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A NOTE ON EULERIAN POLYNOMIALS OF HIGHER ORDER

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Abstract. In this paper we derive some identities on Eulerian polynomials of higher order from non-linear ordinary differential equations. We show that the generating functions of Eulerian polynomials are solutions of our non-linear ordinary differential equations.

1. Introduction

It is well known that the generating function $F(t,x)$ of Euler polynomials $E_n(x)$ is given by

$$F(t,x) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.1}$$

In the special case, $x=0$, $E_n(0)=E_n$ is the n-th Euler number.
From (1.1), we note that

$$E_0 = 1, \quad (E+1)^n + E_n = 0, \quad \text{if } n>0, \tag{1.2}$$

with the usual convention of replacing E^n by E_n. The generating function $F_u(t,x)$ of Eulerian polynomials $H_n(x|u)$ are defined by

$$F_u(t,x) = \frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} H_n(x|u)\frac{t^n}{n!}, \tag{1.3}$$

where $u \in \mathbb{C}$ with $u \neq 1$ (see [1,2]). In the special case, $x=0$, $H_n(0|u) = H_n(u)$ is called the n-th Eulerian number (see [2]). Sometimes that is called the n-th Frobenius-Euler number (see [3-7]).
From (1.1) and (1.3), we note that $H_n(x|-1) = E_n(x)$.

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By (1.3), we get

$$\begin{aligned}\frac{1-u}{e^t-u}e^{xt} &= \left(\sum_{l=0}^{\infty} H_l(u)\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty}\frac{x^k t^k}{k!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}x^{n-l}H_l(u)\right)\frac{t^n}{n!}.\end{aligned} \tag{1.4}$$

From (1.3) and (1.4), we get

$$H_n(x|u) = \sum_{l=0}^{n}\binom{n}{l}x^{n-l}H_l(u) = (H(u)+x)^n \tag{1.5}$$

with the usual convention of replacing $H^n(u)$ by $H_n(u)$.
By (1.5), we obtain

$$\begin{aligned}1-u &= \frac{1-u}{e^t-u}e^t - \frac{1-u}{e^t-u}u \\ &= \sum_{n=0}^{\infty}(H(u)+1)^n\frac{t^n}{n!} - \sum_{n=0}^{\infty}uH_n(u)\frac{t^n}{n!}.\end{aligned} \tag{1.6}$$

Thus, we get the recurrence relation for $H_n(u)$ as follows:

$$H_0(u) = 1, \quad H_n(1|u) - uH_n(u) = (1-u)\delta_{0,n}, \tag{1.7}$$

where $\delta_{n,k}$ is Kronecker symbol (see [1-7]).
For $N \in \mathbb{N}$, the n-th Eulerian polynomials $H_n^{(N)}(x|u)$ of order N are defined by generating function as follows:

$$\begin{aligned}F_u^N(t,x) &= \underbrace{\left(\frac{1-u}{e^t-u}\right)\times\left(\frac{1-u}{e^t-u}\right)\times\cdots\times\left(\frac{1-u}{e^t-u}\right)}_{N-\text{times}}e^{xt} \\ &= \sum_{n=0}^{\infty}H_n^{(N)}(x|u)\frac{t^n}{n!}.\end{aligned} \tag{1.8}$$

In the special case, $x=0$, $H_n^{(N)}(0|u) = H_n^{(N)}(u)$ are called the n-th Eulerian numbers of order N (see [1-7]).

In [6], T. Kim introduced important ideas to obtain some new identities between the sums of products of Frobenius-Euler polynomials and Frobenius-Euler polynomials of higher order. The main idea is to construct non-linear ordinary differential equations with respect to t which are closely related to the generating functions of Frobenius-Euler polynomials.

In this paper, we consider non-linear ordinary differential equations with respect to u not t. The purpose of this paper is to give some new identities on the Eulerian polynomials of higher order by using the non-linear ordinary differential equations with respect to u.

2. Constuction of non-linear differential equations

We define that

$$\begin{aligned} F = F(u) &= \frac{1-u}{e^t-u} \quad \text{and} \\ F^N(t,x) &= \underbrace{F\times\cdots\times F}_{N-\text{times}} e^{xt} \quad \text{for } N\in\mathbb{N}. \end{aligned} \tag{2.1}$$

We note that $F(t,x) = F_u(t,x) = Fe^{xt}$.
By (2.1), we get

$$\begin{aligned} F^{(1)} = \frac{dF}{du} &= -\frac{1}{1-u}\frac{1-u}{e^t-u} + \frac{1}{1-u}\left(\frac{1-u}{e^t-u}\right)^2 \\ &= -\frac{1}{1-u}(F-F^2). \end{aligned} \tag{2.2}$$

By (2.2), we obtain

$$\begin{aligned} F^{(1)}(t,x) = F^{(1)}e^{xt} &= -\frac{1}{1-u}\left(F(t,x)-F^2(t,x)\right), \\ (1-u)F^{(1)} + F &= F^2. \end{aligned} \tag{2.3}$$

THEOREM 2.1. *For $u\in\mathbb{C}$ with $u\neq 1$, $N\in\mathbb{N}$,*
$F(u) = \dfrac{1-u}{e^t-u}$ *is a solution of*

$$F^N(u) = \sum_{k=0}^{N-1}\frac{1}{k!}(1-u)^k F^{(k)}(u), \tag{2.4}$$

where $F^{(k)}(u) = \dfrac{d^kF(u)}{du^k}$ *and* $F^N(u) = \underbrace{F(u)\times\cdots\times F(u)}_{N-times}$.

Proof. (Mathematical Induction)
(i) If $N=1$, then it is obvious.

(ii) Assume that (2.4) is true for some $N > 1$. Let us consider the derivative of (2.4).

$$\begin{aligned} NF^{N-1}F^{(1)} &= \sum_{k=0}^{N-1} \frac{1}{k!} \left(-k(1-u)^{k-1}F^{(k)} + (1-u)^k F^{(k+1)}\right) \\ &= \frac{1}{(N-1)!}(1-u)^{N-1}F^{(N)}. \end{aligned} \tag{2.5}$$

Thus, from (2.3) and (2.5), we have

$$\frac{1}{N!}(1-u)^N F^{(N)} = F^{N-1}(1-u)F^{(1)} = F^{N-1}(-F+F^2) \tag{2.6}$$

By (2.5) and (2.6), we get

$$F^{N+1} = F^N + \frac{1}{N!}(1-u)^N F^{(N)} = \sum_{k=0}^{N} \frac{1}{k!}(1-u)^k F^{(k)}.$$

□

COROLLARY 2.2. *For* $u \in \mathbb{C}$ *with* $u \neq 1$, $N \in \mathbb{N}$, $F(t,x) = \frac{1-u}{e^t-u}e^{xt}$ *is a solution of*

$$F^N(t,x) = \sum_{k=0}^{N-1} \frac{1}{k!}(1-u)^k F^{(k)}(t,x). \tag{2.7}$$

Proof. It is proved from the facts $F^N(t,x) = F^N(u)e^{xt}$ and $F^{(k)}(t,x) = \frac{d^k F(u)}{du^k}e^{xt}$. □

3. Identities on the Eulerian numbers and polynomials of higher order

THEOREM 3.1. *For* $N \in \mathbb{N}$, $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, *we have*

$$H_n^{(N)}(u) = \sum_{k=0}^{N-1} \frac{1}{k!}(1-u)^k \frac{d^k H_n(u)}{du^k}.$$

Proof. By (1.8) and (2.1), we get

$$F^N = \underbrace{\frac{1-u}{e^t-u} \times \cdots \times \frac{1-u}{e^t-u}}_{N-\text{times}} = \sum_{n=0}^{\infty} H_n^{(N)}(u)\frac{t^n}{n!}. \tag{3.1}$$

From (1.3) and (2.1), we get

$$F^{(k)} = \frac{d^k F(u)}{du^k} = \sum_{n=0}^{\infty} \frac{d^k H_n(u)}{du^k} \frac{t^n}{n!}. \tag{3.2}$$

By (2.7) and comparing with coefficients of (3.1) and (3.2), we obtain the result of this theorem. □

COROLLARY 3.2. *For $N \in \mathbb{N}$, $n \in \mathbb{Z}_+$, we have*

$$\begin{aligned}&\sum_{l_1+\cdots+l_N=n} \binom{n}{l_1, \cdots, l_N} H_{l_1}(u) H_{l_2}(u) \cdots H_{l_N}(u)\\ &= \sum_{k=0}^{N-1} \frac{1}{k!}(1-u)^k \frac{d^k H_n(u)}{du^k}.\end{aligned}$$

Proof. From (3.1), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} H_n^{(N)}(u) \frac{t^n}{n!} &= \underbrace{\frac{1-u}{e^t-u} \times \cdots \times \frac{1-u}{e^t-u}}_{N-\text{times}}\\ &= \left(\sum_{l_1=0}^{\infty} H_{l_1}(u) \frac{t^{l_1}}{l_1!}\right) \times \cdots \times \left(\sum_{l_N=0}^{\infty} H_{l_N}(u) \frac{t^{l_N}}{l_N!}\right)\\ &= \sum_{n=0}^{\infty} \left(\sum_{l_1+\cdots+l_N=n} \frac{n! H_{l_1}(u) H_{l_2}(u) \cdots H_{l_N}(u)}{l_1! \cdots l_N!}\right) \frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty} \left(\sum_{l_1+\cdots+l_N=n} \binom{n}{l_1, \cdots, l_N} H_{l_1}(u) H_{l_2}(u) \cdots H_{l_N}(u)\right) \frac{t^n}{n!}.\end{aligned} \tag{3.3}$$

Therefore, by Theorem 3.1 and (3.3), it is proved. □

COROLLARY 3.3. *For $N \in \mathbb{N}$, $n \in \mathbb{Z}_+$, we have*

$$H_n^{(N)}(x|u) = \sum_{k=0}^{N-1} \frac{1}{k!}(1-u)^k \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \frac{d^k H_n(u)}{du^k}.$$

Proof. From (1.4) and (3.2), we get

$$\begin{aligned} F^{(k)}(t,x) &= F^{(k)}e^{xt} \\ &= \left(\sum_{n=0}^{\infty} \frac{d^k H_n(u)}{du^k}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{x^n t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}x^{n-l}\frac{d^k H_l(u)}{du^k}\right)\frac{t^n}{n!}. \end{aligned} \tag{3.4}$$

Therefore, by (1.8), (2.7) and (3.4), it is proved. □

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