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# GALOIS ACTIONS OF A CLASS INVARIANT OVER QUADRATIC NUMBER FIELDS WITH DISCRIMINANT $D \equiv 64 \pmod{72}$

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Abstract. A class invariant is the value of a modular function that generates a ring class field of an imaginary quadratic number field such as the singular moduli of level 1. In this paper, we compute the Galois actions of a class invariant from a generalized Weber function $\mathfrak{g}_1$ over imaginary quadratic number fields with discriminant $D \equiv 64 \pmod{72}$.

## 1. Introduction

Let $K$ be an imaginary quadratic number field with discriminant $D$ and the ring of integers $\mathcal{O} = \mathbb{Z}[\theta]$ where

$$\theta := \begin{cases} \frac{\sqrt{D}}{2}, & \text{if } D \equiv 0 \pmod{4}; \\ \frac{-1+\sqrt{D}}{2}, & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

Then the theory of complex multiplication states that the modular invariant $j(\mathcal{O}) = j(\theta)$ generates the *ring class field* $H_{\mathcal{O}}$ over $K$ with degree $[H_{\mathcal{O}} : K] = h(\mathcal{O})$, the class number of $\mathcal{O}$, and the conjugates of $j(\theta)$ under the action of $Gal(H_{\mathcal{O}}/K)$ are *singular moduli* $j(\tau)$, where $\tau := \tau_Q$ is the Heegner point determined by $Q(\tau_Q, 1) = 0$ for a positive definite integral primitive binary quadratic forms

$$Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$$

with discriminant $D = b^2 - 4ac$.

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In his Lehrbuch der Algebra [10], H. Weber calls the value of a modular function $f(\theta)$ a *class invariant* if we have

$$K(f(\theta)) = K(j(\theta)).$$

Despite a long history of the problem, one began to treat class invariants in a systemic and algorithmic way only after Shimura Reciprocity Law [8] became available. The reciprocity law provides not only a method of systematically determining whether $f(\theta)$ is a class invariant but also a description of the Galois conjugates of $f(\theta)$ under $Gal(H_{\mathcal{O}}/K)$. This tool is well illustrated in several works by Alice Gee and Peter Stevenhagen in [2, 3, 4, 9]. The author[5, 6, 7] compute the Galois actions of certain class invariants over some cases of quadratic number fields.

Gee determine the class invariants from a generalized Weber function $\mathfrak{g}_1$ by using the Shimura Reciprocity Law as follows:

Theorem 1.1. [3, p.73, Theorem 1] *Let $K$ be an imaginary quadratic number field with discriminant $D \equiv 64 \pmod{72}$ and the ring of integers $\mathcal{O} = \mathbb{Z}[\theta]$ where $\theta = \frac{\sqrt{D}}{2}$. Then $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)$ gives an integral generator for the ring class field $H_{\mathcal{O}}$ over $K$ where $\zeta_n$ is a primitive $n$-th root of unity for a positive integer $n$.*

In this paper, we compute the Galois actions of the class invariant $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)$ under $Gal(H_{\mathcal{O}}/K)$.

## 2. Preliminary

Let $\mathcal{Q}_D^0$ be the set of primitive quadratic forms and $C(D) = \mathcal{Q}_D^0/\Gamma(1)$ denote the form class group of discriminant $D$. Since $Gal(H_{\mathcal{O}}/K)$ is isomorphic to $C(D)$, it suffices to compute the action of a primitive quadratic form $Q = [a, b, c]$ on the class invariant $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)$.

Theorem 2.1. [1, 2] *Let $\mathbb{Z}[\theta]$ be the ring of integers of an imaginary quadratic number field $K$ with discriminant $D \equiv 0 \pmod 4$ and let $Q = [a, b, c]$ be a primitive quadratic form with discriminant $D$. Let $\theta = \frac{\sqrt{D}}{2}$ and $\tau_Q = \frac{-b+\sqrt{D}}{2a}$. Let $M = M_{[a,b,c]} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be given as follows:*

$$M \equiv \begin{cases} \begin{pmatrix} a & \frac{b}{2} \\ 0 & 1 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \nmid a; \\ \begin{pmatrix} -\frac{b}{2} & -c \\ 1 & 0 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \mid a \text{ and } p \nmid c; \\ \begin{pmatrix} -\frac{b}{2}-a & -\frac{b}{2}-c \\ 1 & -1 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \mid a \text{ and } p \mid c, \end{cases} \tag{2.1}$$

*where $p$ runs over all prime factors of $N$ and $p^{r_p}||N$. Then the Galois action of the class of $[a,-b,c]$ in $C(D)$ with respect to the Artin map is given by*

$$f(\theta)^{[a,-b,c]} = f^M(\tau_Q)$$

*for any modular function $f$ of level $N$ such that $f(\theta) \in H_{\mathcal{O}}$. Here $f^M$ denote the image of $f$ under the action of $M$.*

The action of $M$ depends only on $M_{p^{r_p}}$ for all primes $p|N$ where $M_{p^{r_p}} \in \mathrm{GL}_2(\mathbb{Z}/p^{r_p}\mathbb{Z})$ is the reduction modulo $p^{r_p}$ of $M$. Every $M_{p^{r_p}}$ with determinant $x$ decomposes as $M_{p^{r_p}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/p^{r_p}\mathbb{Z})$. Since $\mathrm{SL}_2(\mathbb{Z}/p^{r_p}\mathbb{Z})$ is generated by $S_{p^{r_p}} \equiv \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T_{p^{r_p}} \equiv \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, it suffices to find the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)_{p^{r_p}}$, $S_{p^{r_p}}$ and $T_{p^{r_p}}$ on $f$ for all $p|N$, where $f$ is a modular function of level $N$ whose Fourier coefficients belong to $\mathbb{Q}(\zeta_N)$. For $\left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)_{p^{r_p}}$, the action on $f$ is given by lifting the automorphism of $\mathbb{Q}(\zeta_N)$ determined by

$$\zeta_{p^{r_p}} \mapsto \zeta_{p^{r_p}}^x \quad \text{and} \quad \zeta_{q^{r_q}} \mapsto \zeta_{q^{r_q}}$$

for all prime factors $q|N$ with $q \neq p$. In order that the actions of the matrices at different primes commute with each other, we lift $S_{p^{r_p}}$ and $T_{p^{r_p}}$ to matrices in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ such that they reduce to the identity matrix in $\mathrm{SL}_2(\mathbb{Z}/q^{r_q}\mathbb{Z})$ for all $q \neq p$.

The Dedekind-eta function

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad \text{with} \quad q = e^{2\pi i z}$$

is holomorphic and non-zero for $z$ in the complex upper half plane $\mathbb{H}$ and $\Delta(z) = \eta^{24}(z)$ is modular form of weight 12 with no poles or zeros on $\mathbb{H}$. Then we have generalized Weber functions as follows:

(2.2)
$$\mathfrak{g}_0(z) = \frac{\eta(\frac{z}{3})}{\eta(z)},\ \mathfrak{g}_1(z) = \zeta_{24}^{-1}\frac{\eta(\frac{z+1}{3})}{\eta(z)},\ \mathfrak{g}_2(z) = \frac{\eta(\frac{z+2}{3})}{\eta(z)},\ \mathfrak{g}_3(z) = \sqrt{3}\frac{\eta(3z)}{\eta(z)}.$$

Note that the functions in (2.2) are modular of level 72. For the generating matrices $S, T \in SL_2(\mathbb{Z})$ given by $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, the transformation rules $\eta \circ S(z) = \sqrt{-iz}\,\eta(z)$ and $\eta \circ T(z) = \zeta_{24}\eta(z)$ hold. Hence

(2.3)
$$\begin{aligned}(\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ S &= (\mathfrak{g}_3, \zeta_{24}^{-2}\mathfrak{g}_2, \zeta_{24}^{2}\mathfrak{g}_1, \mathfrak{g}_0),\\ (\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ T &= (\mathfrak{g}_1, \zeta_{24}^{-2}\mathfrak{g}_2, \mathfrak{g}_0, \zeta_{24}^{2}\mathfrak{g}_3).\end{aligned}$$

## 3. Results

In this section, we compute the action of a primitive quadratic form $[a,-b,c]$ on the class invariant $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)$. For that we need to find the action of $M_m \in GL_2(\mathbb{Z}/m\mathbb{Z})$ with $m=8,9$. Combining Lemma 6 of [2] and the transformation rule (2.3), we obtain the following:

LEMMA 3.1. *The actions of* $\left(\begin{smallmatrix}1&0\\0&x\end{smallmatrix}\right)_m$, $S_m$ *and* $T_m$ $(m=8,9)$ *on* $\mathfrak{g}_i^2$ $(i=0,1,2,3)$ *are given by*

| | $\mathfrak{g}_0^2$ | $\mathfrak{g}_1^2$ | $\mathfrak{g}_2^2$ | $\mathfrak{g}_3^2$ |
|---|---|---|---|---|
| $\left(\begin{smallmatrix}1&0\\0&x\end{smallmatrix}\right)_8$ | $\mathfrak{g}_0^2$ | $\mathfrak{g}_1^2$ | $\mathfrak{g}_2^2$ | $\mathfrak{g}_3^2$ |
| $S_8$ | $-\mathfrak{g}_0^2$ | $-\mathfrak{g}_1^2$ | $-\mathfrak{g}_2^2$ | $-\mathfrak{g}_3^2$ |
| $T_8$ | $-\mathfrak{g}_0^2$ | $-\mathfrak{g}_1^2$ | $-\mathfrak{g}_2^2$ | $-\mathfrak{g}_3^2$ |
| $\left(\begin{smallmatrix}1&0\\0&x\end{smallmatrix}\right)_9$, $x=-3k+1$ | $\mathfrak{g}_0^2$ | $\zeta_3^{2k}\mathfrak{g}_1^2$ | $\zeta_3^{k}\mathfrak{g}_2^2$ | $\mathfrak{g}_3^2$ |
| $\left(\begin{smallmatrix}1&0\\0&x\end{smallmatrix}\right)_9$, $x=-3k-1$ | $\mathfrak{g}_0^2$ | $\zeta_3^{2k}\mathfrak{g}_2^2$ | $\zeta_3^{k}\mathfrak{g}_1^2$ | $\mathfrak{g}_3^2$ |
| $S_9$ | $-\mathfrak{g}_3^2$ | $\zeta_3\mathfrak{g}_2^2$ | $\zeta_3^2\mathfrak{g}_1^2$ | $-\mathfrak{g}_0^2$ |
| $T_9$ | $-\mathfrak{g}_1^2$ | $\zeta_3\mathfrak{g}_2^2$ | $-\mathfrak{g}_0^2$ | $\zeta_3^2\mathfrak{g}_3^2$ |

Theorem 2.1 gives a matrix $M \in \mathrm{GL}_2(\mathbb{Z}/72\mathbb{Z})$ that satisfies

$$\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]} = (\zeta_3^2\zeta_4\mathfrak{g}_1^2)^M(\tau_Q).$$

Also

$$(\zeta_3^2\zeta_4\mathfrak{g}_1^2)^M = (\zeta_3^2(\zeta_4\mathfrak{g}_1^2)^{M_8})^{M_9}.$$

By Lemma 3.1, we have

$$(\zeta_4\mathfrak{g}_1^2)^{M_8} = u\mu_4\mathfrak{g}_1^2,$$

where $u=(-1)^{a+\frac{b-2}{2}}$ and $\mu_4=\zeta_4^{a+(a+1)(\frac{b^2}{4}+c)}$. Using this, together with Lemma 3.1, we have the following theorems.

THEOREM 3.2. *Let* $D\equiv 64 \pmod{72}$ *be a discriminant of an order* $\mathcal{O}=[\theta,1]$ *in an imaginary quadratic field. Let* $\theta=\frac{\sqrt{D}}{2}$, $\tau_Q=\frac{-b+\sqrt{D}}{2a}$, $u=(-1)^{a+\frac{b-2}{2}}$ *and* $\mu_4=\zeta_4^{a+(a+1)(\frac{b^2}{4}+c)}$. *If* $Q=[a,b,c]$ *is a reduced primitive quadratic form with discriminant* $D$, *then the action of* $[a,-b,c]$ *on* $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)$ *is as follows:*

(1) *The case* $3\nmid a$.

a) *If* $b\equiv 1 \pmod 3$, *then* $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

| | $b\equiv 1 \pmod 9$ | $b\equiv 4 \pmod 9$ | $b\equiv 7 \pmod 9$ |
|---|---|---|---|
| $a\equiv 1 \pmod 9$ | $-u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q)$ | $-u\mu_4\mathfrak{g}_0^2(\tau_Q)$ | $-u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q)$ |
| $a\equiv 2 \pmod 9$ | $-u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q)$ | $-u\mu_4\mathfrak{g}_0^2(\tau_Q)$ | $-u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q)$ |

| | | | |
|---|---|---|---|
| $a \equiv 4 \pmod 9$ | $-u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q)$ | $-u\mu_4\mathfrak{g}_0^2(\tau_Q)$ | $-u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q)$ |
| $a \equiv 5 \pmod 9$ | $-u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q)$ | $-u\mu_4\mathfrak{g}_0^2(\tau_Q)$ | $-u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q)$ |
| $a \equiv 7 \pmod 9$ | $-u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q)$ | $-u\mu_4\mathfrak{g}_0^2(\tau_Q)$ | $-u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q)$ |
| $a \equiv 8 \pmod 9$ | $-u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q)$ | $-u\mu_4\mathfrak{g}_0^2(\tau_Q)$ | $-u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q)$ |

b) *If $a+b \equiv 1 \pmod 3$, then $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ is given by the following table:*

| | $b \equiv 0 \pmod 9$ | $b \equiv 3 \pmod 9$ | $b \equiv 6 \pmod 9$ |
|---|---|---|---|
| $a \equiv 1 \pmod 9$ | $u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_1^2(\tau_Q)$ |
| $a \equiv 4 \pmod 9$ | $u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$ |
| $a \equiv 7 \pmod 9$ | $u\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$ |

| | $b \equiv 2 \pmod 9$ | $b \equiv 5 \pmod 9$ | $b \equiv 8 \pmod 9$ |
|---|---|---|---|
| $a \equiv 2 \pmod 9$ | $u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_1^2(\tau_Q)$ |
| $a \equiv 5 \pmod 9$ | $u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$ |
| $a \equiv 8 \pmod 9$ | $u\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$ |

c) *If $b \not\equiv 1 \pmod 3$ and $a+b \not\equiv 1 \pmod 3$, then $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ is given by the following table:*

| | $b \equiv 2 \pmod 9$ | $b \equiv 5 \pmod 9$ | $b \equiv 8 \pmod 9$ |
|---|---|---|---|
| $a \equiv 1 \pmod 9$ | $u\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$ |
| $a \equiv 4 \pmod 9$ | $u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$ |
| $a \equiv 7 \pmod 9$ | $u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_2^2(\tau_Q)$ |

| | $b \equiv 0 \pmod 9$ | $b \equiv 3 \pmod 9$ | $b \equiv 6 \pmod 9$ |
|---|---|---|---|
| $a \equiv 2 \pmod 9$ | $u\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$ |
| $a \equiv 5 \pmod 9$ | $u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$ |
| $a \equiv 8 \pmod 9$ | $u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_2^2(\tau_Q)$ |

(2) *The case $3|a$ and $3 \nmid c$.*

a) *If $b \equiv 2 \pmod 3$, then $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ is given by the following table:*

| | $b \equiv 2 \pmod 9$ | $b \equiv 5 \pmod 9$ | $b \equiv 8 \pmod 9$ |
|---|---|---|---|
| $c \equiv 1 \pmod 9$ | $u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_3^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q)$ |
| $c \equiv 2 \pmod 9$ | $u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_3^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q)$ |
| $c \equiv 4 \pmod 9$ | $u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_3^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q)$ |
| $c \equiv 5 \pmod 9$ | $u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_3^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q)$ |
| $c \equiv 7 \pmod 9$ | $u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_3^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q)$ |
| $c \equiv 8 \pmod 9$ | $u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_3^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q)$ |

b) *If* $b \not\equiv 2 \pmod 3$ *and* $b + c \equiv 0 \pmod 3$, *then*

| | $b \equiv 1 \pmod 9$ | $b \equiv 4 \pmod 9$ | $b \equiv 7 \pmod 9$ |
|---|---|---|---|
| $c \equiv 2 \pmod 9$ | $u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$ |
| $c \equiv 5 \pmod 9$ | $u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_2^2(\tau_Q)$ |
| $c \equiv 8 \pmod 9$ | $u\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$ |

c) *If* $b + c \equiv 2 \pmod 3$, *then*

| | $b \equiv 1 \pmod 9$ | $b \equiv 4 \pmod 9$ | $b \equiv 7 \pmod 9$ |
|---|---|---|---|
| $c \equiv 1 \pmod 9$ | $u\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$ |
| $c \equiv 4 \pmod 9$ | $u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_1^2(\tau_Q)$ |
| $c \equiv 7 \pmod 9$ | $u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\mu_4\mathfrak{g}_1^2(\tau_Q)$ | $u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$ |

(3) *The case* $3|a$ *and* $3|c$.

a) *If* $b \equiv 1 \pmod 9$, *then*

$$\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]} = \begin{cases} -u\mu_4\mathfrak{g}_0^2(\tau_Q) & \text{if } c \equiv 0 \pmod 9; \\ -u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q) & \text{if } c \equiv 3 \pmod 9; \\ -u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q) & \text{if } c \equiv 6 \pmod 9; \end{cases}$$

b) *If* $b \equiv 8 \pmod 9$, *then*

$$\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]} = \begin{cases} u\mu_4\mathfrak{g}_3^2(\tau_Q) & \text{if } a \equiv 0 \pmod 9; \\ u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q) & \text{if } a \equiv 3 \pmod 9; \\ u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q) & \text{if } a \equiv 6 \pmod 9; \end{cases}$$

## References

[1] H. H. Chan, A. Gee, and V. Tan, *Cubic singular moduli, Ramanujan's class invariants $\lambda_n$ and the explicit Shimura reciprocity law*, Pacific J. Math. **208** (2003), 23–37.

[2] A. Gee, *Class invariants by Shimura's reciprocity law*, J. Théor. Nombre Bordeaux **11** (1999), 45–72.
[3] A. Gee, *Class fields by Shimura reciprocity*, Ph. D. Thesis, Universiteit van Amsterdam, 2001.
[4] A. Gee and P. Stevenhagen, *Generating class fields using Shimura reciprocity*, Proceedings of the Third International Symposium on Algorithmic Number Theory, Lecture Notes in Computer Sciences **1423** pp. 441-453, Springer-Verlag, 1998.
[5] D. Jeon, *Formulas of Galois actions of some class invariants over quartic number fields with discriminants* $D \equiv 1\,(mod\,12)$, J. of Chungcheong Math. Soc. **22** (2009), 799–814.
[6] D. Jeon, *Galois actions of a class invariant over quadratic number fields with discriminant* $D \equiv -3\,(mod\,36)$, J. of Chungcheong Math. Soc. **23** (2010), 853–860
[7] D. Jeon, *Galois actions of a class invariant over quadratic number fields with discriminant* $D \equiv 21\,(mod\,36)$, J. of Chungcheong Math. Soc. **24** (2011), 921–925
[8] G. Shimura, *Introduction to the arithmetic theory of automorphic forms*, Princeton University Press, 1971.
[9] P. Stevenhagen, 'Hilbert's 12th problem, complex multiplication and Shimura reciprocity'Class field theory-its centenary and prospect, ed. K. Miyake, *Adv. Studies in pure math.* **30** (2001), 161-176.
[10] H. Weber, *Lehrbuch der Algebra*, dritter Band, Friedrich Vieweg und Sohn, 1908.

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