  $(QF(X), \Phi_X)$  is the minimal quasi-F cover of  $X$  and for any  $A \in Z(X)^\#$ ,  $\Phi_X(\Sigma_A) = A$  ([4]).

We recall that a space  $X$  is called a *weakly Lindelöf space* if for any open cover  $\mathcal{U}$  of  $X$ , there is a countable subfamily  $\mathcal{V}$  of  $\mathcal{U}$  such that  $\cup\{V \mid V \in \mathcal{V}\}$  is a dense subset of  $X$  and that  $X$  is called a *locally weakly Lindelöf space* if every element of  $X$  has a weakly Lindelöf neighborhood.

For any weakly Lindelöf space,  $\beta QF(X) = QF(\beta X)$ , that is, there is a homeomorphism  $h : \beta QF(X) \rightarrow QF(\beta X)$  such that  $\beta_X \circ \Phi_X = \Phi_{\beta X} \circ h \circ \beta_{QF(X)}$  ([5]). And for any locally weakly Lindelöf space,  $QF(X)$  is the subspace of  $\{\alpha \in QF(\beta X) \mid \cap\{A \mid A \in \alpha\} \in X\}$  of  $Q(\beta X)$  ([8]).

Since  $Z(X)^\#$  and  $Z(\beta X)^\#$  are isomorphic,  $QF(\beta X)$  is homeomorphic to the space  $\{\alpha \mid \alpha \text{ is a } Z(X)^\# \text{-ultrafilter}\}$  which is equipped with the

topology for which  $\{\Sigma'_A \mid A \in Z(X)^\#\}$  is a base for closed sets, where  $\Sigma'_A = \{\alpha \mid \alpha \text{ is a } Z(X)^\#\text{-ultrafilter and } A \in \alpha\}$ .

Let  $X$  be a space and  $(QF(\beta X), \Phi_\beta)((QF(vX), \Phi_v)$ , resp.) denote the minimal quasi-F cover of  $\beta X(vX)$ , resp.).

DEFINITION 2.5. Let  $X$  be a space. Then a  $Z(X)^\#\text{-filter } \alpha$  is called *fixed* if  $\cap\{A \mid A \in \alpha\} \neq \emptyset$ .

Suppose that  $QF(X) = \Phi_\beta^{-1}(X)$ , where  $\Phi_\beta^{-1}(X)$  is the subspace of  $QF(\beta X)$ . Then  $QF(X) = \{\alpha \mid \alpha \text{ is a fixed } Z(X)^\#\text{-ultrafilter}\}$  and  $\{\sigma_A \mid A \in Z(X)^\#\}$  is a base for closed sets of  $QF(X)$ , where  $\sigma_A = \{\alpha \mid \alpha \text{ is a fixed } Z(X)^\#\text{-filter and } A \in \alpha\}$ .

LEMMA 2.6. ([8]) *Let  $X$  be a space. Then  $\Phi_\beta^{-1}(X)$  is a quasi-F space if and only if  $QF(X) = \Phi_\beta^{-1}(X)$  and  $\Phi_X = \Phi_{\beta_X}$ , that is,  $\Phi_X(\alpha) = \cap\{A \mid A \in \alpha\}$ , where  $\Phi_{\beta_X} : \Phi_\beta^{-1}(X) \rightarrow X$  is the restriction and corestriction of  $\Phi_\beta$  with respect to  $\Phi_\beta^{-1}(X)$  and  $X$ , respectively.*

Let  $X$  be a space. Then there is a continuous map  $\Phi^\beta : \beta QF(X) \rightarrow \beta X$  such that  $\beta_X \circ \Phi_X = \Phi^\beta \circ \beta_{QF(X)}$ . Since  $\beta QF(X)$  and  $\beta X$  are compact spaces and  $\beta_X : X \rightarrow \beta X$  and  $\beta_{QF(X)} : QF(X) \rightarrow \beta QF(X)$  are dense embeddings,  $\Phi^\beta$  is a covering map. Hence  $(\beta QF(X), \Phi^\beta)$  is a quasi-F cover of  $\beta X$  and there is a covering map  $q_X : \beta QF(X) \rightarrow QF(\beta X)$  such that  $\Phi^\beta = \Phi_\beta \circ q_X$ .

For any space  $X$  and  $x \in X$ , let  $\delta(x) = \{A \in Z(X)^\# \mid x \in \text{int}_X(A)\}$ .

PROPOSITION 2.7. *Let  $X$  be a space such that  $\Phi_\beta^{-1}(X)$  is a quasi-F space and  $\alpha$  a fixed  $Z(X)^\#\text{-ultrafilter}$ . Then we have the following :*

- (1)  $\Phi_X(\alpha) = x$  if and only if  $\delta(x) \subseteq \alpha$ ,
- (2) for any  $A \in Z(X)^\#$ ,  $\Phi_X(\sigma_A) = A$ , and
- (3) for any decreasing sequence  $(A_n)$  in  $Z(X)^\#$ ,  $\Phi_X(\cap\{\sigma_{A_n} \mid n \in N\}) = \cap\{A_n \mid n \in N\}$ .

*Proof.* (1) ( $\Rightarrow$ ) Let  $\Phi_X(\alpha) = x$ . Suppose that there is an  $A \in \delta(x) - \alpha$ . Since  $A \notin \alpha$  and  $\alpha$  is a  $Z(X)^\#\text{-ultrafilter}$ ,  $A' \in \alpha$  and  $\Phi_X(\alpha) = x \in \text{int}_X(A) = X - A'$ . This is a contradiction. Hence  $\delta(x) \subseteq \alpha$ .

( $\Leftarrow$ ) Suppose that  $\delta(x) \subseteq \alpha$ . Note that  $\delta(x)$  is a local base at  $x$  in  $X$ . Let  $A \in \alpha$ . Then for any  $B \in \delta(x)$ ,  $B \in \alpha$  and  $A \cap B \neq \emptyset$ . Hence  $x \in \text{cl}_X(A) = A$  and so  $x \in \cap\{A \mid A \in \alpha\} = \Phi_X(\alpha)$ .

(2) Let  $A \in Z(X)^\#$  and  $\alpha \in \sigma_A$ . Then  $A \in \alpha$  and  $\Phi_X(\alpha) = \cap\{B \mid B \in \alpha\} \in A$ . Hence  $\Phi_X(\sigma_A) \subseteq A$ .

Suppose that  $y \in A$ . Then  $\delta(y) \cup \{A\}$  has the finite meet property such that  $\delta(y) \cup \{A\} \subseteq Z(X)^\#$ . By Zorn's lemma, there is a  $Z(X)^\#$ -ultrafilter  $\gamma$  such that  $\delta(y) \cup \{A\} \subseteq \gamma$ . By (1),  $\Phi_X(\gamma) = y$  and since  $A \in \gamma$ ,  $\gamma \in \sigma_A$ . Hence  $y \in \Phi_X(\sigma_A)$  and  $A \subseteq \Phi_X(\sigma_A)$ .

(3) By Lemma 2.2, it is trivial. □

### 3. Minimal quasi-F covers of $vX$

Iliadis([6])(Vermeer([8]), resp.) showed that every space  $X$  has the minimal extremally disconnected cover  $(E(X), k_X)$ (the minimal basically disconnected cover  $(\Lambda(X), \Lambda_X)$ , resp.) of  $X$ .

We recall that an  $\mathcal{A}$ -ultrafilter  $\alpha$  is called *stable* if  $\cap\{cl_{\beta X}(A) \mid A \in \alpha\} \subseteq vX$ , where  $\mathcal{A}$  is a sublattice of  $R(X)$ .

In [9] and [7], we can find internal characterizations of a space  $X$  which  $E(vX) = vE(X)$  and  $\Lambda(vX) = v\Lambda(X)$ . In fact, for any space  $X$ (for any space  $X$  with  $\beta\Lambda(X) = \Lambda(\beta X)$ , resp.), the following are equivalent :

- (1)  $E(vX) = vE(X)$ ( $\Lambda(vX) = v\Lambda(X)$ , resp.),
- (2) for any decreasing sequence  $(A_n)$  in  $R(X)$ ( $\sigma Z(X)^\#$ , resp.) with  $\cap\{A_n \mid n \in N\} = \emptyset$ ,  $\cap\{cl_{vX}(A_n) \mid n \in N\} = \emptyset$ ,
- (3) for any decreasing sequence  $(A_n)$  in  $R(X)$ ( $\sigma Z(X)^\#$ , resp.),  $cl_{vX}(\cap\{A_n \mid n \in N\}) = \cap\{cl_{vX}(A_n) \mid n \in N\}$ , and
- (4) every stable  $R(X)$ -ultrafilter( $\sigma Z(X)^\#$ -ultrafilter, resp.) has the countable intersection property

where  $\sigma Z(X)^\#$  is the smallest complete Boolean subalgebra of  $R(X)$  such that it is closed under countable meets and  $Z(X)^\# \subseteq \sigma Z(X)^\#$ .

DEFINITION 3.1. (1) A covering map  $f : Y \rightarrow X$  is called  *$z^\#$ -irreducible* if  $f(Z(Y)^\#) = Z(X)^\#$ .

- (2) A subspace  $X$  of a space  $Y$  is called  *$z^\#$ -embedded in  $Y$*  if for any  $A \in Z(X)^\#$ , there is a  $B \in Z(Y)^\#$  such that  $A = B \cap X$ .

It is well-known that for any compact space  $X$ ,  $\Phi_X : QF(X) \rightarrow X$  is  $z^\#$ -irreducible([4]) and that every  $C^*$ -embedded subspace  $W$  of a space  $Z$  is  $z^\#$ -embedded in  $Z$ ([2]).

Let  $f : Y \rightarrow X$  be a covering map. Since  $Z(X)^\# \subseteq f(Z(Y)^\#)$ ,  $f : Y \rightarrow X$  is  $z^\#$ -irreducible if and only if  $f(Z(Y)^\#) \subseteq Z(X)^\#$ . Using this, we have the following :

PROPOSITION 3.2. *Let  $f : Y \rightarrow X$  and  $g : X \rightarrow W$  be covering maps. Then  $g \circ f$  is  $z^\#$ -irreducible if and only if  $f : Y \rightarrow X$  and  $g : X \rightarrow W$  are  $z^\#$ -irreducible.*

Let  $X$  be a space. Since  $vX$  is a realcompact space, there is a continuous map  $r_X : vQF(X) \rightarrow vX$  such that  $v_X \circ \Phi_X = r_X \circ v_{QF(X)}$ .

THEOREM 3.3. *Let  $X$  be a space such that  $\beta QF(X) = QF(\beta X)$ . Suppose that every stable  $Z(X)^\#$ -ultrafilter has the countable intersection property. Then we have the following :*

- (1) for any  $x \in vX$ ,  $\Phi_\beta^{-1}(x) = r_X^{-1}(x)$ , and
- (2)  $r_X : vQF(X) \rightarrow vX$  is a covering map.

*Proof.* (1) Since  $\beta QF(X) = QF(\beta X)$ ,  $vQF(X)$  is a  $C^*$ -embedded subspace of  $QF(\beta X)$ .

Let  $x \in vX$ . Since  $\Phi_\beta$  is an onto map,  $\Phi_\beta^{-1}(x) \neq \emptyset$ . Suppose that there is an  $\alpha \in \Phi_\beta^{-1}(x) - vQF(X)$ . Then there is a zero-set  $Z$  in  $\beta QF(X)$  such that  $\alpha \in Z$  and  $Z \cap vQF(X) = \emptyset$  ([2]) and there is a real-valued continuous map  $f$  on  $X$  such that  $f^{-1}(0) = Z$ . For any  $n \in N$ , let  $Z_n = cl_{\beta QF(X)}(int_{\beta QF(X)}(f^{-1}([\frac{1}{n}, \frac{1}{n}])))$ . Then  $(Z_n)$  is a decreasing sequence in  $Z(\beta QF(X))^\#$  such that  $Z = \cap\{Z_n \mid n \in N\}$  and  $\alpha \in int_{\beta QF(X)}(Z_n)$  for all  $n \in N$ . Moreover,  $\cap\{Z_n \mid n \in N\} \cap QF(X) = \emptyset$ . Since  $\Phi_\beta$  is  $z^\#$ -irreducible,  $(\Phi_\beta(Z_n))$  is a decreasing sequence in  $Z(\beta X)$ . By Lemma 2.1,  $\alpha_X = \{A \cap X \mid A \in \alpha\}$  is a  $Z(X)^\#$ -ultrafilter.

Let  $n \in N$ . Since  $\Phi_\beta$  is  $z^\#$ -irreducible, there is an  $A_n \in Z(\beta X)$  such that  $\alpha \in \sigma_{A_n} = Z_n$ . Hence  $\Phi_\beta(\alpha) \in A_n = \Phi_\beta(Z_n)$ . Since  $A_n \in \alpha$ ,  $\Phi_\beta(Z_n) \in \alpha$  and  $\Phi_\beta(Z_n) \cap X \in Z(X)^\#$ . Note that  $x \in \cap\{cl_{\beta X}(A) \mid A \in \alpha\} = \cap\{cl_{\beta X}(B) \mid B \in \alpha_X\}$ . Since  $x \in vX$ ,  $\alpha_X$  is a stable  $Z(X)^\#$ -ultrafilter and by the assumption,  $\alpha_X$  has the countable intersection property. Thus  $\cap\{\Phi_\beta(Z_n) \cap X \mid n \in N\} \neq \emptyset$ . Since  $\beta QF(X) = QF(\beta X)$ ,  $QF(X) = \Phi_\beta^{-1}(X)$  and  $\{Z_n \cap \Phi_\beta^{-1}(x) \mid n \in N\}$  is a family of closed sets in  $\Phi_\beta^{-1}(x)$  with the finite intersection property. Since  $\Phi_\beta^{-1}(x)$  is compact,  $\cap\{Z_n \mid n \in N\} \neq \emptyset$ . This is a contradiction and so  $r_X^{-1}(x) = \Phi_\beta^{-1}(x) \subseteq vQF(X)$ .

- (2) By (1),  $r_X : v\Lambda X \rightarrow vX$  is an onto, compact map.

Let  $F$  be a closed set in  $vQF(X)$  and  $x \in vX - r_X(F)$ . Then  $r_X^{-1}(x) \cap F = \emptyset$  and since  $r_X^{-1}(x)$  is a compact subset of  $vQF(X)$  and  $vQF(X)$  is a subspace of  $QF(\beta X)$ , there is an  $A \in Z(\beta X)^\#$  such that  $r_X^{-1}(x) \cap \Sigma_A = \emptyset$  and  $F \subseteq \Sigma_A$ , because  $\{\Sigma_A \mid A \in Z(\beta X)^\#\}$  is a base for closed sets in  $\beta QF(X) = QF(\beta X)$ . Since  $\Phi_\beta(\Sigma_A) = A$ ,  $\Phi_\beta(F) = r_X(F) \subseteq A$ . Then

$x \notin A$  and since  $A$  is a closed set in  $\beta X$ ,  $cl_{vX}(r_X(F)) \subseteq A \cap vX$ . Hence  $x \notin cl_{vX}(r_X(F))$  and  $cl_{vX}(r_X(F)) \subseteq r_X(F)$ . Thus  $r_X$  is a closed map and  $r_X$  is a covering map.  $\square$

PROPOSITION 3.4. *Let  $X$  be a space. Then the following are equivalent :*

- (1) *for any decreasing sequence  $(A_n)$  in  $Z(X)^\#$  with  $\bigcap\{A_n \mid n \in N\} = \emptyset$ ,  $\bigcap\{cl_{vX}(A_n) \mid n \in N\} = \emptyset$ ,*
- (2) *for any decreasing sequence  $(A_n)$  in  $Z(X)^\#$ ,  $cl_{vX}(\bigcap\{A_n \mid n \in N\}) = \bigcap\{cl_{vX}(A_n) \mid n \in N\}$ , and,*
- (3) *every stable  $Z(X)^\#$ -ultrafilter has the countable intersection property.*

*Proof.* (1)  $\Rightarrow$  (2) Clearly,  $cl_{vX}(\bigcap\{A_n \mid n \in N\}) \subseteq \bigcap\{cl_{vX}(A_n) \mid n \in N\}$ . Let  $x \in \bigcap\{cl_{vX}(A_n) \mid n \in N\} - cl_{vX}(\bigcap\{A_n \mid n \in N\})$ . Then there is a  $B \in Z(vX)^\#$  such that  $x \in int_{vX}(B)$  and  $B \cap (\bigcap\{A_n \mid n \in N\}) = \emptyset$ . Let  $C = B \cap X$ . Then by Lemma 2.1,  $C \in Z(X)^\#$  and  $\{C \wedge A_n \mid n \in N\}$  is a decreasing sequence in  $Z(X)^\#$  with  $\bigcap\{C \wedge A_n \mid n \in N\} = \emptyset$ . By (1),  $\bigcap\{cl_{vX}(C \wedge A_n) \mid n \in N\} = \emptyset$ .

Let  $U$  be a neighborhood of  $x$  in  $vX$  and  $n \in N$ . Then  $U \cap int_{vX}(B) \cap A_n \neq \emptyset$ . Note that

$$\begin{aligned} & int_{vX}(B) \cap A_n \\ &= (int_{vX}(B) \cap X) \cap A_n \\ &\subseteq int_X(B \cap X) \cap A_n \\ &= int_X(C) \cap A_n \\ &\subseteq cl_X(int_X(C) \cap int_X(A_n)) \\ &= C \wedge A_n. \end{aligned}$$

Since  $U \cap int_{vX}(B) \cap A_n \neq \emptyset$ ,  $U \cap (C \wedge A_n) \neq \emptyset$  and  $x \in C \wedge A_n$ . Hence  $x \in \bigcap\{cl_{vX}(C \wedge A_n) \mid n \in N\}$ . This is a contradiction. Thus  $\bigcap\{cl_{vX}(A_n) \mid n \in N\} \subseteq cl_{vX}(\bigcap\{A_n \mid n \in N\})$ .

(2)  $\Rightarrow$  (3) Let  $\alpha$  be a stable  $Z(X)^\#$ -ultrafilter and  $(A_n)$  a sequence in  $\alpha$ . For any  $n \in N$ ,  $B_n = \bigwedge\{A_i \mid i = 1, 2, \dots, n\}$ . Then  $(B_n)$  is a decreasing sequence in  $Z(X)^\#$ . Since  $\alpha$  is stable, there is an  $x \in vX$  such that  $x \in \bigcap\{cl_{\beta X}(A_n) \mid n \in N\}$  and so  $x \in \bigcap\{cl_{vX}(A_n) \mid n \in N\}$ . By (2),  $x \in cl_{vX}(\bigcap\{A_n \mid n \in N\})$  and  $\bigcap\{A_n \mid n \in N\} \neq \emptyset$ . Hence  $\alpha$  has the countable intersection property.

(3)  $\Rightarrow$  (1) Let  $(A_n)$  be a decreasing sequence in  $Z(X)^\#$  such that  $\bigcap\{A_n \mid n \in N\} = \emptyset$ . Suppose that  $\bigcap\{cl_{vX}(A_n) \mid n \in N\} \neq \emptyset$ . Pick  $x \in \bigcap\{cl_{vX}(A_n) \mid n \in N\}$ . Let  $\alpha_0 = \{B \cap X \mid B \in Z(vX)^\#, x \in$

$int_{vX}(B)\} \cup \{A_n \mid n \in N\}$ . Then clearly,  $\alpha_0$  is a subset of  $Z(X)^\#$  with the finite meet property and by Zorn's lemma, there is a  $Z(X)^\#$ -ultrafilter  $\alpha$  such that  $\alpha_0 \subseteq \alpha$ . Since  $\{B \in Z(vX)^\# \mid x \in int_{vX}(B)\}$  is a local base at  $x$  in  $vX$ ,  $x \in \cap\{cl_{vX}(A) \mid A \in \alpha\}$  and  $\alpha$  is a stable  $Z(X)^\#$ -ultrafilter. By (3),  $\alpha$  has the countable intersection property and since  $\{A_n \mid n \in N\} \subseteq \alpha$ ,  $\cap\{A_n \mid n \in N\} \neq \emptyset$ . It is a contradiction and thus  $\cap\{cl_{vX}(A_n) \mid n \in N\} = \emptyset$ .  $\square$

Let  $X$  be a space with  $QF(\beta X) = \beta QF(X)$ . Then  $\Phi_X : QF(X) \rightarrow X$  is  $z^\#$ -irreducible([5]).

**THEOREM 3.5.** *Let  $X$  be a space with  $QF(\beta X) = \beta QF(X)$ . Suppose that every stable  $Z(X)^\#$ -ultrafilter has the countable intersection property. Then  $QF(vX) = vQF(X)$ , that is, there is a homeomorphism  $h_X : vQF(X) \rightarrow QF(vX)$  such that  $r_X = \Phi_v \circ h_X$ .*

*Proof.* By Theorem 3.3,  $r_X : v\Lambda X \rightarrow vX$  is a covering map. Since  $(vQF(X), r_X)$  is a quasi-F cover of  $vX$ , there is a covering map  $h_X : vQF(X) \rightarrow QF(vX)$  such that  $r_X = \Phi_v \circ h_X$ . Since  $QF(\beta X) = \beta QF(X)$ ,  $\Phi_X$  is  $z^\#$ -irreducible. Note that  $r_X \circ vQF(X) = vX \circ \Phi_X$  and  $vQF(X), vX$  are  $C^*$ -embedded in  $vQF(X)$  and  $vX$ , respectively. Hence  $r_X$  is  $z^\#$ -irreducible and by Proposition 3.2,  $h_X$  is  $z^\#$ -irreducible.

Let  $p \neq q$  in  $vQF(X)$ . Then there are  $A, B$  in  $Z(vQF(X))^\#$  such that  $p \in A, q \in B$  and  $A \cap B = \emptyset$ . Hence  $A \wedge B = \emptyset$  and by Lemma 2.1,  $\emptyset = h_X(A \wedge B) = h_X(A) \wedge h_X(B)$ . Since  $h_X$  is  $z^\#$ -irreducible,  $h_X(A) \in Z(QF(vX))^\#$  and  $h_X(B) \in Z(QF(vX))^\#$ . Since  $QF(vX)$  is a quasi-F space,  $h_X(A) \wedge h_X(B) = h_X(A) \cap h_X(B) = \emptyset$ . Since  $h_X(p) \in h_X(A)$  and  $h_X(q) \in h_X(B)$ ,  $h_X(p) \neq h_X(q)$  and  $h_X$  is one-to-one. Hence  $vQF(X) = QF(vX)$ .  $\square$

For any space  $X$ ,  $Z(X)^\# \subseteq \sigma Z(X)^\# \subseteq R(X)$ . By Theorem 3.3, Proposition 3.4, and Theorem 3.5, we have the following :

**COROLLARY 3.6.** *Let  $X$  be a space such that  $QF(\beta X) = \beta QF(X)$ . If  $E(vX) = vE(X)$  or  $\Lambda(vX) = v\Lambda(X)$ , then  $QF(vX) = vQF(X)$ .*

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