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LOGARITHMIC COMPOSITION INEQUALITY IN BESOV SPACES

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Abstract. A logarithmic composition inequality in Besov spaces is derived which generalizes Vishik's inequality:

$$\|f\circ g\|_{B^s_{p,1}} \lesssim \big(1+\log(\|\nabla g\|_{\mathbf{L}^\infty}\|\nabla g^{-1}\|_{\mathbf{L}^\infty})\big)\|f\|_{B^s_{p,1}},$$

where g is a volume-preserving diffeomorphism on $\mathbb{R}^n$.

1. The main discussion

M. Vishik[6] derived a logarithmic inequality in order to prove the global in time vorticity existence of the 2-D Euler equations in critical Besov spaces $B^s_{p,1}$ with $sp=2$. It can be explicitly displayed as follows: for a volume-preserving bi-Lipschitz homeomorphism $g:\mathbb{R}^n\to\mathbb{R}^n$ and $f\in B^0_{\infty,1}(\mathbb{R}^n)$, we have $f\circ g^{-1}\in B^0_{\infty,1}(\mathbb{R}^n)$ and

$$\|f\circ g^{-1}\|_{B^0_{\infty,1}} \le C\left(1+\log(\|g\|_{\mathrm{Lip}}\|g^{-1}\|_{\mathrm{Lip}})\right)\|f\|_{B^0_{\infty,1}}$$

for some constant $C=C(n)$ independent of f, g and

$$\|g\|_{\mathrm{Lip}} := \sup_{x\neq x'}\frac{|g(x)-g(x')|}{|x-x'|}.$$

D. Chae later discussed a similar result on Triebel-Lizorkin spaces[1]. This paper generalizes Vishik's inequality on $B^0_{\infty,1}(\mathbb{R}^n)$ to more general Besov spaces $B^s_{p,1}(\mathbb{R}^n)$. Here is the main result:

Theorem 1.1. *Let $f\in B^s_{p,1}(\mathbb{R}^n)$ with $1\le p\le\infty$ and $|s|<1$. Suppose $g:\mathbb{R}^n\to\mathbb{R}^n$ is a volume-preserving diffeomorphism belonging to (homogeneous) Sobolev space $\dot W^{1,\infty}(\mathbb{R}^n)$. Then $f\circ g\in B^s_{p,1}(\mathbb{R}^n)$ and*

$$\|f\circ g\|_{B^s_{p,1}} \lesssim \big(1+\log(\|\nabla g\|_{\mathbf{L}^\infty}\|\nabla g^{-1}\|_{\mathbf{L}^\infty})\big)\|f\|_{B^s_{p,1}}.$$

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It is worth while pointing out that this result on Besov spaces can be discussed in general Triebel-Lizorkin spaces and that some other types of estimates for composition mapping can be found in [5](see page 209).

One of the typical examples of the volume-preserving diffeomorphisms g in Theorem 1.1 is the *particle trajectory mapping* $X(\cdot,t)$ which is often discussed in the theory of fluid mechanics. In fact, if $u(\cdot,t)$ is a divergence free vector field and $\{X(x,t)\}$ is the solution of the ordinary differential equation:

$$\begin{cases} \dfrac{\partial}{\partial t}X(x,t) &= u(X(x,t),t), \\ X(x,0) &= x, \end{cases} \tag{1.1}$$

then it can be noted that $X(\cdot,t)$ is a volume-preserving diffeomorphism. Theorem 1.1 can be applied to the 2-D vorticity equation corresponding to the incompressible Euler equations given by

$$\frac{\partial}{\partial t}\,\omega + (u,\nabla)\omega = 0, \tag{1.2}$$

where $\omega :=$ curl u with the initial vorticity $\omega_0 :=$ curl u_0. It is well-known that the solution $\omega(x,t)$ of the 2-D vorticity equation can be represented by

$$\omega(x,t) = \omega_0(X^{-1}(x,t)), \quad x \in \mathbb{R}^2. \tag{1.3}$$

Therefore by virtue of Theorem 1.1, it can be said that

$$\|\omega(t)\|_{B^s_{p,1}} \lesssim \big(1+\log(\|\nabla_x X(\cdot,t)\|_{L^\infty}\|\nabla_x X^{-1}(\cdot,t)\|_{L^\infty})\big)\|\omega_0\|_{B^s_{p,1}}.$$

Here are some notations which will be used throughout this paper. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing functions. Take a nonnegative radial function $\chi \in \mathcal{S}(\mathbb{R}^n)$ satisfying supp $\chi \subset \{\xi \in \mathbb{R}^n : |\xi| \le \frac{5}{6}\}$, and $\chi = 1$ for $|\xi| \le \frac{3}{5}$. Set $h_j(\xi) := \chi(2^{-j-1}\xi) - \chi(2^{-j}\xi)$, and it can be easily seen that

$$\chi(\xi) + \sum_{j=0}^{\infty} h_j(\xi) = 1 \ \text{ for } \xi \in \mathbb{R}^n.$$

Let φ_j and Φ be functions defined by $\varphi_j := \mathcal{F}^{-1}(h_j)$, $j \ge 0$ and $\Phi := \mathcal{F}^{-1}(\chi)$, where $\mathcal{F}$ represents the Fourier transform on $\mathbb{R}^n$ defined by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

Note that φ_j is a mollifier of φ_0, that is, $\varphi_j(x) := 2^{jn}\varphi_0(2^j x)$ (or $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j}\xi)$). One can readily check that

$$\Phi(x) + \sum_{j=0}^{k-1} \varphi_j(x) = 2^{kn}\Phi(2^k x) \text{ for } k \geq 1.$$

For $f \in \mathcal{S}'(\mathbb{R}^n)$, denote $\Delta_j f \equiv h_j(D)f = \varphi_j * f$ if $j \geq 0$, $\Delta_{-1} f \equiv \Phi * f$ and $\Delta_j f = 0$ if $j \leq -2$. The partial sums are also defined: $S_k f := \sum_{j=-\infty}^{k} \Delta_j f$ for $k \in \mathbb{Z}$. Assume $s \in \mathbb{R}$, and $1 \leq p, q \leq \infty$. The Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are defined by

$$f \in B^s_{p,q}(\mathbb{R}^n) \Leftrightarrow \{\|2^{js}\Delta_j f\|_{L^p}\}_{j\in\mathbb{Z}} \in l^q.$$

Notation Throughout this paper, the notation $X \lesssim Y$ means that $X \leq CY$, where C is a fixed but unspecified constant. Unless explicitly stated otherwise, C may depend on the dimension n and various other parameters such as exponents, but not on the functions or variables $(u, v, f, g, x_i, \cdots)$ involved.

2. The proof

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a volume-preserving diffeomorphism with $g(x) = (g_1(x), g_2(x), \cdots, g_n(x))$ and $f \in B^s_{p,q}(\mathbb{R}^n)$. Then f can be written as

$$f = \sum_{m=-1}^{\infty} \Delta_m f.$$

By plugging this representation into the definition of the Besov space $B^s_{p,q}(\mathbb{R}^n)$, we have

$$\begin{aligned}\|f \circ g\|_{B^s_{p,1}} &\leq \sum_{j=-1}^{\infty}\sum_{m=-1}^{\infty} 2^{js}\left\|\Delta_j(\Delta_m f)\circ g\right\|_{L^p} \\ &= \sum_{m=-1}^{\infty}\sum_{j=-1}^{\infty} 2^{js}\left\|\Delta_j(\Delta_m f)\circ g\right\|_{L^p}.\end{aligned}$$

Choose arbitrary $N \geq 1$ (the explicit choice will be made later) and consider three cases: $j - m > N$, $m - j > N$, and $|m - j| \leq N$ to get

$$\|f \circ g\|_{B^s_{p,1}} \leq \sum_{m=-1}^{\infty} \left(\sum_{j<m-N} + \sum_{j>m+N} + \sum_{|j-m|\leq N} \right) 2^{js} \|\Delta_j((\Delta_m f) \circ g)\|_{L^p}.$$

It suffices to estimate $\|\Delta_j((\Delta_m f) \circ g)\|_{L^p}$. The following partition of $\hat{\varphi}$ can be used:

$$\hat{\varphi}(\xi) = \sum_{k=1}^{n} i\xi_k \hat{\theta}_k(\xi), \qquad \hat{\theta}_k(\xi) = \frac{1}{in\xi_k}\hat{\varphi}(\xi). \tag{2.1}$$

Here $\hat{\varphi}(\xi) \in C_0^\infty(\mathbb{R}^n)$ and supp $\hat{\theta}_k \subset \{\xi \in \mathbb{R}^n | \frac{3}{5} \leq |\xi| \leq \frac{5}{3}\}$ for $k = 1, 2, \cdots, n$. For any $f \in S'(\mathbb{R}^n)$ and $j \geq 0$, we define

$$\tilde{\Delta}_{jk} f = \hat{\theta}_k(2^{-j}D)f = 2^{jn}\theta_k(2^j \cdot) * f, \quad \text{for } k = 1, 2, \cdots, n.$$

Then (2.1) implies that

$$\Delta_j = 2^{-j} \sum_{k=1}^{n} \partial_k \circ \tilde{\Delta}_{jk}, \qquad j \geq 0,$$

which is essential in the following proof.

We now look at the three cases separately. In case of $m > N + j$, we have

$$\begin{aligned}
&\Delta_j((\Delta_m f) \circ g)(x) \\
&= 2^{-m} \sum_{k=1}^{n} \Delta_j(\partial_k(\Delta_{mk} f) \circ g)(x) \\
&= 2^{nj-m} \sum_{k=1}^{n} \int_{\mathbb{R}^n} \varphi(2^j(x-y))(\partial_k \tilde{\Delta}_{mk} f)(g(y))dy \\
&= 2^{nj-m} \sum_{k=1}^{n} \int_{\mathbb{R}^n} \varphi(2^j(x - g^{-1}(z)))(\partial_k \tilde{\Delta}_{mk} f)(z)dz \\
&= -2^{(j-m)+nj} \sum_{k,l=1}^{n} \int_{\mathbb{R}^n} \partial_{z_l}\varphi(2^j(x - g^{-1}(z)))\tilde{\Delta}_{mk} f(z) \partial_{z_l} g_l^{-1}(z)dz.
\end{aligned}$$

From this we get that

$$\|\Delta_j((\Delta_m f) \circ g)\|_{L^p} \lesssim 2^{j-m} \sum_{k=1}^{n} \|\tilde{\Delta}_{mk} f\|_{L^p} \|\nabla g^{-1}\|_{L^\infty},$$

or we get

$$
\begin{aligned}
&2^{js}\|\Delta_j((\Delta_m f)\circ g)\|_{L^p} \\
&\lesssim 2^{(j-m)(s+1)}\left(\sum_{k=1}^{n} 2^{ms}\|\tilde{\Delta}_{mk} f\|_{L^p}\right)\|\nabla g^{-1}\|_{L^\infty}.
\end{aligned}
\tag{2.2}
$$

For the case of $m < j - N$, we can write

$$
\begin{aligned}
&\Delta_j((\Delta_m f)\circ g)(x) \\
&= 2^{(n-1)j}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\partial_{x_k}\theta_k(2^j(x-y))(\Delta_m f)(g(y))dy \\
&= 2^{(n-1)j}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\theta_k(2^j(x-y))\partial_k((\Delta_m f)(g(y)))dy \\
&= 2^{(m-j)+nj}\sum_{k,l=1}^{n}\int_{\mathbb{R}^n}\theta_k(2^j(x-y))(\Delta_m\partial_l f(g(y)))\partial_k g_l(y)dy.
\end{aligned}
$$

Therefore, if $j - m > N$, then we get

$$
\begin{aligned}
\|\Delta_j((\Delta_m f)\circ g)\|_{L^p} &\lesssim 2^{-j}\|\nabla\Delta_m f\|_{L^p}\|\nabla g\|_{L^\infty} \\
&\lesssim 2^{m-j}\|\Delta_m f\|_{L^p}\|\nabla g\|_{L^\infty}.
\end{aligned}
$$

Hence we obtain

$$
2^{js}\|\Delta_j((\Delta_m f)\circ g)\|_{L^p} \lesssim 2^{(m-j)(1-s)}\left(2^{ms}\|\Delta_m f\|_{L^p}\right)\|\nabla g\|_{L^\infty}. \tag{2.3}
$$

Finally, for $|j-m| \le N$, we use the integral representation

$$
\Delta_j((\Delta_m f)\circ g)(x) = 2^{nj}\int_{\mathbb{R}^n}\varphi(2^j(x-y))(\Delta_m f)\circ g dy
$$

to reach to the estimate

$$
\|\Delta_j((\Delta_m f)\circ g)\|_{L^p} \lesssim \|\Delta_m f\|_{L^p}. \tag{2.4}
$$

Now combine the estimates (2.2), (2.3) and (2.4) together to get:

$$\begin{aligned}\|f\circ g\|_{B^s_{p,1}} &\lesssim \|\nabla g^{-1}\|_{\mathbf{L}^\infty} 2^{-N(s+1)} \sum_{k=1}^{n}\sum_{m=0}^{\infty} 2^{ms}\|\tilde{\Delta}_{mkf}\|_{L^p} \\ &\quad + \|\nabla g\|_{\mathbf{L}^\infty} 2^{-N(1-s)} \sum_{m=-1}^{\infty} 2^{ms}\|\Delta_m f\|_{L^p} \\ &\quad + (2N-1)\sum_{m=-1}^{\infty} 2^{ms}\|\Delta_m f\|_{L^p} \\ &\lesssim \Big(2^{-N(1-s)}\|\nabla g\|_{\mathbf{L}^\infty} + 2^{-N(1-s)}\|\nabla g^{-1}\|_{\mathbf{L}^\infty} + N\Big)\|f\|_{B^s_{p,1}}. \qquad (2.5)\end{aligned}$$

Now we choose

$$N = \left[\frac{1}{1-s}\log_2(\|\nabla g\|_{\mathbf{L}^\infty}\|\nabla g^{-1}\|_{\mathbf{L}^\infty})\right] + 1$$

so that inequality (2.5) leads to the statement of the theorem. Notice that $\|\nabla g^{\pm 1}\|_{\mathbf{L}^\infty} \geq 1$ since $g^{\pm 1}$ is volume preserving.

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