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DETERMINATION OF MINIMUM LENGTH OF SOME LINEAR CODES

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Abstract. Hamada ([8]) and Maruta ([17]) proved the minimum length $n_3(6,d) = g_3(6,d)+1$ for some ternary codes. In this paper we consider such minimum length problem for $q \geq 4$, and we prove that $n_q(6,d) = g_q(6,d)+1$ for $d = q^5 - q^3 - q^2 - 2q + e$, $1 \leq e \leq q$. Combining this result with Theorem A in [4], we have $n_q(6,d) = g_q(6,d)+1$ for $q^5 - q^3 - q^2 - 2q + 1 \leq d \leq q^5 - q^3 - q^2$ with $q \geq 4$. Note that $n_q(6,d) = g_q(6,d)$ for $q^5 - q^3 - q^2 + 1 \leq d \leq q^5$ by Theorem 1.2.

1. Introduction

Let $\mathbb{F}_q$ denote the Galois field of q elements and $\mathbb{F}_q^n$ denote the n-dimensional vector space over $\mathbb{F}_q$, where q is a prime power. For a vector $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$, the weight of x denoted by $w(x)$ is the number of nonzero coordinates of x, that is, $w(x) = |\{i \mid x_i \neq 0\}|$.

An $[n,k,d]_q$ linear code C is a k-dimensional subspace of $\mathbb{F}_q^n$ over $\mathbb{F}_q$ with minimum distance d. One of the central problems in coding theory is to determine optimal linear codes. This is to optimize one of the parameters n, k and d for given the other two as follows; (1) Find the smallest length n, denoted by $n_q(k,d)$, for which there exists an $[n,k,d]_q$ code for given k and d. (2) Find the largest minimum distance d, denoted by $d_q(n,k)$, for which there exists an $[n,k,d]_q$ code for given k and n. (3) Find the largest dimension k, denoted by $k_q(n,d)$, for which there exists an $[n,k,d]_q$ code for given n and d.

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A code of length $n_q(k,d)$ [resp. minimum distance $d_q(n,k)$, dimension $k_q(n,d)$] is said to be length-optimal [resp. distance-optimal, dimension-optimal]. Note that a length-optimal code is both distance-optimal and dimension-optimal. So we concentrate on the length-optimal codes. The following is an important lower bound on $n_q(k,d)$ which is called the Griesmer bound.

Theorem 1.1 ([9]). (**Griesmer bound**) *For an $[n,k,d]_q$ linear code, we have $n \geq g_q(k,d)$, where $g_q(k,d) = d + \left\lceil \frac{d}{q} \right\rceil + \left\lceil \frac{d}{q^2} \right\rceil + \cdots + \left\lceil \frac{d}{q^{k-1}} \right\rceil$.*

By Theorem 1.1, we note $n_q(k,d) \geq g_q(k,d)$ for all k and d. It is natural to ask whether there exists a $[g_q(k,d),k,d]_q$ code for given d and k. The following theorem gives a large class of linear codes meeting the Griesmer bound which we call Griesmer codes.

Theorem 1.2 ([9]). *Let $s = \lceil \frac{d}{q^{k-1}} \rceil$ and $d = sq^{k-1} - \sum_{i=1}^{p} q^{u_i - 1}$ with $k > u_1 \geq u_2 \geq \cdots \geq u_p$ and $u_i > u_{i+q-1}$ for $1 \leq i \leq p-q+1$. If*

$$\sum_{i=1}^{\min\{s+1,p\}} u_i \leq sk,$$

then $n_q(k,d) = g_q(k,d)$.

Theorem 1.2 provides a starting point for finding $n_q(k,d)$. For $k=1$ and 2, we have $n_q(k,d) = g_q(k,d)$ for all d. Thus we are interested in $k \geq 3$.

From Theorem 1.2, we have the following:

Corollary 1.3 ([5]). *We have $n_q(k,d) = g_q(k,d)$ for d satisfying one of the following:*

(a) $q^{k-1} - q^{k-1-t} - q^t + 1 \leq d \leq q^{k-1} - q^{k-1-t}$

with $1 \leq t \leq \left\lfloor \frac{k-1}{2} \right\rfloor - 1$ *for* $k \geq 5$,

(b) $q^{k-1} - q^{k-1-t} - q^t + 1 \leq d \leq q^{k-1}$ *with* $t = \left\lfloor \frac{k-1}{2} \right\rfloor$ *for* $k \geq 3$,

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Much research on $n_q(k,d)$ has been done for small dimension k and small q by various methods. For $k = 3,4,5$ and $q = 3,4,5$, we can find tables of the values of $n_q(k,d)$ in [11] and [16].

To find the value of $n_q(k,d)$ for general q or k is more interesting. For minimum distance d with $q^{k-1} - q^{k-1-t} - q^t - sq + 1 \leq d \leq q^{k-1} -$

$q^{k-1-t} - q^t - (s-1)q$ for $1 \le t \le \lfloor \frac{k-1}{2} \rfloor$ and $1 \le s \le q-1$, which is just below the values for d in Corollary 1.3, it is known that there is no Griesmer code with
(1) $t = 1$ and $s = 1$ for $q \ge 3, k \ge 5$ in [3, 4, 13, 14, 15],
(2) $t = 1$ and $2 \le s \le q-1$ in [15],
(3) $s = 1$ and $t = \lfloor \frac{k-1}{2} \rfloor$ for $k \ge 5$ in [3], $2 \le t \le \lfloor \frac{k-1}{2} \rfloor - 1$ for $k \ge 7$ in [5].

Naturally we can ask the cases $t \ge 2$ or $s \ge 2$. When $t = 2$ and $s = 2$, there is a result for the question for $k = 5$ in [2].

In this paper, we consider the case $t = 2$ and $s = 2$ for $k = 6$. In other words, we consider the problem whether Griesmer codes with minimum distance d with $q^5 - q^3 - q^2 - 2q + 1 \le d \le q^5 - q^3 - q^2 - q$ exist or not for $q \ge 4$. For $q = 2$ or 3, we note that $n_2(6, d) = g_2(6, d)$ with $d = 17, 18$ ([6]) and $n_3(6, d) = g_3(6, d) + 1$ with $d = 202, 203, 204$ ([8, 17]).

As the first step to determine the exact value of $n_q(6, d)$ with $d = q^5 - q^3 - q^2 - 2q + \alpha$, $1 \le \alpha \le q$ and $q \ge 4$, we need to prove the following.

Theorem A. *There does not exist a* $[g_q(6, d), 6, d]_q$ *code with* $d = q^5 - q^3 - q^2 - 2q + 1$ *for* $q \ge 4$.

In Section 3, we give a proof of Theorem A and in Section 2, we recall some results needed to prove Theorem A.

Recall that the existence of an $[n, k, d]_q$ code with $d \ge 2$ implies the existence of an $[n-1, k, d-1]_q$ code. Therefore, by Theorem A, we have the following.

Theorem B. *For* $q \ge 4$, *we have* $n_q(6, d) \ge g_q(6, d) + 1$ *with* $q^5 - q^3 - q^2 - 2q + 1 \le d \le q^5 - q^3 - q^2 - q$.

If we let $k = 6$ in Theorem 16 in [3], then we have the following.

Theorem 1.4 ([3]). *For* $q \ge 3$, *there exists a* $[g_q(6, d) + 1, 6, d]_q$ *code for* $q^5 - q^3 - 2q^2 + 1 \le d \le q^5 - q^3 - q^2$.

By Theorem B and Theorem 1.4, we conclude the next theorem.

Theorem C. *For* $q \ge 4$, *we have* $n_q(6, d) = g_q(6, d) + 1$ *with* $q^5 - q^3 - q^2 - 2q + 1 \le d \le q^5 - q^3 - q^2 - q$.

Finally, combining the result of [4] with Theorem C, for $q \ge 4$, we have the following:

$$n_q(6, d) = g_q(6, d) + 1 \text{ for } q^5 - q^3 - q^2 - 2q + 1 \le d \le q^5 - q^3 - q^2.$$

2. Preliminaries

Let $PG(r,q)$ be the r-dimensional projective space over $\mathbb{F}_q$ and let θ_r be the number of points in $PG(r,q)$. Then $\theta_r = q^r + q^{r-1} + \cdots + q + 1$ for a positive integer r. For convenience, we let $\theta_0 = 1$ and $\theta_r = 0$ if $r < 0$. We call a subspace of dimension j in $PG(r,q)$ a j-flat. In particular, we call a subspace of dimension 0 [resp. 1, 2, $r-1$] a point [resp. a line, a plane, a hyperplane].

Let C be a projective $[n,k,d]_q$ linear code with a generator matrix of G. Then no two columns of G are linearly dependent. Each column of G can be considered as a point of $PG(k-1,q)$. Let C_1 be the set of all columns of G and let $C_0 = C_1^c$, the complement of C_1 in $PG(k-1,q)$. For a subset S in $PG(k-1,q)$, we use the following notation;

$$c_0(S) = |S \cap C_0|, \qquad c(S) = |S \cap C_1| \quad \text{and} \quad c_0 = |C_0|.$$

In particular, for a projective $[n,6,d]_q$ linear code C, we have $n = |C_1|$, $c_0 = \theta_5 - n$ and $d = n - \max\{c(H) \mid H \text{ is a 4-flat in } PG(5,q)\}$.

Now we recall theorems which play an important role to prove Theorem A.

For a subset S in the r-dimensional affine space $AG(r,q)$ over $\mathbb{F}_q$, S is a t-fold blocking set with respect to hyperplanes if every hyperplane in $AG(r,q)$ meets S in at least t points.

Theorem 2.1 ([1]). *A t-fold blocking set S with respect to hyperplanes in $AG(r,q)$ satisfies*

$$|S| \geq (r+t-1)(q-1)+1.$$

A subset F of $PG(r,q)$ with $|F| = f$ is called an $\{f,t;r,q\}$-minihyper if every hyperplane meets F in at least t points. Hamada ([7]) showed that for $k \geq 3$ and $1 \leq d < q^{k-1}$, there is a one to one correspondence between the set of all nonequivalent $[n,k,d]_q$ Griesmer codes and the set of all $\{\theta_{k-1} - n, \theta_{k-2} - n + d; k-1, q\}$-minihypers. Thus an $[n,6,d]_q$ Griesmer code C with $d < q^5$ corresponds to a $\{\theta_5 - n, \theta_4 - n + d; 5, q\}$-minihyper. The following is a characterization of some minihypers.

Theorem 2.2 ([7]). *Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be positive integers with $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_m \leq t$ and $1 \leq m \leq t$.*

(a) *In the case $m = 1$, S is a $\{\theta_{\lambda_1}, \theta_{\lambda_1 - 1}; t, q\}$-minihyper if and only if S is a λ_1-flat in $PG(t,q)$.*

(b) *In the case $m \geq 2$ and $t \geq \lambda_m + \lambda_{m-1} + 1$, S is a $\{\sum_{i=1}^{m} \theta_{\lambda_i}, \sum_{i=1}^{m} \theta_{\lambda_i - 1}; t, q\}$-minihyper if and only if S consists of disjoint union of λ_i-flats in $PG(t,q)$.*

(c) *In the case* $m \geq 2$ *and* $t \leq \lambda_m + \lambda_{m-1}$, *there is no* $\{\sum_{i=1}^{m} \theta_{\lambda_i}, \sum_{i=1}^{m} \theta_{\lambda_i - 1}; t, q\}$*-minihyper.*

Let $m_{r,q}(s)$ denote the minimum value of f for which an $\{f, \theta_{r-2} + s; r, q\}$-minihyper exists for $r \geq 3$ and $1 \leq s \leq q-1$. If we let $r = 4$ and $s = 1$ or $s = 2$ in Theorem 2.4 in [15], then we have the following.

THEOREM 2.3 ([15]). *For* $q \geq 3$, *we have*

(a) $m_{4,q}(1) \geq \theta_3 + \theta_1 + q$,
(b) $m_{4,q}(2) \geq \theta_3 + 2\theta_1 + q$.

In [14], Maruta proved the nonexistence of a $[g_q(5,d), 5, d]_q$ code with $q^4 - 2q^2 - q + 1 \leq d \leq q^4 - 2q^2$ for $q \geq 3$. Landjev and Maruta [resp. Cheon *et al.*] proved the nonexistence of a $[g_q(5,d), 5, d]_q$ code with $q^4 - 2q^2 - 2q + 1 \leq d \leq q^4 - 2q^2 - q$ for $q = 4$ [resp. for $q \geq 5$] in [12] [resp. [2]]. Since those two intervals of d-values are consecutive, we conclude that there does not exist a Griesmer code with $q^4-2q^2-2q+1 \leq d \leq q^4-2q^2$ for $q \geq 4$. Here we express the above results with the notion of minihyper respectively.

THEOREM 2.4 ([14]). *For* $q \geq 3$ *and* $0 \leq e \leq q-1$, *there does not exist a* $\{2\theta_2 + e, 2\theta_1; 4, q\}$*-minihyper.*

THEOREM 2.5 ([2, 12]). *For* $q \geq 4$ *and* $0 \leq e \leq q-1$, *there does not exist a* $\{2\theta_2 + \theta_1 + e, 2\theta_1 + 1; 4, q\}$*-minihyper.*

For a Griesmer code, the following holds:

THEOREM 2.6 ([14]). *Let* C *be a* $[g_q(k,d), k, d]_q$ *code and let* $\gamma_j := \sum_{i=0}^{j} \left\lceil \frac{d}{q^{k-1-i}} \right\rceil$ *for* $0 \leq j \leq k-2$. *Then there exist* j*-flats* Δ_j *with* $c(\Delta_j) = \gamma_j$ *such that* $\Delta_0 \subsetneq \Delta_1 \subsetneq \cdots \subsetneq \Delta_{k-2}$ *and that* Δ_j *gives a* $[\gamma_j, j+1, \gamma_j - \gamma_{j-1}]$ *Griesmer code for* $1 \leq j \leq k-2$.

If we let $t = 1$ in Theorem 7 in [10], then we have the following.

THEOREM 2.7. *For an integer* $r \geq 0$ *and* $q \geq r+1$, *let* S *be a subset in* $PG(m,q)$ *with* $|S| \leq \theta_{m-1} + r\theta_{m-3}$. *If* $|S \cap l| \geq 1$ *for any line* l *in* $PG(m,q)$ *then* S *contains a hyperplane.*

3. Main theorem

In this section, we prove Theorem A. On the contrary, we assume that for $q \geq 4$, there exists a $[g_q(6,d), 6, d]_q$ code C with $d = q^5-q^3-q^2-2q+1$.

Since C is a Griesmer code, by Theorem 2.6, we have the following:

$$c_0 = \theta_3 + \theta_2 + 2q,$$
$$c_0(H) \geq \theta_2 + \theta_1 + 1 \text{ for any 4-flat } H \text{ in } PG(5,q),$$
$$c_0(\Delta) \geq \theta_1 + 1 \text{ for any 3-flat } \Delta \text{ in } PG(5,q),$$
$$c_0(\delta) \geq 1 \text{ for any 2-flat } \delta \text{ in } PG(5,q).$$

Let H_0 be a 4-flat in $PG(5,q)$ with $c_0(H_0) = \theta_2 + \theta_1 + 1$. Then H_0 is a $\{\theta_2 + \theta_1 + 1, \theta_1 + 1; 4, q\}$-minihyper. By Theorem 2.2 (b), we note that $H_0 \cap C_0$ is a disjoint union of a plane δ_0, a line l_0 and a point P_0. For any 3-flat Δ in H_0, we have

(3.1) $c_0(\Delta) = \theta_2 + 2, \quad \theta_2 + 1, \quad 2\theta_1 + 1, \quad 2\theta_1, \quad \theta_1 + 2, \quad \text{or} \quad \theta_1 + 1.$

For any 4-flat H in $PG(5,q)$, we have $c_0(H_0 \cap H) \leq \theta_2 + 2$ by (3.1). Since $c_0(H) \geq \theta_2 + \theta_1 + 1$ for any 4-flat H, we have

$$\begin{aligned} c_0 &= c_0(H) + \sum_{H_0 \cap H \subseteq H' \neq H} c_0(H') - qc_0(H_0 \cap H) \\ &\geq c_0(H) + q(\theta_2 + \theta_1 + 1) - q(\theta_2 + 2) \\ &= c_0(H) + q^2, \end{aligned}$$

which implies $c_0(H) \leq \theta_3 + 2\theta_1 + q - 1$.

Therefore, we conclude that

(3.2) $\theta_2 + \theta_1 + 1 \leq c_0(H) \leq \theta_3 + 2\theta_1 + q - 1$ for any 4-flat H in $PG(5,q)$.

Now we will derive a contradiction in two steps as follows: In Step I, we prove that there is no 4-flat H such that

(3.3) $$2\theta_2 + 1 \leq c_0(H) \leq \theta_3 + 2\theta_2 + q - 1.$$

Then, by (3.2), we conclude that $\theta_2 + \theta_1 + 1 \leq c_0(H) \leq 2\theta_2$ for any 4-flat H in $PG(5,q)$. In Step II, we will prove that it is impossible.

Step I. We divide the interval (3.3) into five small intervals, which we refer to as Case 1, …, Case 5 and we prove the nonexistence of a 4-flat H with $c_0(H)$ belonging to each small interval.

When we prove them we use the following computation frequently: For a 4-flat H_1 in $PG(5,q)$, let Δ be a 3-flat in H_1. Then we have

$$\begin{aligned} c_0 &= c_0(H_1) + \sum_{\Delta \subseteq H \neq H_1} c_0(H) - qc_0(\Delta) \\ &\geq c_0(H_1) + q(\theta_2 + \theta_1 + 1) - qc_0(\Delta), \end{aligned}$$

and hence

$$c_0(\Delta) \geq \frac{c_0(H_1) - c_0}{q} + \theta_2 + \theta_1 + 1. \tag{3.4}$$

Case 1. There is no 4-flat H with $\theta_3 + 2\theta_1 \leq c_0(H) \leq \theta_3 + 2\theta_1 + q - 1$.

Proof. Suppose that there exists a 4-flat H_1 with $c_0(H_1) = \theta_3 + 2\theta_1 + q - 1 - f$, $0 \leq f \leq q - 1$. Let Δ be a 3-flat in H_1. Using (3.4), we have $c_0(\Delta) \geq \theta_2 + 2 - \frac{f}{q}$. Since $0 \leq f \leq q - 1$, we have

$$c_0(\Delta) \geq \theta_2 + 2 \text{ for all 3-flat } \Delta \subset H_1.$$

Furthermore, if we let $\Delta = H_0 \cap H_1$, then $c_0(\Delta) = \theta_2 + 2$ since $c_0(H_0 \cap H_1) \leq \theta_2 + 2$ by (3.1). Thus $C_0 \cap H_1$ is a $\{\theta_3 + 2\theta_1 + q - 1 - f, \theta_2 + 2; 4, q\}$-minihyper, which contradicts Theorem 2.3 (b). □

Case 2. There is no 4-flat H with $\theta_3 + \theta_1 + 1 \leq c_0(H) \leq \theta_3 + \theta_1 + q$.

Proof. Suppose that there exists a 4-flat H_1 with $c_0(H_1) = \theta_3 + \theta_1 + q - f$, $0 \leq f \leq q - 1$. Let Δ be a 3-flat in H_1. Using (3.4), we have $c_0(\Delta) \geq \theta_2 + 1$ since $0 \leq f \leq q - 1$.

Suppose that $c_0(\Delta) \geq \theta_2 + 2$ for all 3-flat $\Delta \subseteq H_1$. By Theorem 2.3 (b), we have $c_0(H_1) \geq \theta_3 + 2\theta_1 + q$ which is a contradiction. Thus there exists a 3-flat Δ with $c_0(\Delta) = \theta_2 + 1$. Then $C_0 \cap H_1$ is a $\{\theta_3 + \theta_1 + q - f, \theta_2 + 1; 4, q\}$-minihyper. By Theorem 2.3 (a), we obtain $f = 0$, that is, $C_0 \cap H_1$ is a $\{\theta_3 + \theta_1 + q, \theta_2 + 1; 4, q\}$-minihyper. Here, we prove the following claim.

Claim. $C_0 \cap H_1$ contains a 3-flat.

Proof of Claim: To prove Claim, it suffices to prove that for any line l in H_1, $|(C_0 \cap H_1) \cap l| \geq 1$ by Theorem 2.7. Suppose that there is a line l_1 in H_1 with $|(C_0 \cap H_1) \cap l_1| = 0$. Consider a 3-flat Δ in H_1 containing l_1. Then there is a 3-flat Δ' containing l_1 with $|(C_0 \cap H_1) \cap \Delta'| = \theta_2 + 1$ since $c_0(H_1) = \theta_3 + \theta_1 + q$ and

$$c_0(H_1) = \frac{1}{\theta_1} \left(\sum_{l_1 \subset \Delta \subset H_1} |(C_0 \cap H_1) \cap \Delta| \right) < \frac{1}{\theta_1} \cdot \theta_2 \cdot (\theta_2 + 2).$$

On the other hand, we have

$$\begin{aligned} c_0 &= c_0(H_1) + \sum_{\Delta' \subseteq H, H \neq H_1} c_0(H) - q c_0(\Delta') \\ &\geq c_0(H_1) + q(\theta_2 + \theta_1 + 1) - q(\theta_2 + 1) = \theta_3 + \theta_2 + 2q. \end{aligned}$$

Thus we note $c_0(H) = \theta_2 + \theta_1 + 1$ for any 4-flat $H(\neq H_1)$ containing Δ'. Let H_2 be a 4-flat containing Δ' with $c_0(H_2) = \theta_2 + \theta_1 + 1$. Then by Theorem 2.2 (b), $C_0 \cap H_2$ consists of a plane, a line and a point. Since $|(C_0 \cap H_2) \cap H_1| = |(C_0 \cap H_1) \cap \Delta'| = \theta_2 + 1$, it holds that $(C_0 \cap H_1) \cap \Delta'$ consists of a plane δ' and a point P'. Then $|(C_0 \cap H_1) \cap l_1| \geq |\delta' \cap l_1| \geq 1$, which is a contradiction to the choice of l_1. Thus Claim is proved.

By Claim, $C_0 \cap H_1$ contains a 3-flat, say Δ_1. Let $S = (C_0 \cap H_1) - \Delta_1$. We note $|S| = \theta_1 + q$. On the other hand, since $c_0(\Delta) \geq \theta_2 + 1$ for any 3-flat Δ in H_1, $|S \cap \Delta| \geq 1$ for any 3-flat $\Delta \neq \Delta_1$ in H_1. Thus S can be considered as 1-fold blocking set with respect to hyperplanes in $AG(4, q)$. By Theorem 2.1, we have $|S| \geq (4+1-1)(q-1)+1 = 4q-3$ which is a contradiction since $q \geq 4$. □

Case 3. There is no 4-flat H with $2\theta_2 + \theta_1 + q \leq c_0(H) \leq \theta_3 + \theta_1$.

Proof. Suppose that there exists a 4-flat H_1 with $c_0(H_1) = \theta_3 + \theta_1 - eq - f$, $0 \leq f \leq q-1$, $0 \leq e \leq q^2 - q - 3$. By (3.4), we have $c_0(\Delta) \geq \theta_2 - e$. Suppose there is a 3-flat Δ_1 with $c_0(\Delta_1) = \theta_2 - e$. Since $0 \leq e \leq q^2 - q - 3$, we have $2\theta_1 + 2 \leq c_0(\Delta_1) \leq \theta_2$. By (3.1), we note that $c_0(H) \geq \theta_2 + \theta_1 + 2$ for any 4-flat H containing Δ_1. Thus we have

$$\begin{aligned} c_0 &= c_0(H_1) + \sum_{\Delta_1 \subseteq H, H \neq H_1} c_0(H) - qc_0(\Delta_1) \\ &\geq c_0(H_1) + q(\theta_2 + \theta_1 + 2) - q(\theta_2 - e) = \theta_3 + \theta_2 + 2q + \theta_1 - (f+1), \end{aligned}$$

which is a contradiction since $0 \leq f \leq q-1$. Thus $c_0(\Delta) \geq \theta_2 - e + 1$ for any 3-flat Δ in H_1.

On the other hand, H_1 corresponds to an $[n_1, 5, d_1]_q$ linear code with $n_1 = \theta_4 - c_0(H_1) = \theta_4 - \theta_3 - \theta_1 + eq + f$ and $d_1 \geq q^4 - q^3 - q + eq + f - e$.

Applying the Griesmer bound, we have

$$\begin{aligned} g_q(5, d_1) \geq\ & q^4 - q^3 - q + eq + f - e \\ & + q^3 - q^2 - 1 + e + \left\lceil \frac{f-e}{q} \right\rceil \\ & + q^2 - q + \left\lceil \frac{eq - q + f - e}{q^2} \right\rceil \end{aligned}$$

$$
\begin{aligned}
& +q-1+\left\lceil\frac{eq-q+f-e}{q^3}\right\rceil \\
& +1 \\
\geq \quad & n_1+\left\lceil\frac{f-e}{q}\right\rceil+\left\lceil\frac{eq-q+f-e}{q^2}\right\rceil+\left\lceil\frac{eq-q+f-e}{q^3}\right\rceil .
\end{aligned}
$$

Let

$$T=\left\lceil\frac{f-e}{q}\right\rceil+\left\lceil\frac{eq-q+f-e}{q^2}\right\rceil+\left\lceil\frac{eq-q+f-e}{q^3}\right\rceil .$$

Then $T \leq 0$ by the Griesmer bound. Now we prove the following claim.

Claim. In the set of pairs (e, f) with $0 \leq e \leq q^2-q-3$ and $0 \leq f \leq q-1$, we have the following:

$$T \leq 0 \text{ if and only if } (e,f)=(0,0),(1,0), \text{ or } (1,1).$$

Moreover, in this case $T=0$.

Proof of Claim: We prove

$$\begin{cases} T=0, & \text{if } (e,f)=(0,0),(1,0), \text{ or } (1,1), \\ T>0, & \text{otherwise.} \end{cases}$$

When $(e,f)=(0,0),(1,0)$ or $(1,1)$, we note that $T=0$. Hence we consider the other case.

For $f \geq 2$, since $0 \leq e \leq q^2-q-3<q^2$, we have

$$
\begin{aligned}
T &\geq \left\lceil\frac{f-e}{q}+\frac{eq-q+f-e}{q^2}+\frac{eq-q+f-e}{q^3}\right\rceil \\
&= \left\lceil\frac{(f-1)\theta_2+1-e}{q^3}\right\rceil>0.
\end{aligned}
$$

Now, consider the case $f=0$ or 1.

Assume $f=1$. We have $eq-q+f-e=(e-1)(q-1)$. For $e=0$, we have $T=\left\lceil\frac{1}{q}\right\rceil+\left\lceil\frac{-(q-1)}{q^2}\right\rceil+\left\lceil\frac{-(q-1)}{q^3}\right\rceil=1>0$. For $2 \leq e \leq q$, we have $T=\left\lceil\frac{1-e}{q}\right\rceil+\left\lceil\frac{(e-1)(q-1)}{q^2}\right\rceil+\left\lceil\frac{(e-1)(q-1)}{q^3}\right\rceil=0+1+1=2>0$. For $tq+1 \leq e \leq (t+1)q$ with $1 \leq t \leq q-2$, we have $\left\lceil\frac{1-e}{q}\right\rceil=-t$, $t \leq\left\lceil\frac{(e-1)(q-1)}{q^2}\right\rceil \leq t+1$ and $\left\lceil\frac{(e-1)(q-1)}{q^3}\right\rceil=1$, and hence $T>0$.

Finally, assume $f=0$. For $2 \leq e \leq q-1$, we get $T=\left\lceil\frac{-e}{q}\right\rceil+\left\lceil\frac{e(q-1)-q}{q^2}\right\rceil+\left\lceil\frac{e(q-1)-q}{q^3}\right\rceil=0+1+1=2>0$. For $tq \leq e \leq (t+1)q-1$

with $1 \le t \le q-2$, we have $\left\lceil \frac{-e}{q} \right\rceil = -t$, $t \le \left\lceil \frac{e(q-1)-q}{q^2} \right\rceil \le t+1$ and $\left\lceil \frac{e(q-1)-q}{q^3} \right\rceil = 1$, and hence $T > 0$. Thus the claim is proved.

By the claim, there remain only three possibilities for $c_0(H_1)$.

If $(e,f) = (0,0)$, then $C_0 \cap H_1$ is a $\{\theta_3+\theta_1, \theta_2+1; 4, q\}$-minihyper, which is impossible by Theorem 2.2 (c).

Assume $(e,f) = (1,0)$. Then $C_0 \cap H_1$ is a $\{\theta_3+1, \theta_2; 4, q\}$-minihyper. By Theorem 2.2 (b), the minihyper $C_0 \cap H_1$ consists of a 3-flat Δ_1 and a point P_1. From $c_0 = c_0(H_1) + \sum_{\Delta_1 \subseteq H, H \ne H_1} c_0(H) - qc_0(\Delta_1)$, since $c_0(\Delta_1) = \theta_3$, we have

$$\sum_{\Delta_1 \subseteq H, H \ne H_1} c_0(H) = q(\theta_3+\theta_1+2).$$

Hence there is a 4-flat H containing Δ_1 with $c_0(H) \ge \theta_3+\theta_1+2$, which contradicts Case 1 and Case 2.

If $(e,f) = (1,1)$ then $C_0 \cap H_1$ is a $\{\theta_3, \theta_2; 4, q\}$-minihyper, i.e., a 3-flat. Then $c_0(H_0 \cap H_1) = \theta_2$ or θ_3, which is a contradiction by (3.1). □

Case 4. There is no 4-flat H with $2\theta_2+\theta_1 \le c_0(H) \le 2\theta_2+2q$.

Proof. Suppose that there exists a 4-flat H_1 with $c_0(H_1) = 2\theta_2 + 2q - f$, $0 \le f \le q-1$. For any 3-flat Δ in H_1, by (3.4) we have $c_0(\Delta) \ge 2\theta_1+1$.

Suppose that $c_0(\Delta) \ge 2\theta_1+2$ for all $\Delta \subseteq H_1$. Then H_1 is an $[n_1, 5, d_1]_q$ code with $n_1 = \theta_4 - 2\theta_2 - 2q + f$ and $d_1 \ge q^4 - 2q^2 - 2q + f + 2$, which contradicts the Griesmer bound. Thus there is a 3-flat Δ in H_1 with $c_0(\Delta) = 2\theta_1+1$. Then $C_0 \cap H_1$ is a $\{2\theta_2+2q-f, 2\theta_1+1; 4, q\}$-minihyper, which contradicts Theorem 2.5. □

Case 5. There is no 4-flat H with $2\theta_2+1 \le c_0(H) \le 2\theta_2+q$.

Proof. Suppose that there exists a 4-flat H_1 with $c_0(H_1) = 2\theta_2+q-f$, $0 \le f \le q-1$. For any 3-flat Δ in H_1, we have $c_0(\Delta) \ge 2\theta_1$ by (3.4). Suppose that $c_0(\Delta) \ge 2\theta_1+1$ for any Δ in H_1. Then H_1 corresponds to an $[n_1, 5, d_1]_q$ linear code with $n_1 = \theta_4 - 2\theta_2 - q + f$ and $d_1 \ge q^4 - 2q^2 - q + f + 1$, which contradicts the Griesmer bound. Thus there exists a 3-flat Δ in H_1 with $c_0(\Delta) = 2\theta_1$, and hence $C_0 \cap H_1$ is a $\{2\theta_2+q-f, 2\theta_1; 4, q\}$-minihyper. For $1 \le f \le q-1$, by Theorem 2.4, such a minihyper $C_0 \cap H_1$ does not exist. Thus we have $f = 0$ and

$C_0 \cap H_1$ is a $\{2\theta_2 + q, 2\theta_1; 4, q\}$-minihyper. Let Δ_1 be a 3-flat in H_1 with $c_0(\Delta_1) = 2\theta_1$. Then we have

$$\begin{aligned} c_0 &= c_0(H_1) + \sum_{\Delta_1 \subseteq H, H \neq H_1} c_0(H) - qc_0(\Delta_1) \\ &= 2\theta_2 + q + \sum_{\Delta_1 \subseteq H, H \neq H_1} c_0(H) - q \cdot 2\theta_1, \end{aligned}$$

and hence

$$\sum_{\Delta_1 \subseteq H, H \neq H_1} c_0(H) = q(\theta_2 + \theta_1 + 1).$$

Thus we conclude $c_0(H) = \theta_2 + \theta_1 + 1$ for all 4-flat H ($\neq H_1$) containing Δ_1, say $H_2, \ldots, H_{q+1}$. By Theorem 2.2 (b), for $2 \leq i \leq q+1$, $H_i \cap C_0$ consists of a plane δ_i, a line l_i and a point Q_i, respectively. Let $C_0 \cap \Delta_1 = C_0 \cap H_1 \cap H_i = l_i \cup m_i$, $2 \leq i \leq q+1$, where $m_i = \delta_i \cap H_1$ is a line. Since $c_0(\Delta_1) = 2\theta_1$ and $q \geq 4$, there exist H_i and H_j such that $m_i = m_j$, $2 \leq i < j \leq q+1$. Then we note that $\delta_i \cap \delta_j = m_i$. Consider the linear span of δ_i and δ_j, denoted by $\langle \delta_i, \delta_j \rangle$. Then $c_0(\langle \delta_i, \delta_j \rangle) \geq \theta_2 + q^2$. Thus we have

$$\begin{aligned} c_0 &= \sum_{\langle \delta_i, \delta_j \rangle \subseteq H} c_0(H) - qc_0(\langle \delta_i, \delta_j \rangle) \\ &\leq \sum_{\langle \delta_i, \delta_j \rangle \subseteq H} c_0(H) - q(\theta_2 + q^2), \end{aligned}$$

and hence $\sum_{\langle \delta_i, \delta_j \rangle \subseteq H} c_0(H) \geq 3\theta_3 + 2q - 1$. Therefore, there exists a 4-flat H containing $\langle \delta_i, \delta_j \rangle$ with $c_0(H) \geq 3q^2 + 5$. However, from Case 1, 2, 3, 4, there is no 4-flat H with $2\theta_2 + \theta_1 \leq c_0(H) \leq \theta_3 + 2\theta_1 + q - 1$. Thus we have a contradiction since $q \geq 4$. □

Step II. By Step I and (2) we conclude that

(3.5) $\quad \theta_2 + \theta_1 + 1 \leq c_0(H) \leq 2\theta_2 \quad$ for any 4-flat H in $PG(5, q)$.

On the other hand, since $H_0 \cap C_0$ is a disjoint union of a plane δ_0, a line l_0 and a point P_0, the linear span $\Delta_0 = \langle \delta_0, P_0 \rangle$ is 3-flat with $c_0(\Delta_0) = \theta_2 + 2$. Then we have

$$\begin{aligned} c_0 &= c_0(H_0) + \sum_{\Delta_0 \subseteq H, H \neq H_0} c_0(H) - qc_0(\Delta_0) \\ &= \theta_2 + \theta_1 + 1 + \sum_{\Delta_0 \subseteq H, H \neq H_0} c_0(H) - q(\theta_2 + 2), \end{aligned}$$

and hence $\sum_{\Delta_0 \subseteq H, H \neq H_0} c_0(H) = 2\theta_3 + 3q - 3$. Thus there is a 4-flat $H(\neq H_0)$ containing Δ_0 with $c_0(H) \geq 2\theta_2 + 3$, which contradicts (3.5). Thus the proof is completed. □

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