TOPOLOGY FIELDS, TOPOLOGICAL FLOWS AND TOPOLOGICAL ORGANISMS

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ABSTRACT. Topology may described a pattern of existence of elements of a given set X. The family $\tau(X)$ of all topologies given on a set X form a complete lattice. We will give some topologies on this lattice $\tau(X)$ using a topology on X and regard $\tau(X)$ a topological space.

A topology τ on X can be regarded a map from X to $\tau(X)$ naturally. Such a map will be called *topology field*. Similarly we can also define *pe-topology field*. If X is a topological flow group with acting group T, then naturally we can get a another topological flow $\tau(X)$ with same acting group T. If the topological flow X is minimal, we can prove $\tau(X)$ is also minimal.

The disjoint unions of the topological spaces can describe some topological systems (topological organisms). Here we will give a definition of topological organism. Our purpose of this study is to describe some properties concerning patterns of relationship between topology fields and topological organisms.

1. Introduction

Let X be a topological space with topology τ and p be a point of X. The set of all nbds of p will be denoted by $N(p)$. We can easily verify that $N(p) \cup \{\emptyset\}$ become a topology on X for any point p in X. Moreover the collection of all open nbds of p and empty set, that is, $\{V \in \tau | p \in V\} \cup \{\emptyset\}$ becomes a topology on X for any point $p \in X$. We will denote such a topology by τ_p and call *localized topology* of τ at p. Clearly τ_p is weaker than $N(p) \cup \{\emptyset\}$. We will denote the localized topology of the discrete topology $\mathcal{P}(X)$ on X at p by 1_p.

It is easy to show that (X, τ) is T_0 if and only if the localized topologies of τ at any two points are different with each other. We may regard X as the subset $\{1_p | \ p \in X\}$ of $\tau(X)$ since the localized topologies of the discrete topology at distinct point are different. If we denote τ_p^e by $\tau_p^e = \{U \in \tau | p \notin U\} \cup \{X\}$, then it is also a topology on X. Moreover $\tau_p \cap \tau_p^e = \{\emptyset, X\}$ and $\tau_p \cup \tau_p^e = \tau$. We will call it *point p excluded topology* or *pe-topology* of τ at p.

THEOREM 1.1. *Let τ and η be two topologies on X . Then $\tau = \eta$ if and only if the localized topologies τ_p, η_p are equal for every point p in X. Moreover $(\tau_p)_q = (\tau_q)_p$.*

This Theorem 1.1 tells us that if some points of the set X change their nbds, that is, change their localized topologies, then the "entire" topology of X will be changed. The family $\tau(X)$ would consist of all topologies on a given *fixed* set X. Here we want to study some conditions concerning the family $\tau(X)$ of all the topologies of the given set X.

2. Topology fields and pe-topology fields on X.

The family $\tau(X)$ of all topologies on X form a complete lattice, that is, given any collection of topologies on X, there is a smallest (respectively largest) topology on X containing(contained in) each member of the collection. Of course, the partial order $\leq$ on $\tau(X)$ is defined by inclusion $\subset$ naturally.

Given a collection of topologies $\{ \tau_j \mid j \in J \} \subset \tau(X)$, we write $\wedge_{j \in J} \tau_j$ and $\vee_{j \in J} \tau_j$ for their infimum and supremum with respect to this lattice. Of course, $\wedge_{j \in J} \tau_j$ is just $\cap_{j \in J} \tau_j$, whereas $\vee_{j \in J} \tau_j$ has $\cup_{j \in J} \tau_j$ as a subbasis . The smallest topology in this lattice $\tau(X)$ is $\{\emptyset, X\}$ and the largest one is $\mathcal{P}(X)$. These topologies will sometimes be denoted by 0 and 1 respectively.

Denote $\tau_p(X) = \{\eta_p \mid \eta \in \tau(X) \}$ for a point $p \in X$. Since $\tau(X)$ is a complete lattice, we can easily find that $\tau_p(X)$ is a sublattice of $\tau(X)$. The smallest element of this sublattice $\tau_p(X)$ is $0_p = 0$, the largest element is $\mathcal{P}(X)_p = 1_p \neq 1$. We will call this sublattice $\tau_p(X)$ as *sublattice of all localized topologies* at p in X. Now we will regard any member τ of $\tau(X)$ as a map from X to $\cup_p \tau_p(X) \subset \tau(X)$ defined by $\tau(p) = \tau_p$. Hence this map τ acts like a vector field on X. Such a map $f : X \to \tau(X)$ defined by $f(p) \in \tau_p(X)$ will be called *topology field* on X[7].

Since $\tau(X)$ is a complete lattice, every topology field is a member of $\tau(X)^X$ consisting of all mappings from X to $\tau(X)$. Hence every topology field acts as a $\tau(X)-$fuzzy subset on X.

If f is a map from X to $\tau(X)$ defined by $f(p)\in\tau_p(X)$, can we find a topology ζ such that $\zeta_p = f(p)$? In general, we can not find the correct topology ζ such that $\zeta_p = f(p)$.[7]

Moreover, for every point p of X, we can have a map $pr_p : \tau(X) \to \tau_p(X)$ defined by $pr_p(\tau) = \tau_p$. This map pr_p also will be called *projection from $\tau(X)$ to $\tau_p(X)$* [7]. And we can have naturally inclusion map $i_n : \tau_p(X) \to \tau(X)$ defined by $i_n(\zeta_p) = \zeta_p \in \tau(X)$. It is easy to find that the projections $pr_p:\tau(X)\to\tau_p(X)$ are monotone.

Similarly denote $\tau_p^e(X) = \{\eta_p^e \mid \eta \in \tau(X) \}$ for a point $p \in X$. Since $\tau(X)$ is a complete lattice, we can easily observe that $\tau_p^e(X)$ is a sublattice of $\tau(X)$. The smallest element of this sublattice $\tau_p^e(X)$ is $0_p^e=0$, but the largest element is $\mathcal{P}(X)_p^e=1_p^e\neq1$. We will call this sublattice $\tau_p^e(X)$ as *sublattice of all pe-localized topologies* at p in X.
Now we regard any member τ of $\tau(X)$ as a map from X to $\cup_p\tau_p^e(X) \subset \tau(X)$ defined by $\tau(p) = \tau_p^e$. Hence this map τ acts also like a vector field on X. In order to avoid the confusion with topology field, we will use symbol τ^e instead of this map τ. Such a map g from X to $\cup_p\tau_p^e(X) \subset \tau(X)$ defined by $g(p)\in \tau_p^e(X) \subset \tau(X)$ will be called *pe-topology field* on X. And now we can have a map pr_p^e from $\tau(X)$ onto $\tau_p^e(X) \subset \tau(X)$ defined by $pr_p^e(\tau) = \tau_p^e$ for every point p of X. This map pr_p^e also will be called *pe $-$ projection from $\tau(X)$ to $\tau_p^e(X)$*.

Let (X,τ) be a topological space. We want to find some topologies on $\tau(X)$ induced by the given topology τ which provides the continuity of topology fields and projections.

DEFINITION 2.1. [7] Let (X,τ) be a topological space, and $G \in \tau$. Let $i(G)=\{\zeta\in\tau(X) \mid G\in\zeta\}$ and denote $\epsilon=\{i(G)|G\in\tau\}$, a family of subset of $\tau(X)$. Then there is exactly one topology In_τ on $\tau(X)$ with ϵ as a subbasis. We will call this topology as inner topology induced by the topology τ. And the relative topology of In_τ in $\tau_p(X)$ will be denoted by $\mathcal{R}In_\tau$ briefly. Note that In_1 need not be the discrete topology in $\tau(X)$.

THEOREM 2.2. $(\tau(X), In_1)$ *is T_0 space.*

Proof. Let $\eta, \zeta \in \tau(X)$ with $\eta \neq \zeta$. There there exists at least one $G \in 1$ which is included in the one of them and is excluded from the other one. Say, $G \in \eta$ and $G \notin \zeta$. Hence $\eta \in i(G)$ and $\zeta \notin i(G)$. $\square$

If $\zeta \leq \eta$, then $\forall\, G \in \zeta$, $G \in \eta$. That is, if $i(G) \ni \zeta$, then $i(G) \cap \{\eta\} \neq \emptyset$. This implies $\zeta \in \overline{\{\eta\}}$. Conversely $\zeta \in \overline{\{\eta\}}$ implies $\zeta \leq \eta$. If this relation holds we say that ζ is a *specialization* of η [9]. For any $\eta \in \tau(X)$ we will denote the subset $\{\zeta \in \tau(X) | \zeta \geq \eta\}$ by $\uparrow(\eta)$. (We shall also use later the notation $\downarrow(\eta)$ for $\{\zeta \in \tau(X)|\zeta \leq \eta\}$. Then since $i(G) = \{\,\zeta \in \tau(X) \mid G \in \zeta\,\}$, $i(G) = \uparrow(\{\emptyset, X, G\})$. Hence $\zeta \in \overline{\{\eta\}}$ iff $\zeta \leq \eta$. Since *Alexandrov topology* Υ on $\tau(X)$ is the collection of all *upper sets* in $\tau(X)$ (i.e. sets U such that $\eta \in U$ and $\eta \leq \zeta$ imply $\zeta \in U$) [9], $i(G) \in \Upsilon$. Hence we have the following result

THEOREM 2.3. *If $\tau \leq \zeta \leq 1$, then $In_\tau \leq In_\zeta \leq In_1 \leq \Upsilon$.*

Proof. Let $i(G) \in In_\tau$. Then $G \in \tau \leq \zeta$. Hence $G \in \zeta$. Consequently $i(G) = \{\alpha \in \tau(X)|G \in \alpha\} \in In_\zeta$. This implies $In_\tau \leq In_\zeta$. Since every subbasic open set $i(G)$ is a upper set, every open set must be an upper set(cf. proposition 1.8 II [9]). This completes the proof. $\square$

Now we consider In as a map from $\tau(X)$ to $\tau(\tau(X))$ defined by $In(\eta) = In_\eta$. Then we have some results:

THEOREM 2.4. *$In : (\tau(X), \Upsilon) \to (\tau(\tau(X)), \Upsilon)$ is continuous.*

Proof. Let $\zeta \in \tau(X)$ and K is a neighborhood of $In(\zeta) = In_\zeta$. Then K is a upper set in $\tau(\tau(X))$. On the other hand the upper set $\uparrow(\zeta)$ in $\tau(X)$ is a nbd of ζ. We will show that $In(\uparrow(\zeta)) \subset K$. Let $\delta \in\, \uparrow(\zeta)$. Then $\delta \geq \zeta$ and $In_\delta \geq In_\zeta$. Hence we have $In_\delta \in K$. $\square$

THEOREM 2.5. *$((\tau(X), In_\tau),\ \wedge)$ and $((\tau(X), \Upsilon),\ \wedge)$ are topological monoids.*

Proof. We will prove only the case of $((\tau(X), In_\tau),\ \wedge)$. The other case is similar. First closeness of the operation $\wedge$ is trivial. Moreover we can easily get the associative law. The largest element 1 in $\tau(X)$ is the identity element for this operation $\wedge$. Now we will prove that the binary operation $\wedge : \tau(X) \times \tau(X) \to \tau(X)$ is continuous. Let $i(G)$ be an nbd of $\zeta \wedge \eta$. Then $G \in \zeta$ and $G \in \eta$. Thus $i(G) \times i(G)$ is a nbd of (ζ, η) and $\wedge(i(G) \times i(G)) \subset i(G)$. This completes the theorem. $\square$

THEOREM 2.6. [7] *Topology field ζ:$(X,\tau) \to (\tau(X),\ In_\tau)$ and projection pr_p:$(\tau(X),\ In_\tau) \to (\tau_p(X),\ \mathcal{R}In_\tau)$ are continuous. Moreover inclusion $i_n : (\tau_p(X), RIn_\tau) \to (\tau(X), In_\tau)$ is continuous.*

Proof. Let $p \in X$ and $i(G)$ be a subbasic open nbd of $\zeta(p) = \zeta_p$. Then $G \in \zeta_p$. This implies $p \in G \in \zeta$. Moreover since $G \in \tau$, G is a nbd of p. Hence if $q \in G$, $\zeta(q) = \zeta_q \in i(G)$, so that $\zeta(G) \subset i(G)$. This shows that topology

field ζ is continuous.   Now let $\eta \in \tau(X)$ and $i(G) \cap \tau_p(X)$ be a subbasic open nbd of $pr_p(\eta) = \eta_p$. Then $G \in \eta_p$. Consequently $p \in G \in \eta$. This means that $i(G)$ is a nbd of η. Moreover $pr_p(i(G)) \subset i(G) \cap \tau_p(X)$. In fact, if $\zeta \in i(G)$, then $G \in \zeta$. Since $p \in G$, $G \in \zeta_p$. Hence $\zeta_p \in i(G)$. This means that projection pr_p is continuous. Finally the continuity of inclusion is natural. $\square$

Let $f:(X, \tau) \to (Y, \eta)$ be a continuous surjective map. If we define a map $f_*:\tau(X) \to \tau(Y)$ by $f_*(w)=\{U \subset Y | f^{-1}(U) \in w\}$, then $f_*(0) = 0$ and $f_*(1) = 1$. Let $\omega \in \tau(X)$. For any subbasic open neighborhood $i(G)$ of $f_*(\omega)$, $G \in f_*(\omega)$. Thus $f^{-1}(G) \in \omega$. Hence $\omega \in i(f^{-1}(G))$. So that $i(f^{-1}(G))$ is an open neighborhood of ω. Conversely, if $\zeta \in i(f^{-1}(G))$ then $f^{-1}(G) \in \zeta$, $G \in f_*(\zeta)$, $f_*(\zeta) \in i(G)$, and $\zeta \in f_*^{-1}(i(G))$. Consequently we have

$$f_*^{-1}(i(G)) = i(f^{-1}(G)).$$

Note that $i(G) \cup \{0\}$ is also complete sublattice of $\tau(X)$ for a $G \in \tau$. We will denote this sublattice $i(G) \cup \{0\}$ by $i_F(G)$. Then naturally we can restrict domain of definition of this map f_* to $i_F(H)$ for some open H in X. Hence we can have $f_* : i_F(f^{-1}(G)) \to i_F(G)$ for each open G in (Y, η).

THEOREM 2.7. *Let $f:(X, \tau) \to (Y, \eta)$ be a continuous bijective map. Then the induce map $f_* : i_F(f^{-1}(G)) \to i_F(G)$ is bijective for each open G in (Y, η).*

Proof. In fact, since $f : X \to Y$ is continuous, for each open G in (Y, η), $f^{-1}(G) \in \tau$. If $\zeta \in i_F(f^{-1}(G))$, then $\zeta = 0$ or $f^{-1}(G) \in \zeta$. Hence $f_*(0) = 0 \in i_F(G)$ or $G \in f_*(\zeta)$. This implies $f_*(\zeta) \in i_F(G)$. Now let $f_*(\alpha) = f_*(\beta)$ for $\alpha, \beta \in i_F(f^{-1}(G))$. Then for all $U \in \alpha$, $U = f^{-1}(f(U)) \in \alpha$. So that $f(U) \in f_*(\alpha) = f_*(\beta)$. Hence $U \in \beta$. So that $\alpha = \beta$. This implies that f_* is injective. Now let $\delta(\neq 0) \in i_F(G)$. Then $\vartheta = \{f^{-1}(U) | U \in \delta\} \in i_F(f^{-1}(G))$. And $f_*(\vartheta) = \delta$. This completes the proof. $\square$

THEOREM 2.8. *Let $f:(X, \tau) \to (Y, \eta)$ be a continuous map. If we define a map $f_*:\tau(X) \to \tau(Y)$ by $f_*(w)=\{U \subset Y | f^{-1}(U) \in w\}$, then the map f_* is continuous. If $\gamma \le \delta$, then $f_*(\gamma) \le f_*(\delta)$ and $f_*(\tau) \ge \eta$. And for any topology field ζ, the diagram*

$$\begin{array}{ccc} (X, \tau) & \stackrel{f}{\longrightarrow} & (Y, \eta) \\ \downarrow \zeta & & \downarrow f_*(\zeta) \end{array}$$

$$(\tau(X), In_\tau) \qquad \xrightarrow{f_*} \qquad (\tau(Y), In_\eta)$$

$$\downarrow pr_p \qquad\qquad\qquad \downarrow pr_{f(p)}$$

$$(\tau_p(X), \mathcal{R}In_\tau) \qquad \xrightarrow{f_*} \qquad (\tau_{f(p)}(Y), \mathcal{R}In_\eta)$$

commutes. If, furthermore, (Z, θ) is a topological space and $g: (Y, \eta)$ $\to (Z, \theta)$ is a map, then

$$(g \circ f)_* = g_* \circ f_*.$$

Finally, if $f:(X, \tau) \to (X, \tau)$ is the identity homeomorphism, then so is f_*.

Proof. Let $\omega \in \tau(X)$. For any subbasic open neighborhood $i(G)$ of $f_*(\omega)$, $G \in f_*(\omega)$. Thus $f^{-1}(G) \in \omega$. Hence $\omega \in i(f^{-1}(G))$. So that $i(f^{-1}(G)$ is a open neighborhood of ω. Conversely if $\zeta \in i(f^{-1}(G))$, then $f^{-1}(G) \in \zeta$. $G \in f_*(\zeta)$. $f_*(\zeta) \in i(G)$. $\zeta \in f_*^{-1}(i(G))$. Consequently we have

$$f_*^{-1}(i(G)) = i(f^{-1}(G)).$$

This means f_* is continuous. Moreover

$$\begin{aligned} f_*(\zeta_p) &= \{U | f^{-1}(U) \in \zeta_p\} = \{U | p \in f^{-1}(U) \in \zeta\} \\ &= \{U | f(p) \in U, f^{-1}(U) \in \zeta\} = \{U | U \in f_*(\zeta), f(p) \in U\} \\ &= f_*(\zeta)_{f(p)}. \end{aligned}$$

This implies upper diagram commutes. And similarly we can have the lower diagram commutes. All other statements follow directly from the definitions. $\square$

Consequently for a fixed topology ζ we can have a continuous map defined by $\wedge_\zeta : \tau(X) \to \tau(X), \wedge_\zeta(\eta) = \zeta \wedge \eta$. Let $f:(X, \tau) \to (Y, \eta)$ be a continuous map. If we define a map $f_*:\tau(X) \to \tau(Y)$ by $f_*(w) = \{U \subset Y | f^{-1}(U) \in w\}$, then

$$f^*(\zeta \wedge \eta) = \{U | f^{-1}(U) \in \zeta \wedge \eta\} = \{U | f^{-1}(U) \in \zeta\} \wedge \{U | f^{-1}(U) \in \eta\}$$

$$= f^*(\zeta) \wedge f^*(\eta).$$

Thus we have the following commutative diagram

$$(\tau(X), In_\tau) \qquad \xrightarrow{f_*} \qquad (\tau(Y), In_\eta)$$

$$\downarrow \wedge_\zeta \qquad\qquad\qquad \downarrow \wedge_{f^*(\zeta)}$$

$$(\tau(X), In_\tau) \qquad \xrightarrow{f_*} \qquad (\tau(Y), In_\eta).$$

Using the same method we can also have next theorem

COROLLARY 2.9. [7] *Let $f{:}(X,\,\tau) \to (Y,\,\eta)$ be a continuous map. If we define a map $f_*^{\eta}{:}\tau(X) \to \tau(Y)$ by $f_*^{\eta}(w){=}\{U \in \eta | f^{-1}(U) \in w\}$, then the map f_*^{η} is continuous and $f_*^{\eta}(\tau){=}\eta$. Moreover, for any topology field ζ, the diagram*

$$
\begin{array}{ccc}
(X,\tau) & \xrightarrow{\;f\;} & (Y,\eta) \\[2pt]
\downarrow \zeta & & \downarrow f_*^{\eta}(\zeta) \\[6pt]
(\tau(X), In_\tau) & \xrightarrow{\;f_*^{\eta}\;} & (\tau(Y), In_\eta) \\[2pt]
\downarrow pr_p & & \downarrow pr_{f(p)} \\[6pt]
(\tau_p(X), \mathcal{R}In_\tau) & \xrightarrow{\;f_*^{\eta}\;} & (\tau_{f(p)}(Y), \mathcal{R}In_\eta)
\end{array}
$$

commutes. If, furthermore, $(Z,\,\theta)$ is a topological space and $g{:}\,(Y,\,\eta) \to (Z,\,\theta)$ is a map, then

$$(g \circ f)_*^{\theta} = g_*^{\theta} \circ f_*^{\eta}.$$

Now we will give another topology on $\tau(X)$ which provide the continuity of pe-topology fields and pe-projections.

DEFINITION 2.10. Let (X,τ) be a topological space, and $G \in \tau$. Let $o(G){=}\{\zeta{\in}\tau(X) \,|G{\notin}\zeta\}$ and denote $\epsilon_o{=}\{o(G)|G{\in}\tau\}$, a family of subset of $\tau(X)$ with $o(X){=}o(\emptyset){=}\emptyset$. Then there is exactly one topology Out_τ on $\tau(X)$ with ϵ_o as a subbasis. We will call this topology outer topology induced by the topology τ. And the relative topology of Out_τ in $\tau_p(X)$ will be denoted by $\mathcal{R}Out_\tau$ briefly.

Using the notation $\uparrow$ we can describe $o(G)$ by $\tau(X) - i(G){=}\tau(X) - \uparrow (\{\emptyset, X, G\})$. Note that $o(G) \cup \{1\}$ is also complete lattice for a $G \in \tau$. Also we will denote this sublattice $o(G) \cup \{1\}$ by $o_F(G)$.

THEOREM 2.11. *$(\tau(X), Out_1)$ is T_0 space.*

Proof. Let $\eta, \zeta \in \tau(X)$ with $\eta \neq \zeta$. There there exists at least one $G \in 1$ which is included in one of them and is excluded from the other one. Say, $G \in \eta, G \notin \zeta$. Hence $\eta \notin o(G), \zeta \in o(G)$. $\qquad\square$

If $\zeta \leq \eta$, then $\forall\, G \notin \eta,\, G \notin \zeta$. That is, if $o(G) \ni \eta$, then $o(G) \cap \{\zeta\} \neq \emptyset$. This implies $\eta \in \overline{\{\zeta\}}$. Conversely $\eta \in \overline{\{\zeta\}}$ implies $\zeta \leq \eta$. Hence $\eta \in \overline{\{\zeta\}}$ iff $\zeta \leq \eta$. Since $o(G) = \{\, \zeta \in \tau\,(X) \mid G \notin \zeta \,\}$, $o(G) = \tau(X) - \uparrow (\{\emptyset, X, G\})$ is a *lower set*(cf. 2.1 I [9]). Now we will define a topology $\perp$ to be the collection of all lower sets in $\tau(X)$. It is clearly a topology, since it is closed under arbitrary unions and intersections. Hence we have

THEOREM 2.12. *If $\tau \leq \zeta \leq 1$, then $Out_\tau \leq Out_\zeta \leq Out_1 \leq \perp$.*

Proof. We will prove the case of $Out_\tau \leq Out_\zeta$. Let $o(G) \in Out_\tau$. Then $G \in \tau$ and consequently $G \in \zeta$. Hence $o(G) = \{\alpha \in \tau(X)|G \notin \alpha\} \in Out_\zeta$. Since every subbasic open set $o(G)$ is a lower set, every open set must be an lower set. This completes the proof. $\square$

Now we consider Out as a map from $\tau(X)$ to $\tau(\tau(X))$ defined by $Out(\eta) = Out_\eta$. Then we have

THEOREM 2.13. *$In : (\tau(X), \perp) \rightarrow (\tau(\tau(X)), \perp)$ is continuous.*

Proof. Let $\zeta \in \tau(X)$ and K be a neighborhood of $Out(\zeta) = Out_\zeta$. Then K is a lower set in $\tau(\tau(X))$. On the other hand the lower set $\downarrow (\zeta)$ in $\tau(X)$ is a nbd of ζ. We will show that $Out(\downarrow (\zeta)) \subset K$. Let $\delta \in \downarrow (\zeta)$. Then $\delta \leq \zeta$ and $Out_\delta \leq Out_\zeta$. Hence we have $Out_\delta \in K$. $\square$

THEOREM 2.14. *$((\tau(X), Out_\tau), \vee)$ and $((\tau(X), \perp), \vee)$ are topological monoids.*

Proof. We will prove only the case of $((\tau(X), Out_\tau), \vee)$. The other case is similar. First closeness of the operation $\vee$ is trivial. Moreover we can easily get the associative law. The smallest element 0 in $\tau(X)$ is the identity element for this operation $\vee$. Now we will prove that the binary operation $\vee : \tau(X) \times \tau(X) \rightarrow \tau(X)$ is continuous. Let $o(G)$ be an nbd of $\zeta \vee \eta$. Then $G \notin \zeta$ and $G \notin \eta$. Thus $o(G) \times o(G)$ is a nbd of (ζ, η) and $\vee(o(G) \times o(G)) \subset o(G)$. This completes the theorem. $\square$

Consequently for a fixed topology ζ we can have a continuous map defined by $\vee_\zeta : \tau(X) \rightarrow \tau(X), \vee_\zeta(\eta) = \zeta \vee \eta$.

THEOREM 2.15. *Let $\tau \leq \delta$. Then the pe-topology field $\delta^e : (X, \tau) \rightarrow (\tau(X), Out_\tau)$ and pe-projection $pr_p^e : (\tau(X), Out_\tau) \rightarrow (\tau_p^e(X), \mathcal{R}Out_\tau)$ are also continuous.*

Proof. Let $p \in X$ and $o(G)$ be a subbasic open nbd of $\delta^e(p) = \delta_p^e$. Then $G \notin \delta_p^e = \{A \in \delta | p \notin A\} \cup \{X\}$. This implies $p \in G$. Since $G \in \tau$, G is a nbd of p. If $q \in G$, $G \notin \delta^e(q) = \delta_q^e$. Hence $\delta_q^e \in o(G)$. This implies that $\delta^e(G) \subset o(G)$. It implies that the pe-topology field δ^e is continuous.

Now let $\eta \in \tau(X)$ and $o(G) \cap \tau_p^e(X)$ be a subbasic open nbd of $pr_p^e(\eta) = \eta_p^e$. Hence we get $p \in G$. For any $\zeta \in o(G)$ we have $G \notin \zeta$ and $G \notin \zeta_p^e$. This implies $\zeta_p^e \in o(G) \cap \tau_p^e(X)$. Consequently $pr_p^e(o(G)) \subset o(G) \cap \tau_p^e(X)$. This means that pe-projection pr_p^e is continuous. $\square$

Let $f:(X, \tau) \to (Y, \eta)$ be a continuous surjective map. If we define a map $f_*:\tau(X) \to \tau(Y)$ by $f_*(w)=\{U \subset Y|f^{-1}(U) \in w\}$, then $f_*(0) = 0$ and $f_*(1) = 1$. Similarly as in the case of inner topology we can restrict domain of definition of this map f_* to $o_F(G)$ for some open G in X. We then have $f_* : o_F(f^{-1}(H)) \to o_F(H)$ for each open H in (Y,η). In fact, since $f : X \to Y$ is continuous, for each open H in (Y,η), $f^{-1}(H) \in \tau$. If $\zeta \in o_F(f^{-1}(H))$, then $\zeta = 1$ or $f^{-1}(H) \notin \zeta$. Hence $f_*(1) = 1 \in o_F(H)$ or $H \notin f_*(\zeta)$. This implies $f_*(\zeta) \in o_F(H)$. Moreover if $f:(X, \tau) \to (Y, \eta)$ is injective, $f_* : o_F(f^{-1}(H)) \to o_F(H)$ is surjective. For $\delta(\neq 1) \in o_F(H)$, $\vartheta = \{f^{-1}(U)|U \in \delta\} \in o_F(f^{-1}(H))$. And $f_*(\vartheta) = \delta$. Hence we have

THEOREM 2.16. *Let $f:(X, \tau) \to (Y, \eta)$ be a continuous bijective map. Then the induce map $f_* : i_F(f^{-1}(H)) \to i_F(H)$ is bijective for each open H in (Y,η).*

THEOREM 2.17. *Let $\tau \leq \zeta$. Let $f:(X, \tau) \to (Y, \eta)$ be a continuous map. If we define a map $f_*:\tau(X) \to \tau(Y)$ by $f_*(w)=\{U \subset Y|f^{-1}(U) \in w\}$, then the map f_* is continuous and $f_*(\tau)\geq\eta$. Moreover, for any topology field ζ, the diagram*

$$
\begin{array}{ccc}
(X,\tau) & \xrightarrow{f} & (Y,\eta) \\
\downarrow \zeta_p^e & & \downarrow f_*(\zeta)_{f(p)}^e \\
(\tau(X), Out_\tau) & \xrightarrow{f_*} & (\tau(Y), Out_\eta) \\
\downarrow pr_p^e & & \downarrow pr_{f(p)}^e \\
(\tau_p(X), ROut_\tau) & \xrightarrow{f_*} & (\tau_{f(p)}(Y), ROut_\eta)
\end{array}
$$

commutes. If, furthermore, (Z, θ) is a topological space and $g: (Y, \eta) \to (Z, \theta)$ is a map, then

$$(g \circ f)_* = g_* \circ f_*.$$

Finally, if $f:(X, \tau) \to (X, \tau)$ is the identity homeomorphism, then so is f_*.

Proof. Let $\omega \in \tau(X)$. For any subbasic open neighborhood $o(G)$ of $f_*(\omega)$, $G \notin f_*(\omega)$. Thus $f^{-1}(G) \notin \omega$. Hence $\omega \in o(f^{-1}(G))$. So that $o(f^{-1}(G)$ is a open neighborhood of ω. Conversely if $\zeta \in o(f^{-1}(G))$, then $f^{-1}(G) \notin \zeta$. $G \notin f_*(\zeta)$. $f_*(\zeta) \in o(G)$. $\zeta \in f_*^{-1}(o(G))$. Consequently we have

$$f_*^{-1}(o(G)) = o(f^{-1}(G)).$$

This means f_* is continuous. And we can easily see that the diagram commutes. Moreover the other statements follow directly from the definitions. □

Using the same method in above theorem we can have

COROLLARY 2.18. *Let $\tau \leq \zeta$. Let $f:(X, \tau) \to (Y, \eta)$ be a continuous map. If we define a map $f_*^{\eta}:\tau(X) \to \tau(Y)$ by $f_*^{\eta}(w)=\{U \in \eta|f^{-1}(U) \in w\}$, then the map f_*^{η} is continuous and $f_*^{\eta}(\tau)=\eta$. Moreover, for any topology field ζ, the diagram*

$$(X,\tau) \qquad \xrightarrow{f} \qquad (Y,\eta)$$

$$\downarrow \zeta_p^e \qquad\qquad\qquad \downarrow f_*^{\eta}(\zeta)_{f(p)}^e$$

$$(\tau(X), Out_\tau) \quad \xrightarrow{f_*^{\eta}} \quad (\tau(Y), Out_\eta)$$

$$\downarrow pr_p^e \qquad\qquad\qquad \downarrow pr_{f(p)}^e$$

$$(\tau_p(X), ROut_\tau) \quad \xrightarrow{f_*^{\eta}} \quad (\tau_{f(p)}(Y), ROut_\eta)$$

commutes. If, furthermore, (Z, θ) is a topological space and $g: (Y, \eta) \to (Z, \theta)$ is a map, then

$$(g \circ f)_*^{\theta} = g_*^{\theta} \circ f_*^{\eta}.$$

Finally, if $f:(X ,\tau) \to (X, \tau)$ is the identity homeomorphism, then so is f_*.

Let $f:(X, \tau) \to (Y, \eta)$ be a continuous map. Then we have

$$f^*(\zeta \vee \eta) = \{U|f^{-1}(U) \in \zeta \vee \eta\} \geq \{U|f^{-1}(U) \in \zeta\} \vee \{U|f^{-1}(U) \in \eta\}$$

$$= f^*(\zeta) \vee f^*(\eta).$$

Thus if $\tau \leq \zeta$, then we have the following commutative diagram

$$(\tau(X), Out_\tau) \qquad \xrightarrow{f_*} \qquad (\tau(Y), Out_\eta)$$

$$\downarrow \vee_\zeta \qquad\qquad\qquad \downarrow \vee_{f^*(\zeta)}$$

$$(\tau(X), Out_\tau) \qquad \xrightarrow{f_*} \qquad (\tau)(Y), Out_\eta).$$

3. Topological flow (X, T, π) and topologies on X

A topological flow(or transformation group) is a triple (X, T, π), where X is a topological space(phase space), T is a topological group(acting group), and π is a continuous map from $X \times T$ to X satisfying

$(i) \pi(x, e) = x$ ($x \in X, e$ the identity of T)

$(ii) \pi(\pi(x, s), t) = \pi(x, st)$ ($x \in X, s, t \in T$).

Each $t \in T$ defines a continuous map π^t from X to X by $\pi^t(x) = \pi(x, t)$. If $t, s \in T$, then it is immediate that $\pi^s \pi^t = \pi^{ts}$; in particular, $\pi^t \pi^{t^{-1}} = \pi^e$, the identity map of X, so each π^t is a homeomorphism of X onto itself, with $(\pi^t)^{-1} = \pi^{t^{-1}}$. In general, we will identify an element $t \in T$ with the homeomorphism π^t of X. Thus T may be regard as a subgroup of the total homeomorphism group of X. Moreover, the topology of acting group is really not important. For this reason we will frequently assume that the acting group T has the discrete topology.

THEOREM 3.1. *If (X, T, π) is a flow with discrete acting group T, we have a flow $(\tau(X), T, \pi_*)$ with the same acting group T.*

Proof. Here we will prove the theorem only in the case of $\tau(X)$ which has outer topology In_τ induced by τ. The other case is due to the same method. Since each $t \in T$ defines a continuous map(homeomorphism) $\pi^t(= t)$ of X to X by $\pi^t(x)(= t(x)) = \pi(x, t) = xt$, we can get induced continuous map(homeomorphism) t_* of $\tau(X)$ to itself by theorem 2.8. Thus we can define a map π_*

$$\pi_* : \tau(X) \times T \longrightarrow \tau(X), \quad \pi_*(\zeta, t) = \zeta t = t_*(\zeta) = \{U \subset X | t^{-1}(U) \in \zeta\}.$$

Then this map π_* is well defined and continuous. Indeed, for a open subset G of X, $t^{-1}(G)$ is also open. Moreover, by theorem 2.8, we can get $\pi_*(i(t^{-1}(G)), t) \subset i(G)$. This implies that the map π_* is continuous. Finally we can easily find that $\pi_*(\zeta, e) = \zeta$ and $\pi_*(\pi_*(\zeta, s), t) = \pi_*(\zeta, st)$. $\square$

THEOREM 3.2. [7] *Let (X, T, π) be a flow with discrete acting group T and $(\tau(X), T, \pi_*)$ be the induced flow with the same acting group T. Then, between the projection maps $pr_{\pi(x,t)} = pr_{xt}$ and pr_x, the equation*

$$pr_{\pi(x,t)}(\pi_*(\zeta, t)) = t_*(\zeta)_{xt} = t_*(\zeta_x) = \pi_*(\zeta_x, t)$$

holds.

A *fixed point* of the flow (X, T, π) is a point x_0 such that $x_0 t = x_0$, for all $t \in T$. By the definition of $\pi_* : \tau(X) \times T \to \tau(X)$, we can have $\pi_*(\mathcal{P}(X), t) = t_*(\mathcal{P}(X)) = \mathcal{P}(X)$, and $\pi_*(\{\emptyset, X\}, t) = \{\emptyset, X\}$ for every $t \in T$.

THEOREM 3.3. *Let $(\tau(X), T, \pi_*)$ be the induced flow with the same acting group T by (X, T, π). Then the phase space topology τ on X is a fixed point of $(\tau(X), T, \pi_*)$.*

Proof. Since $\pi^t = t : (X, \tau) \to (X, \tau)$ is a homeomorphism for every $t \in T$, $\pi_*(\tau, t) = t_*(\tau) = \{U \subset X | t^{-1}(U) \in \tau\} = \tau$. This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In any mathematical system, one is interested in the maps which respect the structure of the system. The appropriate maps in topological dynamics are those which are continuous and equivariant. To be precise, let (X, T) and (Y, T) be flows with same acting group. A *homomorphism* from X to Y is a continuous map $\phi : X \longrightarrow Y$ such that

$$\phi(xt) = \phi(x)t \quad (x \in X, \;\; t \in T).$$

Hence from the above theorems in section 2 and definitions, we have

THEOREM 3.4. *Let (X, T, π) be a flow with discrete acting group T and $(\tau(X), T, \pi_*)$ be the induced flow with the same acting group T. Then the topology field $\zeta : (X, \tau) \longrightarrow (\tau(X), In_\tau)$ is a homomorphism.*

Proof. Since t is a bijective, we can easily have

$$\begin{aligned}
\zeta(xt) &= \{U | U \in \zeta, xt \in U\} \cup \{\emptyset\} \\
&= \{U | U \in \zeta, x \in t^{-1}(U)\} \cup \{\emptyset\} \\
&= \{U | x \in t^{-1}(U) \in \zeta\} \cup \{\emptyset\} \\
&= \{U | t^{-1}(U) \in \zeta_x\} \\
&= t_*(\zeta_x) = t_*(\zeta(x)) = \pi_*(\zeta(x), t) = \zeta(x)t.
\end{aligned}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$$

The point x and y of a flow X are said to be *proximal* [2] if there exists a net $\{t_i\}$ of elements of T such that $\lim xt_i = \lim yt_i$. The points x and y are *distal* if either $x = y$ or x and y are not proximal. A homomorphism $\phi : X \to Y$ is said to be *distal(proximal)* provided that whenever $\phi(x_1) = \phi(x_2)$, with $x_1 \neq x_2$, then x_1 and x_2 are distal(proximal) [2].

THEOREM 3.5. *If x and y are proximal, then $\eta(x)$ and $\eta(y)$ are also proximal. Hence every topology field preserves the proximity. Similarly we can have every co topology field preserves the proximity.*

Proof. We will prove theorem for case of the topology field. The other case follows with the same method. Let $G \in \tau$ be any open. Let $\lim \pi_*(\eta(x), t_i) = \lim t_{i*}(\eta_x) = \lim\{U \in \tau | t_i^{-1}(U) \in \eta_x\} \in i(G)$, where $\{t_i\}$ is a net of elements of T. Then for almost all i, $\{U \in \tau | t_i^{-1}(U) \in$

$\eta_x\} \in i(G)$. Hence for almost all i, $G \in \{U \in \tau | t_i^{-1}(U) \in \eta_x\}$. That is, for almost all i, $t_i^{-1}(G) \in \eta_x$. Consequently $x \in t_i^{-1}(G), t_i(x) \in G$ for almost all i. Since x and y are proximal, $t_i(y) \in G$ for almost all i. This implies $y \in t_i^{-1}(G)$ for almost all i. Hence $\lim \pi_*(\eta(y), t_i) \in i(G)$. $\qquad\square$

By virtue of equation in the theorem 3.2 we have the following corollary.

COROLLARY 3.6. *If η and ζ are proximal, then $\eta(x)$ and $\zeta(x)$ are also proximal. Hence every projection preserves the proximity. Similarly we can have every co projection preserves the proximity.*

Given a flow X, we can look at the family of closed and non-empty subsets of X. Each of these is called a subflow. We say that a flow is minimal if it has no proper subflow.

THEOREM 3.7. *Let (X, T) be a minimal flow. Let $\eta : X \to \tau(X)$ be a topology field. Then the induced flow $(\tau(X), T, \pi_*)$ has minimal sets $\overline{\eta_{x_0} T}$. Moreover if $\eta_{x_0} = \zeta_{x_0}$, the minimal sets $\overline{\eta(x_0) T}$ and $\overline{\zeta(x_0) T}$ are equal.*

Proof. Let $\eta \in \tau(X)$ be a topology field. Then $\overline{\eta_{x_0} T}$ is closed and non-vacuous. Moreover since t_* is a homeomorphism, $\pi_*(\overline{\eta_{x_0} T}, t) = t_*(\overline{\eta_{x_0} T}) = \overline{t_*(\eta_{x_0} T)} = \overline{(\eta_{x_0} T) t} = \overline{\eta_{x_0} (T t)} = \overline{\eta_{x_0} T}$. And we have $\overline{\eta_{x_0} T} \subset \overline{\eta T}$. In fact, if $\zeta \in \overline{\eta_{x_0} T}$, for any subbasic nbd $i(G)$ of ζ, there exists $t \in T$ such that $\pi_*(\eta_{x_0}, t) = t_*(\eta_{x_0}) \in i(G)$. Thus $t^{-1}(G) \in \eta_{x_0}$. Consequently $t^{-1}(G) \in \eta$. Hence $t_*(\eta) \in i(G)$. This implies $\zeta \in \overline{\eta T}$. Hence $\overline{\eta_{x_0} T}$ is minimal. Moreover if $\eta_{x_0} = \zeta_{x_0}$, then $\overline{\eta_{x_0} T} = \overline{\zeta_{x_0} T}$. $\qquad\square$

THEOREM 3.8. *Let (X, τ) be a distal minimal flow. And Assume that $(\tau(X), Out_\tau)$ is distal. If $\eta : X \to \tau(X)$ is a T_0 topology field, then it is a distal and proper homomorphism.*

Proof. First by Theorem 3.4, we see that η is a homomorphism. Let $x_1, x_2 \in X$, with $x_1 \neq x_2$ such that $\eta_{x_1} = \eta_{x_2}$. Since η is T_0, there are $G_1 \in \eta$ such that $x_1 \in G_1, x_2 \notin G_1$. This implies $G_1 \in \eta_{x_1}, G_1 \notin \eta_{x_2}$. This contradict to $\eta_{x_1} = \eta_{x_2}$. Hence there are no pairs of $x_1, x_2 \in X$, with $x_1 \neq x_2$ such that $\eta_{x_1} = \eta_{x_2}$. This means that every T_0 topology field η is distal. And since η is not onto, by virtue of Corollary in [1], we see that the topology field $\eta : X \to \tau(X)$ is proper. $\qquad\square$

4. Topology fields and topological organism

Let X be state space, U the input space and Y the output space. A deterministic dynamic system [8] is a complex

$$\mathcal{S} = \{X, U, Y, \delta, \gamma\}$$

where $\delta : X \times U \to X$ is the dynamics $\qquad x_{t+1} = \delta(x_t, u_t)$
and $\gamma : X \to Y$ is the output map. Here we will give a generalization of this system whose output maps are topology fields.

DEFINITION 4.1. Let I be index set and X_t state space at $t \in I$. Let U be an input space. Let $X = \coprod_t X_t = \{\coprod x_t | x_t \in X_t\}$ be the total space, disjoint sum of spaces X_t. Denote $\tau(X) = \coprod \tau(X_t) = \{\tau_t | \tau \in \tau(X_t)\}$. The total topology τ on the total space X is induced by $\tau(X)$

$$\tau = \{G \mid G = \coprod G_t | G_t \in \tau_t\}.$$

Hence we can get inner topology In_τ on $\tau(X)$ induced by τ like as in section 2. A topological organism(or topological organic system) is a complex

$$\mathcal{O}g = \{X, U, \tau(X), \delta, \gamma\}$$

where $\delta : X \times U \to X$ is a dynamics and γ is a topology field ("will be called output map") from (X, τ) to $(\tau(X).In_\tau)$.

Here $x_t \in (X_t, \tau_t)$ is the *local states* in the *topological states* (X_t, τ_t) *at time* t. In this organic system the current state and input determine the next state and output.

This definition is a generalization of the definition of system [8], considering not only the states in the topological spaces, but also the outputs as beings topologies. Hence we have called this system *topological organism(or topological organic system)*.

EXAMPLE 4.2. *Let* $I = R_0^+$ *be the set of nonnegative real numbers. Set* $X_t = \{(x, y) \in R^2 | x^2 + y^2 = t^2 , y \geq 0\}$ *be the subspace of Euclidean plane* R^2, $t \in R_0^+$. *Let* $X = \coprod_t X_t$ *and* $\tau(X) = \coprod \tau(X_t)$. *Then the total space may be represented by the sum of all half circles and point* $(0, 0)$ *in the upper plane* $\{(x, y) \in R^2 \mid y \geq 0\}$. *We give the total topology as above. Then the open set on total space can be represented by* $G \cap X = \coprod_t (G \cap X_t)$ *for some open* G *in* R^2.
Define $\delta : X \times R_0^+ \to X$ *by* $\delta((x, \sqrt{t^2 - x^2}), s) = (x, \sqrt{(t + s)^2 - x^2})$. *It slides point* $(x, \sqrt{t^2 - x^2}) \in X_t$ *to the point* $(x, \sqrt{(t + s)^2 - (x)^2}) \in X_{t+s}$ *vertically. Hence if* $s = 1$, *it slide the horizonal line* $\{(x, 0) | x \in R\}$ *to*

the curve $\{(x, \sqrt{2x+1})|x \in R\}$ *vertically. Moreover for a fixed state* $(0,0) \in X_0$, $\delta((0,0),s) = \{(0,s)\}$. *Hence the image* $\delta((0,0), R_0^+)$ *is the vertical half line* $\{(0,s)|s \geq 0\}$. *Finally if we define a natural topology field* $\gamma : (X,\tau) \to (\tau(X), Out_\tau)$ *by* $\gamma(x_t) = \tau_{x_t}$, *the we can get topological organism*

$$\mathcal{O}g = \{X, R_0^+, \tau(X), \delta, \gamma\}.$$

In this example, the input space R_0^+ is not a group and hence the dynamics is not a flow.

Considering a fixed input s, we can build maps

$$\delta_s : X \to X, \delta_s(x_t) = \delta(x_t, s),$$

$$\delta_{s*}^\tau : \tau(X) \to \tau(X), \delta_{s*}^\tau(\tau_t) = \{G \in \tau|\delta_s^{-1}(G) \in \tau_t\}$$

and will be called *input map* induced by input s and *induced map by input map*, respectively.

Similarly consider a fixed state $x_t \in (X_t, \tau_t)$, we can build maps

$$\bar{\delta}_t : U \to X, \quad \bar{\delta}_t(s) = \delta(x_t, s),$$

$$\bar{\gamma}_t : U \to \tau(X), \quad \bar{\gamma}_t(s) = \gamma_{x_{t+s}}$$

and will be called *input map* and *output map* induced by state x_t respectively.

Therefore, $\bar{\delta}_t(s)$ and $\bar{\gamma}_t(s)$ are computed by starting the system in state $x_t \in (X_t, \tau_t)$, feeding in the input sequence s and looking at the final state.

DEFINITION 4.3. The topological organism $\mathcal{O}_g$ is called totally reachable from the state $x_t \in (X_t, \tau_t)$ if the map $\bar{\delta}_t$ is onto, that is $\delta(x_t, U)$ $= X$. And the topological organism $\mathcal{O}_g$ is called almost reachable from the state $x_t \in (X_t, \tau_t)$ if $\overline{\delta(x_t, U)} = X$. And topological organism $\mathcal{O}_g$ is called globally reachable from the state (X_t, τ_t) if $\delta(X_t, U) = X$ and almost globally reachable from the state (X_t, τ_t) if $\overline{\delta(X_t, U)} = X$. Hence (almost) totally reachable organism is (almost) globally reachable.

Above concept of almost reachable is closely related to the concepts of almost periodic and minimal in the topological dynamics and is central concept of the topological dynamics. Let x be an almost periodic point of (X, R) where X is the plane and R is reals. Then the Poincare-Bendixon theorem state $\overline{xR}$ is either a point or a circle.

DEFINITION 4.4. The topological organism $\mathcal{O}_g$ is called observable if the map $\bar{\gamma}_t$ is one to one.

The topological organism defined in Example 1 is observable and not totally reachable from the one any state $x_t \in (X_t, \tau_t)$. Moreover it is not globally reachable from the any state (X_t, τ_t).

The next example tells us there exist almost globally reachable from the any state (X_t, τ_t).

EXAMPLE 4.5. *Let R be reals and $X_t = \{(e^{2\pi i t}, e^{2\pi i u}) | u \in R\}$ for $t \in R$. Let $X = \coprod_t X_t$ and $\tau(X) = \coprod \tau(X_t)$. Then the total space may be representant by torus $S^1 \times S^1$ in the Euclidean space R^3. We give the total topology as above. Then the open set on total space can be represented by $G \cap X = \coprod_t (G \cap X_t)$ for some open G in R^3. Consider rationals Q as input space. Define $\delta : X \times Q \to X$ by $\delta((e^{2\pi t}, e^{2\pi i u}), s) = (e^{2\pi (t+s)}, e^{2\pi i u})$. It slides point $(e^{2\pi i t}, e^{2\pi i u})) \in X_t$ to the point $(e^{2\pi (t+s)}, e^{2\pi i u}) \in X_{t+s}$ along the circle in the torus. Hence if $s = 1$, it slide the circle X_t to the circle X_{t+s} along the horizontal circle. Moreover for a fixed state $(0,0) \in X_0$, $\delta((0,0), s) = \{(e^{2\pi s}, 0)\}$. Hence the image $\delta((0,0), Q)$ is the dense subset $\{(e^{2\pi i s}, 0) | s \in Q\}$ of the horizontal circle $\{(e^{2\pi i r}, 0) | r \in R\}$. Moreover $\overline{\delta((X_t, \tau), Q)} = X$. Finally we can get a natural topology field $\gamma : (X, \tau) \to (\tau(X), In_\tau)$ by $\gamma(e^{2\pi t}, e^{2\pi i u}) = \tau_{(e^{2\pi t}, e^{2\pi i u})}$. Consequently we can have a topological organism*

$$\mathcal{O}g = \{X, Q, \tau(X), \delta, \gamma\}.$$

Thus this organism is almost globally reachable.

In this example, the input space Q is a additive group and hence the dynamics is a flow. We will give a totally reachable and globally reachable example.

EXAMPLE 4.6. *For the same index space and state spaces in the above Example, consider another input space $R \times R$. Define $\delta : X \times R \times R \to X$ by $\delta((e^{2\pi t}, e^{2\pi i u}), (s, w)) = (e^{2\pi (t+s)}, e^{2\pi i (u+w)})$. It changes any local state $(e^{2\pi i t}, e^{2\pi i u}) \in X_t$ to the local state $(e^{2\pi i (t+s)}, e^{2\pi i (u+w)})$ in X_{t+s} along the circular helix in the torus. Hence if $s = 1, w = 1$, it changes the local state $(e^{2\pi i t}, e^{2\pi i u}) \in X_t$ to the local state $(e^{2\pi (t+1)}, e^{2\pi i (u+1)})$ on the circle X_{t+s} along the circular helix. Moreover for a fixed local state $(0,0) \in X_0$, $\delta((0,0), (s, w)) = (e^{2\pi i s}, e^{2\pi i w})$. Hence the image $\delta((0,0), R \times R)$ is the space $S^1 \times S^1$. And if $s = 1, w = 0$, it changes the local state $(e^{2\pi i t}, e^{2\pi i u})) \in X_t$ to the local state $(e^{2\pi (t+1)}, e^{2\pi i u})$ on the circle X_{t+1} along the circle. Moreover for a fixed local state $(1,1) \in X_0$, $\delta((1,1), (s, 0)) = (e^{2\pi i s}, 1)$. Hence the image $\delta((1,1), R \times \{0\})$ is the circle $S^1 \times \{1\}$. Finally we get a topology field $\gamma : (X, \tau) \to (\tau(X), Out_\tau)$ by*

$\gamma(e^{2\pi t}, e^{2\pi iu}) = \tau_{(e^{2\pi t}, e^{2\pi iu})}$. *Thus we can get topological organism*

$$\mathcal{O}g = \{X, R \times R, \tau(X), \delta, \gamma\}.$$

Hence this topological organism is totally reachable and globally reachable example.

References

[1] J. Auslander, *Homomorphisms Of Minimal Transformation Groups*, Topology **9** (1970), 195-203

[2] J. Auslander, *Minimal Flows and Their Extensions*, North-Holland Mathematics Studies 153, 1988

[3] R. Ellis, *Lectures on Topological Dynamics*, W. A. Benjamin, Inc., 1969.

[4] D. H. Gottlieb, *Lectures on Vector Fields and the Unity of Mathematics*, Preprint.

[5] G. Grätzer, *General Lattice Theory*, Basel: Birkhäuser, 1978.

[6] J. Hartmanis , *On the lattice of topologies*, Canad. J. Math. **10** (1958), 547-553.

[7] J. R. Kim and M. H. Woo, *Topology Fields and Fixed Points of Flows,* Research and Exposition in Mathematics **27** (2003), 73-81.

[8] C. V. Negoita and D. A. Ralescu, *Applications of Fuzzy Sets to Systems Analysis*, A Halsted Press Book, 1975

[9] P. T. Johnstone, Stone spaces, Cambridge University Press, 1982.

[10] R. E. Larson and S. J. Andima, *The lattice of topologies: A survey*, Rockey mountain Jour. Math. **5** (1975), no. 2, Spring.

[11] A. K. Steiner, *The lattice of topologies: Structure and Complementation*, Trans. Amer. Math. Soc. **3** (1963), 167-171.

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