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ON LINEAR BCI-ALGEBRAS

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Abstract. In this note, we show that every linear BCI-algebra $(X;*,e)$, $e \in X$, has of the form $x*y = x-y+e$ where $x, y \in X$, where X is a field with $|X| \geq 3$.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim introduced the notion of B-algebras ([8, 9]), i.e., (I) $x*x = e$; (II) $x*e = x$; (III) $(x*y)*z = x*(z*(e*y))$, for any $x, y, z \in X$. C. B. Kim and H. S. Kim ([5]) introduced the notion of a BG-algebras which is a generalization of B-algebras, and constructed a BG-algebra from a non-empty set, which is non-group-derived. A. Walendziak ([12]) introduced a new notion, called an BF-algebra, i.e., (I); (II) and (IV) $e*(x*y) = y*x$ for any $x, y \in X$. In ([12]) it was shown that a BF-algebra is a generalizations of B-algebras. In this note, we show that every linear BCI-algebra $(X;*,e)$, $e \in X$, has of the form $x*y = x-y+e$ where $x, y \in X$, where X is a field with $|X| \geq 3$. Moreover, it is shown that every linear BCI-algebra is equivalent to quadratic BF(B, BG, Q)-algebras.

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2. Preliminaries

Definition 2.1. A *BCI-algebra* is an algebra $(X; *, e)$ of type (2,0) satisfying the following axioms:

(I) $x * x = e$,
(II) $x * y = y * x = e$ implies $x = y$,
(III) $(x * (x * y)) * y = e$,
(IV) $((x * y) * (x * z)) * (z * y) = e$ for any $x, y, z \in X$.

A BCK-algebra is a BCI-algebra satisfying the following axiom:

(V) $e * x = e$ for any $x \in X$.

A *BF-algebra* ([12]) is an algebra $(X; *, e)$ of type (2,0) satisfying the following axioms (I), and

(B2) $x * e = x$,
(BF) $e * (x * y) = y * x$ for any $x, y \in X$.

Note that BF-algebras need not be BCI-algebras, since it does not satisfy the anti-symmetry condition. The following example shows a BF-algebra which is not a BCI-algebra.

Example 2.2. ([12]) *Let $(X; *, 0)$ be the algebra where X is the set of all real numbers and $*$ is defined by*

$$x * y = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then X is a BF-algebra, but not a BCI-algebra, since $3 * 4 = 0 = 4 * 3$, but $3 \neq 4$.

3. Constructions of linear BCI-algebras

Let X be a field with $|X| \geq 3$. An algebra $(X; *)$ is said to be *linear* if $x * y$ is defined by $x * y = Ax + By + C$ where $A, B, C \in X$, for any $x, y \in X$. A linear algebra $(X; *)$ is said to be a *linear BCI-algebra* if it satisfies the conditions (I)∼(IV).

Theorem 3.1. *Let X be a field with $|X| \geq 3$. Then every linear BCI-algebra $(X; *, e)$, $e \in X$, has of the form $x * y = x - y + e$ where $x, y \in X$.*

Proof. Define $x * y := Ax + By + C$ where $A, B, C \in X$ for any $x, y \in X$. If we consider (I), then $e = x * x = (A + B)x + C$ for any $x \in X$. Assume $A + B \neq 0$. Then $x = \frac{e-C}{A+B}$ for any $x \in X$, a contradiction. Hence $A + B = 0$ and $C = e$. Hence

$$x * y = A(x - y) + e \tag{3.1}$$

Consider (II). Suppose $x * y = 0 = y * x$. Then by applying (1), we obtain

$$A(x - y) + e = e = A(y - x) + e$$

Hence $A(x - y) = 0 = A(y - x)$. If we assume that $A \neq 0$, then $x = y$. This means that the necessary condition to hold (III) is that $A \neq 0$.

Consider (IV). Given $x, y \in X$,

$$\begin{aligned} e &= (x * (x * y)) * y \\ &= A(x * (x * y)) + By + C \\ &= A^2(1 + B)x + B(AB + 1)y + C(AB + A + 1) \end{aligned}$$

It follows that $A^2(1 + B) = 0, B(AB + 1) = 0, C(AB + A + 1) = e$.

Case 1: $B = 0$. Since $A^2(1 + B) = 0$, $A^2 = 0$ and hence $A = 0$, a contradiction.

Case 2: $B \neq 0$. Since $B(AB + 1) = 0, C(AB + A + 1) = e$, we obtain $AB + 1 = 0, CA = e$, i.e., $A = -\frac{1}{B}$ and $C = -Be$. Hence

$$x * y = -\frac{1}{B}x + By - Be \tag{3.2}$$

Subcase (2.1): $C \neq 0$, i.e., $e \neq 0$. For any $x \in X$,

$$\begin{aligned} e &= x * x \\ &= -\frac{1}{B}x + Bx - Be \\ &= (-\frac{1}{B} + B)x - Be \end{aligned}$$

It follows that $B = -1, A = 1$ and $C = e$. Hence $x * y = x - y + e$. Subcase (2.2): $C = 0$, i.e., $e = 0$. By (2), we have $x * y = -\frac{1}{B}x + By$. If we let $y := x$, then $0 = e = x * x = (B - \frac{1}{B})x$ for any $x \in X$. It follows that $B = \pm 1$. Since $A^2(B + 1) = 0$, we have $B = -1$. In fact, if $B = 1$, then $0 = A^2(1 + B) = 2A^2$, a contradiction, since $A \neq 0$. Thus $x * y = x - y = x - y + 0$, proving the theorem. □

Example 3.2. Let $\mathbf{R}$ be the set of all real numbers. Then $(\mathbf{R}, *, 0)$ is a linear BCI-algebra where $x * y := x - y$, for any $x, y \in \mathbf{R}$.

Proposition 3.3. *Let X be a field with $|X| \geq 3$. If X is a linear BCI-algebra, then $(x * z) * (y * z) = x * y$ for any $x, y, z \in X$.*

Proof. Straightforward. □

Let X be a field with $|X| \geq 3$. An algebra $(X; *)$ is said to be *quadratic* if $x * y$ is defined by $x * y = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$, where $a_1, a_2, a_3, a_4, a_5, a_6 \in X$, for any $x, y \in X$. A quadratic algebra $(X; *)$ is said to be a *quadratic BF-algebra* if it satisfies the condition (I), (B2) and (BF).

Theorem 3.4. *([7]) Let X be a field with $|X| \geq 3$. Then every quadratic BF-algebra $(X; *, e)$, $e \in X$, has of the form $x * y = x - y + e$ where $x, y \in X$.*

H. K. Park and H. S. Kim ([11]) proved that every quadratic B-algebra $(X; *, e)$, $e \in X$, has of the form $x * y = x - y + e$, where X is a field with $|X| \geq 3$. J. Neggers, S. S. Ahn and H. S. Kim ([10]) introduced the notions of $Q-$algebra, and obtained that every quadratic Q-algebra $(X; *, e)$, $e \in X$, has of the form $x * y = x - y + e$, $x, y \in X$, where X is a field with $|X| \geq 3$. Also, H. S. Kim and H. D. Lee ([6]) showed that every quadratic BG-algebra $(X; *, e)$, $e \in X$, has of the form $x * y = x - y + e$, $x, y \in X$, where X is a field with $|X| \geq 3$. We summarize:

Theorem 3.5. *Let X be a field with $|X| \geq 3$. Then the followings are equivalent :*

(1) *$(X; *, e)$ is a quadratic BF-algebra,*
(2) *$(X; *, e)$ is a quadratic BG-algebra,*
(3) *$(X; *, e)$ is a quadratic Q-algebra,*
(4) *$(X; *, e)$ is a quadratic B-algebra.*

Theorem 3.6. *([11]) Let X be a field with $|X| \geq 3$. Then every quadratic B-algebra on X is a BCI-algebra.*

From Theorem 3.4 and Theorem 3.6, we obtain the following.

Corollary 3.7. *Let X be a field with $|X| \geq 3$. Then every quadratic $BF(BG, Q, B)$-algebra on X is a linear BCI-algebra and vice versa.*

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