

A CHANGE OF SCALE FORMULA FOR GENERALIZED WIENER INTEGRALS II

BYOUNG SOO KIM*, TEUK SEOB SONG**, AND IL YOO***

ABSTRACT. Cameron and Storvick discovered change of scale formulas for Wiener integrals on classical Wiener space. Yoo and Skoug extended this result to an abstract Wiener space. In this paper, we investigate a change of scale formula for generalized Wiener integrals of various functions using the generalized Fourier–Feynman transform.

1. Introduction

It has long been known that Wiener measure and Wiener measurability behave badly under the change of scale transformation [3] and under translations [4]. Cameron and Storvick [3] expressed the analytic Feynman integral for a rather large class of functionals as a limit of Wiener integrals. In doing so, they discovered nice change of scale formulas for Wiener integrals on classical Wiener space $(C_0[0, T], m_w)$. In [13, 14], Yoo and Skoug extended these results to an abstract Wiener space (H, B, ν) . In [15], Yoo, Song, Kim and Chang investigated a change of scale formula for Wiener integrals of functions on abstract Wiener space which need not be bounded or continuous. In this paper, we investigate a change of scale formula for generalized Wiener integrals of various functions using the generalized Fourier–Feynman transform on classical Wiener space.

2. Definitions and preliminaries

Let $C_0[0, T]$ denote the Wiener space, that is, the space of \mathbb{R} -valued

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Correspondence should be addressed to Il Yoo, iyoo@yonsei.ac.kr.

continuous functions x on $[0, T]$ with $x(0) = 0$. Let \mathcal{M} denote the class of all Wiener measurable subsets of $C_0[0, T]$ and let m_w denote Wiener measure. $(C_0[0, T], \mathcal{M}, m_w)$ is a complete measure space and we denote the Wiener integral of a functional F by $\int_{C_0[0, T]} F(x) dm_w(x)$.

Let \mathbb{C} , \mathbb{C}_+ and \mathbb{C}_+^\sim denote the set of complex numbers, complex numbers with positive real part, and nonzero complex numbers with non-negative real part, respectively.

A subset E of $C_0[0, T]$ is said to be scale-invariant measurable provided αE is measurable for each $\alpha > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m_w(\alpha N) = 0$ for each $\alpha > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (*s-a.e.*). If two functionals F and G are equal *s-a.e.*, then we write $F \approx G$.

Let F be a \mathbb{C} -valued scale-invariant measurable functional on $C_0[0, T]$ such that

$$(2.1) \quad J(\lambda) = \int_{C_0[0, T]} F(\lambda^{-1/2} Z_h(x, \cdot)) dm_w(x)$$

exists as a finite number for all real $\lambda > 0$ where Z_h is the Gaussian process

$$(2.2) \quad Z_h(x, t) = \int_0^t h(s) \tilde{d}x(s)$$

where h is in $L_2[0, T]$ and $\int_0^t h(s) \tilde{d}x(s)$ denotes the Paley-Wiener- Zygmund(P.W.Z) integral [2]. If there exists an analytic function $J^*(\lambda)$ on \mathbb{C}_+ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the generalized analytic Wiener integral of F over $C_0[0, T]$ with parameter λ , and for $\lambda \in \mathbb{C}_+$ we write

$$(2.3) \quad \int_{C_0[0, T]}^{\text{anw}\lambda} F(Z_h(x, \cdot)) dm_w(x) = I_a^\lambda(F) = J^*(\lambda).$$

Let F be a functional on $C_0[0, T]$ such that $I_a^\lambda(F)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists for nonzero real q , then we call it the generalized analytic Feynman integral of F over $C_0[0, T]$ with parameter q and we write

$$(2.4) \quad I_a^q(F) = \lim_{\lambda \rightarrow -iq} I_a^\lambda(F)$$

where $\lambda \rightarrow -iq$ through \mathbb{C}_+ . When $h \equiv 1$, the generalized analytic Wiener integral and the generalized analytic Feynman integral reduced

to the analytic Wiener integral and the analytic Feynman integral, respectively [6,10].

For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0, T]$, let

$$(2.5) \quad (T_{\lambda,h}(F))(y) = \int_{C_0[0,T]}^{\text{anw}\lambda} F(Z_h(x, \cdot) + y) dm_w(x).$$

DEFINITION 2.1. Let q be a non-zero real number. We define the L_1 analytic generalized Fourier-Feynman transform $T_{q,h}^{(1)}(F)$ of F by

$$(2.6) \quad (T_{q,h}^{(1)}(F))(y) = \lim_{\lambda \rightarrow -iq} (T_{\lambda,h}(F))(y)$$

for s -a.e. $y \in C_0[0, T]$, where $\lambda \rightarrow -iq$ through \mathbb{C}_+ .

The Banach algebra \mathcal{S} [2] consists of functionals on $C_0[0, T]$ expressible in the form

$$(2.7) \quad F(y) = \int_{L_2[0,T]} \exp\{i(u, y)\} d\mu(u)$$

for s -a.e. y in $C_0[0, T]$ where μ is an element of $M(L_2[0, T])$, the space of all \mathbb{C} -valued countably additive Borel measures on $L_2[0, T]$, and (u, y) denotes the P.W.Z. integral $\int_0^T u(t) \tilde{d}y(t)$.

The following theorem is the existence theorem for L_1 analytic generalized Fourier-Feynman transform of functions in the Banach algebra \mathcal{S} introduced by Huffman, Park and Skoug [10].

THEOREM 2.2. ([10]) *Let $F \in \mathcal{S}$ be given by (2.7). Then for each $\lambda \in \mathbb{C}_+$,*

$$(2.8) \quad T_{\lambda,h}(F)(y) = \int_{L_2[0,T]} \exp\left\{i(v, y) - \frac{1}{2\lambda} \|vh\|_2^2\right\} d\mu(v)$$

for s -a.e. $y \in C_0[0, T]$. Moreover L_1 analytic generalized Fourier-Feynman transform of F exists for all real $q \neq 0$ and is given by

$$(2.9) \quad T_{q,h}^{(1)}(F)(y) = \int_{L_2[0,T]} \exp\left\{i(v, y) - \frac{i}{2q} \|vh\|_2^2\right\} d\mu(v)$$

for s -a.e. $y \in C_0[0, T]$.

3. A change of scale formula for functionals in \mathcal{S}

In this section, we discuss a change of scale formula for generalized Wiener integrals of functions in \mathcal{S} using the Fourier–Feynman transform on classical Wiener space.

We next introduce an integration formula which plays an important role in this section. This lemma is obtained by using the similar method as in the proof of Lemma 2 and 3 in [4] and hence we will state it without proof.

LEMMA 3.1. *Let λ be in \mathbb{C}_+ , $h \in L_\infty[0, T]$ with $1/h$ in $L_\infty[0, T]$ and $v \in L_2[0, T]$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a subset in $L_2[0, T]$ such that $\{\alpha_1 h, \alpha_2 h, \dots, \alpha_n h\}$ are orthonormal on $L_2[0, T]$. Then*

$$\begin{aligned} & \int_{C_0[0, T]} \exp\left\{\frac{1-\lambda}{2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2 + i(v, Z_h(x, \cdot) + y)\right\} dm_w(x) \\ &= \lambda^{-n/2} \exp\left\{\frac{\lambda-1}{2\lambda} \sum_{k=1}^n \langle \alpha_k h, v h \rangle^2 - \frac{1}{2} \|v h\|_2^2 + i(v, y)\right\}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $L_2[0, T]$.

Let h be in $L_\infty[0, T]$ with $1/h$ in $L_\infty[0, T]$ and let $Z_h(x, t)$ be given by (2.2). Let $\{\gamma_1, \dots, \gamma_k, \dots\}$ be a complete orthonormal set on $L_2[0, T]$. Now we set

$$(3.1) \quad \alpha_k = \gamma_k/h \text{ for } k = 1, 2, 3, \dots,$$

and then the α_k 's clearly belong to $L_2[0, T]$.

In the following theorem, we give relationships between the generalized Wiener integral and $T_{\lambda, h}(F)(y)$.

THEOREM 3.2. *Let $F \in \mathcal{S}$ be given by (2.7). Let q be a non-zero real number and let $\{\alpha_k\}$ be given as in (3.1). Then for each $\lambda \in \mathbb{C}_+$, we have*

$$(3.2) \quad T_{\lambda, h}(F)(y) = \lim_{n \rightarrow \infty} \lambda^{n/2} \int_{C_0[0, T]} \exp\left\{\frac{1-\lambda}{2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2\right\} \\ \times F(Z_h(x, \cdot) + y) dm_w(x)$$

Proof. Since $F \in \mathcal{S}$ we have

$$F(x) = \int_{L_2[0, T]} \exp\{i(v, x)\} d\mu(v),$$

for some $\mu \in M(L_2[0, T])$. By Fubini theorem and Lemma 3.1, we obtain

$$\begin{aligned}
 (3.3) \quad & \int_{C_0[0, T]} \exp\left\{\frac{1-\lambda}{2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2\right\} F(Z_h(x, \cdot) + y) dm_w(x) \\
 &= \int_{C_0[0, T]} \int_{L_2[0, T]} \exp\left\{\frac{1-\lambda}{2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2 + i(v, Z_h(x, \cdot) + y)\right\} \\
 &\quad \times d\mu(v) dm_w(x) \\
 &= \int_{L_2[0, T]} \int_{C_0[0, T]} \exp\left\{\frac{1-\lambda}{2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2 \right. \\
 &\quad \left. + i(v, Z_h(x, \cdot) + y)\right\} dm_w(x) d\mu(v) \\
 &= \lambda^{-n/2} \int_{L_2[0, T]} \exp\left\{\frac{\lambda-1}{2\lambda} \sum_{k=1}^n \langle \alpha_k h, v h \rangle^2 - \frac{1}{2} \|vh\|_2^2 + i(v, y)\right\} d\mu(v).
 \end{aligned}$$

By the Parseval's relation, we obtain that

$$(3.4) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \langle \alpha_k h, v h \rangle^2 = \|vh\|_2^2.$$

Using the bounded convergence theorem, equations (3.4), (3.3) and (2.8), we obtain

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \lambda^{n/2} \int_{C_0[0, T]} \exp\left\{\frac{1-\lambda}{2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2\right\} \\
 &\quad \times F(Z_h(x, \cdot) + y) dm_w(x) \\
 &= \int_{L_2[0, T]} \exp\left\{i(v, y) - \frac{1}{2\lambda} \|vh\|_2^2\right\} d\mu(v) \\
 &= T_{\lambda, h}(F)(y)
 \end{aligned}$$

which completes the proof. \square

The following is a relationship between generalized Wiener integral and the L_1 analytic generalized Fourier-Feynman transform. It follows from Theorem 3.2 and (2.9).

THEOREM 3.3. *Let $F \in \mathcal{S}$ be given by (2.7). Let q be a non-zero real number and let $\{\alpha_n\}$ be given as in (3.1). Let $\{\lambda_n\}$ be a sequence of*

complex numbers with $\operatorname{Re}\lambda_n > 0$ such that $\lambda_n \rightarrow -iq$. Then

$$(3.5) \quad T_{q,h}^{(1)}(F)(y) = \lim_{n \rightarrow \infty} \lambda_n^{n/2} \int_{C_0[0,T]} \exp\left\{\frac{1-\lambda_n}{2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2\right\} \\ \times F(Z_h(x, \cdot) + y) dm_w(x)$$

for *s-a.e.* $y \in C_0[0, T]$.

The next theorem shows our change of scale formula for generalized Wiener integrals on Wiener space which follows from Theorem 3.2 above.

THEOREM 3.4. *Let F and $\{\alpha_n\}$ be given as in Theorem 3.3. Then for every $\rho > 0$,*

$$(3.6) \quad \int_{C_0[0,T]} F(Z_h(\rho x, \cdot) + y) dm_w(x) \\ = \lim_{n \rightarrow \infty} \rho^{-n} \int_{C_0[0,T]} \exp\left\{\frac{\rho^2-1}{2\rho^2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2\right\} \\ \times F(Z_h(x, \cdot) + y) dm_w(x)$$

for *s-a.e.* $y \in C_0[0, T]$.

Proof. First note that for $\lambda > 0$, $T_{\lambda,h}(F)$ is given by

$$T_{\lambda,h}(F) = \int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}} Z_h(x, \cdot) + y) dm_w(x)$$

by the equation (2.5). Letting $\lambda = \rho^{-2}$ in Theorem 3.2, we can get the equation (3.6). \square

4. A change of scale formula for unbounded functions

Cameron and Storvick [5] introduced a class of functions of the form

$$(4.1) \quad F(x) = G(x) \Psi((\alpha_1, x), (\alpha_2, x), \dots, (\alpha_r, x))$$

for $G \in \mathcal{S}$ and $\Psi = \psi + \phi$ where $\psi \in L_p(\mathbb{R}^r)$, $1 \leq p < \infty$, α_k 's given as in (3.1) in Section 3, and $\phi \in \hat{M}(\mathbb{R}^r)$, the set of functions ϕ defined on \mathbb{R}^r by

$$(4.2) \quad \phi(\vec{s}) = \int_{\mathbb{R}^r} \exp\left\{i \sum_{k=1}^r s_k t_k\right\} d\rho(\vec{t})$$

where ρ is a complex Borel measure of bounded variation on \mathbb{R}^r , $\vec{s} = (s_1, \dots, s_r)$ and $\vec{t} = (t_1, \dots, t_r)$. They showed that the above functions

(4.1) which need not be bounded or continuous are analytic Feynman integrable.

In this section, we establish a change of scale formula for generalized Wiener integrals of functions of the form (4.1) using the Fourier-Feynman transform on classical Wiener space.

To simplify the expressions, we use the following notations:

$$(\vec{\alpha}, x) = ((\alpha_1, x), (\alpha_2, x), \dots, (\alpha_r, x))$$

and

$$(\vec{\alpha}h, x) = ((\alpha_1h, x), (\alpha_2h, x), \dots, (\alpha_rh, x)).$$

The following theorem is the existence theorem for L_1 analytic generalized Fourier-Feynman transform of functions having a type (4.1) above. Using the similar methods as in the proof of Cameron and Storvick's theorem, we obtain the following existence theorem and so we will state it without proof.

THEOREM 4.1. *Let $F(x) = G(x)\psi((\vec{\alpha}, x))$ where $G \in \mathcal{S}$, $\psi \in L_p(\mathbb{R}^r)$ and $1 \leq p < \infty$. Then for each $\lambda \in \mathbb{C}_+$,*

$$(4.3) T_{\lambda, h}(F)(y) = \left(\frac{\lambda}{2\pi}\right)^{r/2} \int_{L_2[0, T]} \int_{\mathbb{R}^r} \exp\left\{\frac{1}{2\lambda} \left[\sum_{k=1}^r (i\lambda u_k + \langle \alpha_k h, v h \rangle)^2 - \|vh\|_2^2 \right] + i(v, y)\right\} \psi(\vec{u} + (\vec{\alpha}, y)) d\vec{u} d\mu(v)$$

for s -a.e. $y \in C_0[0, T]$. Moreover if $p = 1$, the L_1 analytic generalized Fourier-Feynman transform of F exists for all real $q \neq 0$ and is given by

$$(4.4) T_{q, h}^{(1)}(F)(y) = \left(-\frac{iq}{2\pi}\right)^{r/2} \int_{L_2[0, T]} \int_{\mathbb{R}^r} \exp\left\{\frac{i}{2q} \left[\sum_{k=1}^r (qu_k + \langle \alpha_k h, v h \rangle)^2 - \|vh\|_2^2 \right] + i(v, y)\right\} \psi(\vec{u} + (\vec{\alpha}, y)) d\vec{u} d\mu(v)$$

for s -a.e. $y \in C_0[0, T]$.

THEOREM 4.2. *Let $F(x) = G(x)\phi((\vec{\alpha}, x))$ where $G \in \mathcal{S}$ and $\phi \in \hat{M}(\mathbb{R}^r)$. Then for each $\lambda \in \mathbb{C}_+$,*

$$(4.5) T_{\lambda, h}(F)(y) = \int_{L_2[0, T]} \int_{\mathbb{R}^r} \exp\left\{-\frac{1}{2\lambda} \left[\|vh\|_2^2 + \sum_{k=1}^r 2t_k \langle \alpha_k h, v h \rangle + \sum_{k=1}^r t_k^2 \right] + i(v, y) + i \sum_{k=1}^r t_k (\alpha_k, y) \right\} d\rho(\vec{t}) d\mu(v)$$

for s -a.e. $y \in C_0[0, T]$. Moreover, for any $1 \leq p < \infty$, the L_p analytic generalized Fourier-Feynman transform of F exists for all real $q \neq 0$ and is given by

$$(4.6) \quad T_{q,h}^{(p)}(F)(y) = \int_{L_2[0,T]} \int_{\mathbb{R}^r} \exp\left\{-\frac{i}{2q} \left[\|vh\|_2^2 + \sum_{k=1}^r 2t_k \langle \alpha_k h, vh \rangle + \sum_{k=1}^r t_k^2 \right] + i(v, y) + i \sum_{k=1}^r t_k \langle \alpha_k, y \rangle \right\} d\rho(\vec{t}) d\mu(v)$$

for s -a.e. $y \in C_0[0, T]$.

Now we give a relationship between $T_{\lambda,h}(F)$ and generalized Wiener integral.

THEOREM 4.3. *Let $\{\alpha_k\}$ be given as in (3.1). Let $F(x) = G(x) \psi((\vec{\alpha}, x))$ where $G \in \mathcal{S}$ and $\psi \in L_p(\mathbb{R}^r)$, $1 \leq p < \infty$. Then for each $\lambda \in \mathbb{C}_+$, we have*

$$(4.7) \quad T_{\lambda,h}(F)(y) = \lim_{n \rightarrow \infty} \lambda^{n/2} \int_{C_0[0,T]} \exp\left\{ \frac{1-\lambda}{2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2 \right\} \times F(Z_h(x, \cdot) + y) dm_w(x).$$

Proof. Let n be a natural number with $n > r$ and let

$$\Gamma(n) = \int_{C_0[0,T]} \exp\left\{ \frac{1-\lambda}{2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2 \right\} F(Z_h(x, \cdot) + y) dm_w(x).$$

By the Fubini theorem, we obtain

$$(4.8) \quad \begin{aligned} & \Gamma(n) \\ &= \int_{L_2[0,T]} \int_{C_0[0,T]} \exp\left\{ \frac{1-\lambda}{2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2 + i(v, Z_h(x, \cdot) + y) \right\} \\ & \quad \times \psi((\vec{\alpha}_1, Z_h(x, \cdot) + y)) dm_w(x) d\mu(v) \\ &= \left(\frac{\lambda}{2\pi}\right)^{r/2} \lambda^{-n/2} \exp\left\{ \frac{\lambda-1}{2\lambda} \sum_{k=1}^n \langle \alpha_k h, vh \rangle^2 - \frac{1}{2} \|vh\|_2^2 + i(v, y) \right\} \\ & \quad \times \int_{L_2[0,T]} \int_{\mathbb{R}^r} \exp\left\{ \frac{1}{2\lambda} \sum_{k=1}^r (i\lambda u_k + \langle \alpha_k h, vh \rangle)^2 \right\} \\ & \quad \times \psi(\vec{u} + (\vec{\alpha}, y)) d\vec{u} d\mu(v). \end{aligned}$$

Note that, by the Bessel inequality, we have

$$\begin{aligned} & \left| \exp \left\{ \frac{\lambda - 1}{2\lambda} \sum_{k=1}^n \langle \alpha_k h, v h \rangle^2 - \frac{1}{2} \|v h\|_2^2 + i(v, y) \right. \right. \\ & \quad \left. \left. + \frac{1}{2\lambda} \sum_{k=1}^r (i\lambda u_k + \langle \alpha_k h, v h \rangle)^2 \right\} \psi(\vec{u} + (\vec{\alpha}, y)) \right| \\ & \leq \exp \left\{ -\frac{\operatorname{Re} \lambda}{2} \sum_{k=1}^r u_k^2 \right\} |\psi(\vec{u} + (\vec{\alpha}, y))| \end{aligned}$$

and the right hand side of the last inequality is integrable on $L_2[0, T] \times \mathbb{R}^r$, since $\psi \in L_p(\mathbb{R}^r)$ and $\mu \in M(L_2[0, T])$. Hence by the dominated convergence theorem and the Parseval's relation, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda^{n/2} \Gamma(n) &= \left(\frac{\lambda}{2\pi} \right)^{r/2} \int_{L_2[0, T]} \int_{\mathbb{R}^r} \exp \left\{ \frac{1}{2\lambda} \left[\sum_{k=1}^r (i\lambda u_k \right. \right. \\ & \quad \left. \left. + \langle \alpha_k h, v h \rangle)^2 - \|v h\|_2^2 \right] + i(v, y) \right\} \psi(\vec{u} + (\vec{\alpha}, y)) d\vec{u} d\mu(v). \end{aligned}$$

Now by (4.3) the proof is completed. \square

Moreover if $p = 1$, we obtain the following relationships between the L_1 analytic generalized Fourier-Feynman transform and generalized Wiener integral for functionals in (4.1).

THEOREM 4.4. *Let $\{\alpha_k\}$ be given as in Theorem 4.3. Let $F(x) = G(x)\psi((\vec{\alpha}, x))$ where $G \in \mathcal{S}$ and $\psi \in L_1(\mathbb{R}^r)$ and let $\{\lambda_n\}$ be a sequence of complex numbers in \mathbb{C}_+ such that $\lambda_n \rightarrow -iq$. Then*

$$(4.9) \quad T_{q,h}^{(1)}(F)(y) = \lim_{n \rightarrow \infty} \lambda_n^{n/2} \int_{C_0[0, T]} \exp \left\{ \frac{1 - \lambda_n}{2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2 \right\} \\ \times F(Z_h(x, \cdot) + y) dm_w(x).$$

Proof. We can obtain the following equation from the equation (4.8),
(4.10)

$$\begin{aligned} & \lambda_n^{n/2} \int_{C_0[0, T]} \exp \left\{ \frac{1 - \lambda_n}{2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2 \right\} F(Z_h(x, \cdot) + y) dm_w(x) \\ &= \lambda_n^{n/2} \left(\frac{\lambda_n}{2\pi} \right)^{r/2} \lambda_n^{-n/2} \int_{L_2[0, T]} \exp \left\{ \frac{\lambda_n - 1}{2\lambda_n} \sum_{k=1}^n \langle \alpha_k h, v h \rangle^2 - \frac{1}{2} \|v h\|_2^2 \right. \\ & \quad \left. + i(v, y) \right\} \int_{\mathbb{R}^r} \exp \left\{ \frac{1}{2\lambda_n} \sum_{k=1}^r (i\lambda_n u_k + \langle \alpha_k h, v h \rangle)^2 \right\} \psi(\vec{u} + (\vec{\alpha}, y)) d\vec{u} d\mu(v). \end{aligned}$$

Letting $n \rightarrow \infty$ in equation (4.10) and using the dominated convergence theorem and by the equation (4.4), we have that for s -a.e. $y \in C_0[0, T]$,

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left(\frac{\lambda_n}{2\pi} \right)^{r/2} \int_{L_2[0, T]} \exp \left\{ \frac{\lambda_n - 1}{2\lambda_n} \sum_{k=1}^n \langle \alpha_k h, v h \rangle^2 - \frac{1}{2} \|v h\|_2^2 + i(v, y) \right\} \\
& \quad \times \int_{\mathbb{R}^r} \exp \left\{ \frac{1}{2\lambda_n} \sum_{k=1}^r (i\lambda_n u_k + \langle \alpha_k h, v h \rangle)^2 \right\} \\
& \quad \times \psi(\vec{u} + (\vec{\alpha}, y)) d\vec{u} d\mu(v) \\
& = \left(-\frac{i q}{2\pi} \right)^{r/2} \int_{L_2[0, T]} \int_{\mathbb{R}^r} \exp \left\{ \frac{i}{2q} \left[\sum_{k=1}^r (q u_k + \langle \alpha_k h, v h \rangle)^2 - \|v h\|_2^2 \right] \right. \\
& \quad \left. + i(v, y) \right\} \psi(\vec{u} + (\vec{\alpha}, y)) d\vec{u} d\mu(v) \\
& = T_{q, h}^{(1)}(F)(y).
\end{aligned}$$

Therefore, we have the desired result. \square

THEOREM 4.5. *Let $\{\alpha_k\}$ be given as in (3.1). Let $F(x) = G(x) \phi((\vec{\alpha}, x))$ where $G \in \mathcal{S}$ and $\phi \in \hat{M}(\mathbb{R}^r)$. Then equation (4.7) holds.*

Proof. Let n be a natural number with $n > r$ and let $\Gamma(n)$ be the same as in the proof of Theorem 4.3. By the Fubini theorem, we have

$$\begin{aligned}
& \Gamma(n) \\
& = \int_{L_2[0, T]} \int_{\mathbb{R}^r} \int_{C_0[0, T]} \exp \left\{ \frac{1 - \lambda}{2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2 + i(v, Z_h(x, \cdot) + y) \right. \\
& \quad \left. + i \sum_{k=1}^r t_k (\alpha_k, Z_h(x, \cdot) + y) \right\} dm_w(x) d\rho(\vec{t}) d\mu(v) \\
& = \lambda^{-n/2} \int_{L_2[0, T]} \int_{\mathbb{R}^r} \exp \left\{ \frac{\lambda - 1}{2\lambda} \sum_{k=1}^n \langle \alpha_k h, v h \rangle^2 - \frac{1}{\lambda} \sum_{k=1}^r t_k \langle \alpha_k h, v h \rangle \right. \\
& \quad \left. - \frac{1}{2\lambda} \sum_{k=1}^r t_k^2 - \frac{1}{2} \|v h\|_2^2 + i(v, y) + i \sum_{k=1}^r t_k (\alpha_k, y) \right\} d\rho(\vec{t}) d\mu(v).
\end{aligned}$$

Using the Bessel inequality, we have that the exponential of the last expression above is bounded in absolute value by unity. Hence by the

dominated convergence theorem and the Parseval's relation, we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \lambda^{n/2} \Gamma(n) \\ &= \int_{L_2[0,T]} \int_{\mathbb{R}^r} \exp \left\{ -\frac{1}{2\lambda} \left[\|vh\|_2^2 + \sum_{k=1}^r 2t_k \langle \alpha_k h, vh \rangle + \sum_{k=1}^r t_k^2 \right] \right. \\ & \quad \left. + i(v, y) + i \sum_{k=1}^r t_k (\alpha_k, y) \right\} d\rho(\vec{t}) d\mu(v). \end{aligned}$$

By (4.5), the proof is completed. □

Modifying the proof of Theorem 4.5, by replacing “ λ ” and “ λ_k ” whenever it occurs and using (4.6) instead of (4.5) we have the following corollary.

COROLLARY 4.6. *Let $\{\alpha_k\}$ and $\{\lambda_k\}$ be given as in Theorem 4.4 and let F be given as in Theorem 4.5. Then equation (4.9) holds.*

From Theorem 4.3 and Theorem 4.5, and the linearity of the analytic Wiener integral on classical Wiener space, we obtain the following corollaries.

COROLLARY 4.7. *Let $\{\alpha_k\}$ be given as in Theorem 4.3. Let $F(x) = G(x)\Psi((\vec{\alpha}, x))$ where $G \in \mathcal{S}$ and $\Psi = \psi + \phi \in L_p(\mathbb{R}^r) + \hat{M}(\mathbb{R}^r), 1 \leq p < \infty$. Then equation (4.7) holds.*

COROLLARY 4.8. *Let $\{\alpha_k\}$ and $\{\lambda_k\}$ be given as in Theorem 4.4. Let $F(x) = G(x)\Psi((\vec{\alpha}, x))$ where $G \in \mathcal{S}$ and $\Psi = \psi + \phi \in L_1(\mathbb{R}^r) + \hat{M}(\mathbb{R}^r)$. Then equation (4.9) holds.*

Our main result, namely, a change of scale formula for generalized Wiener integrals follows from Corollary 4.7.

THEOREM 4.9. *Let $\{\alpha_k\}$ be given as in Theorem 4.3. Let $F(x) = G(x)\Psi((\vec{\alpha}, x))$ where $G \in \mathcal{S}$ and $\Psi = \psi + \phi \in L_p(\mathbb{R}^r) + \hat{M}(\mathbb{R}^r), 1 \leq p < \infty$. Then for any $\rho > 0$,*

$$\begin{aligned} (4.11) \quad & \int_{C_0[0,T]} F(\rho Z_h(x, \cdot) + y) d\nu(x) \\ &= \lim_{n \rightarrow \infty} \rho^{-n} \int_{C_0[0,T]} \exp \left\{ \frac{\rho^2 - 1}{2\rho^2} \sum_{k=1}^n (\alpha_k, Z_h(x, \cdot))^2 \right\} \\ & \quad \times F(Z_h(x, \cdot) + y) dm_w(x). \end{aligned}$$

Proof. By letting $\lambda = \rho^{-2}$ in (4.7), we have equation (4.11). □

COROLLARY 4.10. *When $h \equiv 1$ in Theorem 4.9, we obtain the change of scale formula for Wiener integrals of functions of the form (4.1) introduced in [11] and [15].*

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School of Liberal Arts
Seoul National University of Science and Technology
Seoul 139-743, Republic of Korea
E-mail: mathkbs@seoultech.ac.kr

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Department of Computer Engineering
Mokwon University
Daejeon 302-729, Republic of Korea
E-mail: teukseob@mokwon.ac.kr

Department of Mathematics
Yonsei University
Kangwondo 220-710, Republic of Korea
E-mail: iyoo@yonsei.ac.kr