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OPTIMAL CONSUMPTION AND INVESTMENT PROBLEM WITH REGIME-SWITCHING AND CARA UTILITY

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ABSTRACT. We use the dynamic programming method to investigate the optimal consumption and investment problem with regimeswitching. We derive the optimal solutions in closed-form with constant absolute risk aversion (CARA) utility.

1. Introduction

After the pioneer work of Merton [3, 4], portfolio optimization problem has been one of the most important and active area in mathematical finance. Recently, portfolio selection problems combined with regimeswitching are widely considered. ([1, 6, 5])

In this paper we investigate the optimal consumption and investment problem with regime-switching under the dynamic programming framework of Karatzas *et al.* [2]. We use the constant absolute risk aversion (CARA) utility function to obtain the optimal solutions in closed-form.

2. The financial market

Let us define a standard Brownian motion B_t and a continuous-time two-state Markov chain ϵ_t on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that B_t and ϵ_t are independent and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by both the Brownian motion B_t and the Markov chain ϵ_t .

We assume that two assets are traded in the financial market: One is a riskless asset with constant interest rate r and the other is a risky asset. We assume that there are two regimes 1, 2 in the market and regime i switches into regime j at the first jump time of an independent

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Yong Hyun Shin

Poisson process with intensity λ_i , for $i, j \in \{1, 2\}$. Here two states of regimes 1, 2 are economically considered as bull market and bear market. So, in regime $i \in \{1, 2\}$, the risky asset price process follows $dS_t/S_t = \mu_i dt + \sigma_i dB_t$. The market price of risk is defined by $\theta_i \triangleq (\mu_i - r)/\sigma_i$, i = 1, 2. Let π_t be the \mathcal{F}_t -progressively measurable portfolio process, the dollar-amount of the agent's wealth invested in the risky asset at time t and c_t be the nonnegative \mathcal{F}_t -progressively measurable consumption process at time t. We assume that they hold the following technical conditions:

$$\int_0^t c_s ds < \infty \quad \text{and} \quad \int_0^t \pi_s^2 ds < \infty, \text{ for all } t \ge 0, \text{ almost surely (a.s.)}.$$

The agent's wealth process X_t at time t follows

$$dX_t = [rX_t + \pi_t(\mu_i - r) - c_t] dt + \sigma_i \pi_t dB_t, \quad X_0 = x > 0, \ i = 1, \ 2.$$

3. The optimization problem

The agent's optimization problem with CARA utility $u(c) := -e^{-\gamma c}/\gamma$, is to maximize her expected utility

(3.1)
$$V_i(x) = \sup_{(c,\pi)\in\mathcal{A}(x)} \mathbb{E}\left[-\int_0^{\tau_i} \frac{e^{-\rho t - \gamma c_t}}{\gamma} dt + e^{-\rho \tau_i} V_j(X_{\tau_i})\right],$$

where τ_i is the first jump time since the beginning of regime $i, \rho > 0$ is a subjective discount rate, $\gamma > 0$ is absolute risk aversion and $\mathcal{A}(x)$ is an admissible set of pairs (c, π) at x, where $i, j \in \{1, 2\}$ and $i \neq j$.

Assumption 3.1.

$$\rho-r+\frac{\theta_i^2}{2}>0,\ i\in\{1,2\}.$$

Next theorem gives our main results.

THEOREM 3.2. The value function is given by

(3.2)
$$V_i(x) = -\frac{M_i}{\gamma r} e^{-\gamma r x}$$

where M_i and M_j are the solutions of the system of algebraic equations

(3.3)
$$\left(\rho + \lambda_i - r + \frac{\theta_i^2}{2}\right) M_i + r M_i \log M_i - \lambda_i M_j = 0,$$

86

where $i, j \in \{1, 2\}$ and $i \neq j$, and the optimal policies (c_i^*, π_i^*) are given by

(3.4)
$$c_i^* = rx - \frac{1}{\gamma} \log M_i$$
 and $\pi_i^* = \frac{\theta_i}{\gamma r \sigma_i}, i = 1, 2.$

Proof. From the optimization problem (3.1), we have the following coupled Bellman equations

(3.5)
$$\max_{(c_i,\pi_i)} \left[\left\{ rx + \pi_i(\mu_i - r) - c_i \right\} V'_i(x) + \frac{1}{2} \sigma_i^2 \pi_i^2 V''_i(x) - (\rho + \lambda_i) V_i(x) + \lambda_i V_j(x) - \frac{e^{-\gamma c_i}}{\gamma} \right] = 0,$$

where $i, j \in \{1, 2\}$ and $i \neq j$. The first-order conditions (FOCs) imply

(3.6)
$$c_i^* = -\frac{1}{\gamma} \log(V_i'(x))$$
 and $\pi_i^* = -\frac{\theta_i V_i'(x)}{\sigma_i V_i''(x)}, i = 1, 2.$

We assume that the optimal consumption $c_i^* = C_i(x)$, i = 1, 2, is a function of wealth. And let $X_i(\cdot)$, i = 1, 2, be the inverse function of $C_i(\cdot)$, i = 1, 2. That is, $X_i(\cdot) = C_i^{-1}(\cdot)$, i = 1, 2. Then the FOCs (3.6) imply

(3.7)
$$V'_i(x) = e^{-\gamma C_i(x)}$$
 and $V''_i(x) = -\frac{\gamma}{X'_i(c_i)}e^{-\gamma C_i(x)}, i = 1, 2.$

Plugging the FOCs (3.6) with (3.7) into the Bellman equation (3.5), then we obtain

(3.8)
$$rX_{i}(c_{i})e^{-\gamma c_{i}} + \frac{\theta_{i}^{2}}{2\gamma}X_{i}'(c_{i})e^{-\gamma c_{i}} - c_{i}e^{-\gamma c_{i}} - (\rho + \lambda_{i})V_{i}(X_{i}(c_{i})) + \lambda_{i}V_{j}(X_{i}(c_{i})) - \frac{e^{-\gamma c_{i}}}{\gamma} = 0,$$

where $i, j \in \{1, 2\}$ and $i \neq j$. Taking derivative of the equation (3.8) with respect to c_i , then we obtain

$$rX'_{i}(c_{i})e^{-\gamma c_{i}} - \gamma rX_{i}(c_{i})e^{-\gamma c_{i}} + \frac{\theta_{i}^{2}}{2\gamma}X''_{i}(c_{i})e^{-\gamma c_{i}} - \frac{\theta_{i}^{2}}{2}X'_{i}(c_{i})e^{-\gamma c_{i}} + \gamma c_{i}e^{-\gamma c_{i}} - (\rho + \lambda_{i})X'_{i}(c_{i})e^{-\gamma c_{i}} + \lambda_{i}X'_{i}(c_{i})e^{-\gamma c_{j}} = 0$$

where $i, j \in \{1, 2\}$ and $i \neq j$, since $x = X_1(c_1) = X_2(c_2)$ implies $V'_j(X_i(c_i)) = V'_j(x) = e^{-\gamma c_j}$, where $i, j \in \{1, 2\}$ and $i \neq j$. Thus we

Yong Hyun Shin

obtain the coupled second order differential equations with respect to c_1 and c_2

(3.9)
$$\frac{\theta_i^2}{2\gamma} X_i''(c_i) - \left(\rho + \lambda_i - r + \frac{\theta_i^2}{2}\right) X_i'(c_i) - \gamma r X_i(c_i) + \gamma c_i + \lambda_i e^{-\gamma (c_j - c_i)} X_i'(c_i) = 0,$$

where $i, j \in \{1, 2\}$ and $i \neq j$. If we conjecture the solution $X_i(c_i)$ of the form

(3.10)
$$X_i(c_i) = \frac{c_i}{r} + \frac{1}{\gamma r} \log M_i$$
 and $c_i = rx - \frac{1}{\gamma} \log M_i$, $i = 1, 2,$

for some constant $M_i > 0$, then $X'_i(c_i) = 1/r$ and $X''_i(c_i) = 0$, i = 1, 2. The equation (3.10) implies

$$c_j - c_i = \frac{1}{\gamma} \log \frac{M_i}{M_j},$$

where $i, j \in \{1, 2\}$ and $i \neq j$. So the equation (3.9) can be reduced into the system of algebraic equations (3.3).

Now we want to show that there exists a unique pair solution (M_1, M_2) to (3.3). Without loss of generality, we may assume that $\theta_i < \theta_j$. Let

$$M_j = f(M_i) := \frac{1}{\lambda_i} \left(\rho + \lambda_i - r + \frac{\theta_i^2}{2} \right) M_i + \frac{r}{\lambda_i} M_i \log M_i > 0,$$

for $M_i > 0$, and let

$$f_1(M_i) := \frac{1}{\lambda_i} \left(\rho + \lambda_i - r + \frac{\theta_i^2}{2} \right) + \frac{r}{\lambda_i} \log M_i > 0,$$

then $f'_1(M_i) = r/(\lambda_i M_i) > 0$. Now we consider the constants \bar{x} and \underline{x} with $\bar{x} > \underline{x}$ as follows:

$$\bar{x} := e^{-\frac{1}{r} \left(\rho - r + \frac{\theta_i^2}{2}\right)} < 1 \quad \text{and} \quad \underline{x} := e^{-\frac{1}{r} \left(\rho + \lambda_i - r + \frac{\theta_i^2}{2}\right)} < 1,$$

where the inequalities are obtained from Assumption 3.1. Then $f_1(\bar{x}) = 1$, $f_1(\underline{x}) = 0$. Since $f_1(M_i) > 0$, $M_i > \underline{x}$.

Now let

$$g(M_i) := \left(\rho + \lambda_j - r + \frac{\theta_j^2}{2}\right) M_i f_1(M_i) + r M_i f_1(M_i) \log M_i f_1(M_i) - \lambda_j M_i,$$

and let

$$g_1(M_i) := \left(\rho + \lambda_j - r + \frac{\theta_j^2}{2}\right) f_1(M_i) + r f_1(M_i) \log M_i f_1(M_i) - \lambda_j.$$

88

Note that $g_1(\bar{x}) = (\theta_i^2 - \theta_i^2)/2 > 0$. Since, by l'Hospital's rule,

$$\lim_{M_i \to \underline{x}+} \frac{\log M_i f_1(M_i)}{1/f_1(M_i)} = \lim_{M_i \to \underline{x}+} \frac{f_1(M_i)(f_1(M_i) + M_i f_1'(M_i))}{-M_i f_1'(M_i)} = 0,$$

 $\lim_{M_i\to\underline{x}+} g_1(M_i) = -\lambda_j < 0$. Thus, by intermediate value theorem, there exists $\overline{M} > 0$ such that $g_1(\overline{M}) = 0$ and $\underline{x} < \overline{M} < \overline{x}$. Taking derivative of $g_1(M_i)$, then

(3.11)
$$\begin{aligned} g_1'(M_i) \\ &= \left(\rho + \lambda_j + \frac{\theta_j^2}{2}\right) f_1'(M_i) + r f_1'(M_i) \log M_i f_1(M_i) + r \frac{f_1(M_i)}{M_i} \\ &= \frac{r}{\lambda_i M_i} h(M_i), \end{aligned}$$

where

$$h(M_i) := \left(2\rho - r + \lambda_i + \lambda_j + \frac{\theta_i^2 + \theta_j^2}{2}\right) + r\log M_i^2 f_1(M_i).$$

Taking derivative of $h(M_i)$, then

$$h'(M_i) = r\left(\frac{2}{M_i} + \frac{f_1'(M_i)}{f_1(M_i)}\right) > 0.$$

Thus $h(M_i)$ is increasing, $\lim_{M_i \to \underline{x}+} h(M_i) = -\infty$ and $h(\overline{x}) = r + \lambda_i + \lambda_j + (\theta_j^2 - \theta_i^2)/2 > 0$. Again, by intermediate value theorem, there exists a unique $x^* > 0$ such that $h(x^*) = 0$ and $\underline{x} < x^* < \overline{x}$. That is, $h(M_1) < 0$ for (\underline{x}, x^*) and $h(M_1) > 0$ for (x^*, ∞) . This implies $g'_1(M_1) < 0$ for (\underline{x}, x^*) and $g'_1(M_1) > 0$ for (x^*, ∞) . Thus $g_1(M_1)$ is decreasing and negative for (\underline{x}, x^*) and $g_1(M_1)$ is increasing for (x^*, ∞) . Therefore \overline{M} with $x^* < \overline{M} < \overline{x}$ is the unique solution to $g_1(M_i) = 0$, and consequently we obtain the unique pair solution (M_1, M_2) to (3.3).

Plugging c_i in (3.10) into (3.7), then we derive

(3.12)
$$V'_i(x) = M_i e^{-\gamma r x} \quad \text{and} \quad V''_i(x) = -\gamma r M_i e^{-\gamma r x}$$

Also plugging (3.12) into the FOCs (3.6), then we obtain the optimal policies (c_i^*, π_i^*) in (3.4). Therefore, from the Bellman equations (3.5), we obtain the value function $V_i(x)$ in (3.2).

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Yong Hyun Shin

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