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# TRANSITIVE SETS WITH $C^1$-STABLY LIMIT SHADOWING

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Abstract. We show that a nontrivial transitive set is $C^1$-stably limit shadowable if and only if the transitive set is hyperbolic.

## 1. Introduction

Let $M$ be a closed $C^\infty$ manifold, and let Diff$(M)$ be the space of diffeomorphisms of $M$ endowed with the $C^1$-topology. Denote by $d$ the distance on $M$ induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle $TM$. Let $f \in$ Diff$(M)$. In the dynamical systems, the shadowing property is very useful notion. Actually, it deals with the stability theorem (see [4]). For instance, Sakai [6] showed that $f$ belongs to the $C^1$-interior of the set of a diffeomorphism having the shadowing property if and only if $f$ is structurally stable. In this paper, we deal with the another shadowing property, that is, the limit shadowing property which was introduced by Lee [1]. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b}(-\infty \le a < b \le \infty)$ in $M$ is called a *$\delta$-pseudo orbit* of $f$ if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$. Let $\Lambda \subset M$ be a closed $f$-invariant set. We say that $f$ has the *limit shadowing property on* $\Lambda$ (or $\Lambda$ *is limit shadowable* for $f$) if there exists a $\delta > 0$ with the following property: if a sequence $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ is a $\delta$-pseudo orbit of $f$ for which relations $d(f(x_i), x_{i+1}) \to 0$ as $i \to +\infty$, and $d(f^{-1}(x_{i+1}), x_i) \to 0$ as

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$i \to -\infty$ hold, then there is a point $y \in M$ such that $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. We say that $\Lambda$ is *locally maximal* if there is a neighborhood $U$ of $\Lambda$ such that $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$. We say that $\Lambda$ is *transitive* if there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$. We say that $\Lambda$ is *nontrivial* if $\Lambda$ is not just a periodic orbit. We say that $f$ has the *$C^1$-stably limit shadowing property* on $\Lambda$ (or $\Lambda$ *is the $C^1$-stably limit shadowable* for $f$) if there are a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ and a compact neighborhood $U$ of $\Lambda$ such that

(1) $\Lambda = \Lambda_f(U) = \bigcap_{n\in\mathbb{Z}} f^n(U)$ (locally maximal),
(2) for any $g \in \mathcal{U}(f)$, $g$ has the limit shadowing property on $\Lambda_g(U)$, where $\Lambda_g(U) = \bigcap_{\in\mathbb{Z}} g^n(U)$ is the continuation of $\Lambda = \Lambda_f(U)$.

We say that $\Lambda$ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a $Df$-invariant splitting $E^s \oplus E^u$ and there exists constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. Moreover, we say that $\Lambda$ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous $Df$-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. In this paper, we study the relations between the $C^1$-stably limit shadowing property and hyperbolic. For the above definition, in [2] the authors showed that $f$ has the $C^1$-stably shadowing property on the chain component $C_f(p)$ containing hyperbolic saddle $p$ if and only if $C_f(p)$ is hyperbolic. In this paper, we use the above definition on the nontrvial transitive set. The following is main theorem in this paper.

THEOREM 1.1. *Let $\Lambda$ be a nontrivial transitive set of $f \in \mathrm{Diff}(M)$. $f$ has the $C^1$-stably limit shadowing property on $\Lambda$ if and only if it is hyperbolic.*

In [1], Lee showed that if $\Lambda$ is hyperbolic then it is limit shadowable. And by the hyperbolicity, $f$ has the $C^1$-stably limit shadowing property. Thus in this paper we show that if $f$ has the $C^1$-stably limit shadowing property on transitive sets, then it is hyperbolic.

## 2. Proof of Theorem 1.1

Let $M$ be as before, and let $f \in \mathrm{Diff}(M)$. The following lemma is obtained by Pugh's closing lemma.

LEMMA 2.1. [5] *Let $\Lambda$ be a nontrivial transitive set. There exist a sequence of diffeomorphisms $\{g_n\}_{n\in\mathbb{N}}$ and periodic orbit $P_n$ of $g_n$ with period $\pi(P_n)$ as $n \to \infty$ such that $g_n \to f$ with $C^1$-topology and $\lim P_n = \Lambda$.*

From the above lemma, if $\Lambda$ is a locally maximal transitive set, then we can take a periodic point $p \in P(f)$ such that $\mathcal{O}_f(p) \subset U$, where $U$ is a compact neighborhood of $\Lambda$. Since $\Lambda$ is locally maximal in $U$, we know that $p \in \Lambda$.

REMARK 2.2. We know that

(a) Let $I$ be the unit interval. If $f : I \to I$ is an identity map, then $f$ does not have the limit shadowing property.
(b) Let $S^1$ be the unit circle. If $f : S^1 \to S^1$ is an irrational rotation then $f$ does not have the limit shadowing property.

The following so-called Franks' lemma will play essential roles in our proof.

LEMMA 2.3. *Let $\mathcal{U}(f)$ be any given $C^1$-neighborhood of $f$. Then there exist $\epsilon > 0$ and a $C^1$-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of $f$ such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \dots, x_N\}$, a neighborhood $U$ of $\{x_1, x_2, \dots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \le \epsilon$ for all $1 \le i \le N$, there exists $g' \in \mathcal{U}(f)$ such that $g'(x) = g(x)$ if $x \in \{x_1, x_2, \dots, x_N\} \cup (M \setminus U)$ and $D_{x_i}g' = L_i$ for all $1 \le i \le N$.*

LEMMA 2.4. *Let $f \in \mathrm{Diff}(M)$, and let $\Lambda$ be a compact $f$-invariant set. Suppose that $f$ has the $C^1$-stably limit shadowing property on $\Lambda$. Then there exists a $C^1$-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of $f$ such that for any $g \in \mathcal{U}_0(f)$, every $p \in \Lambda_g(U) \cap P(g)$ is hyperbolic for $g$, where $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$.*

*Proof.* Since $f$ has the $C^1$-stably limit shadowing property on $\Lambda$, there exist a compact neighborhood $U$ of $\Lambda$ and a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ such that for any $g \in \mathcal{U}(f)$, $g$ has the limit shadowing property on $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$. Let $\epsilon > 0$ and $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ be the corresponding number and $C^1$-neighborhood of $f$ given by Lemma 2.3 with respect to $\mathcal{U}(f)$. Then we obtained a $C^1$ neighborhood $\mathcal{U}_0(f)$. Suppose that there exists a non-hyperbolic periodic point $p \in \Lambda_g(U)$ for some $g \in \mathcal{U}_0(f)$. Note that since $\Lambda$ is locally maximal (reducing $\mathcal{U}_0(f)$ if necessary), we may assume that $q$ is contained in the interior of $U$. To simplify the notions, we may assume that $g(p) = p$. Then by making use of Lemma 2.3, we take a linear isomorphism $L : T_pM \to T_pM$ such that $\|L - D_pg\| < \epsilon$. We can choose $\alpha > 0$ with $B_{4\alpha}(p) \subset U$ and $g_1$ $C^1$-nearby $g$ such that

$$g_1(x) = \begin{cases} \exp_p \circ L \circ \exp_p^{-1} & \text{if } x \in B_\alpha(p), \\ g(x) & \text{if } x \notin B_{4\alpha}(p). \end{cases}$$

Clearly, $g_1(p) = g(p)$. Then if $\lambda$ is real then there is a $g_1$-invariant normally hyperbolic small arc $\mathcal{I}_p$ center at $p$ such that $g_1^k|_{\mathcal{I}_p} = id$ for some $k > 0$. And if $\lambda$ is complex then a $g_1$-invariant normally hyperbolic small circle $\mathcal{S}_p$ with a small diameter center at $p$ such that $g_1|_{\mathcal{S}}$ is conjugated to an irrational rotation map. Since $\mathcal{I}_p$ and $\mathcal{S}_p$ are $g_1$-invariant, we see that $\mathcal{I}_p \subset \Lambda_{g_1}(U)$ and $\mathcal{S}_p \subset \Lambda_{g_1}(U)$. Since $g_1$ has the limit shadowing property on $\Lambda_{g_1}(U)$, both $g_1^k|_{\mathcal{I}_p}$ and $g_1|_{\mathcal{S}_p}$ must have the limit shadowing property. Since $\mathcal{I}_p$ and $\mathcal{S}_p$ are $g_1$-invariant normally hyperbolic, the shadowing point belongs to $\mathcal{I}_p$ and $\mathcal{S}_p$. If not, then we show that a contradiction. Let $y \in M$ be a shadowing point. Since $y \notin \mathcal{I}_p$, by hyperbolicity there is $l \in \mathbb{Z}$ such that for any $\eta > 0$, $f^l(y) \notin B_\eta(\mathcal{I}_p)$. Moreover, if $l > 0$ then for all $i \geq 0$, $f^{l+i}(y) \notin B_\eta(\mathcal{I}_p)$. Therefore, $d(f^i(y), x_i) \not\to 0$ as $i \to +\infty$. This is a contradiction. Thus we know that the shadowing point belongs to $\mathcal{I}_p$. Similarly we get the same result for $\mathcal{S}_p$. By Remark 2.2, this is a contradiction. This completes the proof of Lemma 2.4. □

From Lemma 2.4, the family of periodic sequences of linear isomorphisms of $\mathbb{R}^{\mathrm{dimM}}$ generated by $Dg(g \in \mathcal{U}_0(f))$ along the hyperbolic periodic points $p \in \Lambda \cap P(g)$ is uniformly hyperbolic. That is, there exists $\epsilon > 0$ such that for any $g \in \mathcal{U}_0(f), p \in \Lambda \cap P(g)$, and any sequence of linear maps $L_i : T_{g^i(p)}M \to T_{g^{i+1}(p)}M$ with $\|L_i - D_{g^i(p)}g\| < \epsilon$ for $1 \leq i \leq \pi(p) - 1$, and $\prod_{i=0}^{\pi(p)-1} L_i$ is hyperbolic. Here $\mathcal{U}_0(f)$ is the $C^1$-neighborhood of $f$ given by Lemma 2.4 with respect to $\mathcal{U}(f)$. Thus by Proposition II.1 in [3] and the above Lemma 2.4, we get the following.

Proposition 2.5. *Suppose that $f$ has the $C^1$-stably limit shadowing property on $\Lambda$ and let $\mathcal{U}_0(f)$ as in the Lemma 2.4. Then there are constants $C > 0, \lambda \in (0, 1)$ and $m > 0$ such that*

(a) *for any* $g \in \mathcal{U}_0(f)$, *if* $p \in \Lambda \cap P(g)$ *has a minimum period* $\pi(p) \geq m$, *then*

$$\prod_{i=0}^{k-1} \|D_{g^{im}(p)} g^m|_{E^s_{g^{im}(p)}}\| < C\lambda^k \ \text{ and } \ \prod_{i=0}^{k-1} \|D_{g^{-im}(p)} g^{-m}|_{E^u_{g^{-im}(p)}}\| < C\lambda^k,$$

*where* $k = [\pi(p)/m]$.

(b) $\Lambda$ *admits a dominated splitting* $T_\Lambda M = E \oplus F$ *with* $\dim E = \mathrm{index}(p)$.

It is well known that if $p$ is a hyperbolic periodic point of $f$ with period $k$ then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \quad \text{as} \quad n \to \infty\} \quad \text{and}$$
$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \quad \text{as} \quad n \to \infty\}$$

are $C^1$-injectively immersed submanifolds of $M$. Let $q$ be a hyperbolic periodic point of $f$. We say that $p$ and $q$ are *homoclinically related*, and write $p \sim q$ if

$$W^s(p) \pitchfork W^u(q) \neq \emptyset \quad \text{and} \quad W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

It is clear that if $p \sim q$ then $\mathrm{index}(p) = \mathrm{index}(q)$; that is, $\dim W^s(p) = \dim W^s(q)$. Let $p$ be a hyperbolic periodic point of $f$, and let $\Lambda$ be a transitive set.

Proposition 2.6. *Suppose that $f$ has the $C^1$-stably limit shadowing property on $\Lambda$. Then for any hyperbolic point $q \in \Lambda \cap P(f)$,*

$$\mathrm{index}(p) = \mathrm{index}(q).$$

To prove Proposition 2.6, we need the following lemma.

Lemma 2.7. *Let $\Lambda$ be a transitive set of $f$. Suppose that $f$ has the limit shadowing property on $\Lambda$. Then for any $p, q \in \Lambda \cap P_h(f)$,*

$$W^s(p) \cap W^u(q) \neq \emptyset \ \text{ and } \ W^u(p) \cap W^s(q) \neq \emptyset,$$

*where $P_h(f)$ is the set of hyperbolic periodic points of $f$.*

*Proof.* In this proof, we will show that $W^u(p) \cap W^s(q) \neq \emptyset$. The other case is similar. Let $p, q$ be two hyperbolic periodic points of $f$ in $\Lambda$ and let $\epsilon(p) > 0$ and $\epsilon(q) > 0$ be as before with respect to $p$ and $q$. Fix $\epsilon = \min\{\epsilon(p), \epsilon(q)\}$. To simplify the notions in the proof, we assume that $f(p) = p$ and $f(q) = q$. Let $0 < \delta < \epsilon/2$ be the number of the limit shadowing property of $f|_\Lambda$. Since $\Lambda$ is a transitive set, there is $x \in \Lambda$ such that $\omega(x) = \Lambda$. For the above $\delta > 0$, we can choose $l_1 > 0$ and $l_2 > 0$ such that $d(f^{l_1}(x), p) < \delta/2$ and $d(f^{l_2}(x), q) < \delta/2$. We may assume that $l_2 > l_1 > 0$. Then we can construct a $\delta$-limit pseudo orbit of $f$ as follows: (i) $f^i(p) = x_i$ for $i \geq 0$, (ii) $f^{l_1+i}(x) = x_i$ for $0 \leq i \leq l_2 - 1$, and (iii) $f^i(q) = x_i$ for $l_2 \leq i$. Then we obtained a $\delta$-limit pseudo orbit of $f$,

$$\xi = \{\dots, p, p, x_1, \dots, x_{l_2-1}, q, q, \dots\}.$$

Since $f$ has the limit shadowing property on $\Lambda$, we can choose a point $y \in M$ such that $d(f^n(y), x_n) \to 0$ as $n \to \pm\infty$. Then we choose $n_1 > 0$ sufficiently large such that $f^{-n}(y) \in W^u_\epsilon(p)$ and $f^n(y) \in W^s_\epsilon(q)$, for all $n \geq n_1$. Therefore, $y \in f^n(W^u_\epsilon(p))$ and $y \in f^{-n}(W^s_\epsilon(q))$. Thus $y \in W^u(p) \cap W^s(q)$. Consequently, one can get $W^u(p) \cap W^s(q) \neq \emptyset$. $\square$

If $p \in P(f)$ is hyperbolic, then for any $g \in \text{Diff}(M)$ $C^1$-nearby $f$, there exists a unique hyperbolic periodic point $p_g \in P(g)$ nearby $p$ such that $\pi(p_g) = \pi(p)$ and $\text{index}(p_g) = \text{index}(p)$. Such a $p_g$ is called the *continuation of* $p$. It is well known that a dominated splitting always extends to a neighborhood. More precisely, let $\Lambda$ be a closed $f \in \text{Diff}(M)$-invariant set. Then if $\Lambda$ admits a dominated splitting $T_\Lambda M = E \oplus F$ such that $\dim E_x (x \in \Lambda)$ is constant, then there are a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ and a compact neighborhood $U$ of $\Lambda$ such that for any $g \in \mathcal{U}(f), \bigcap_{n\in\mathbb{Z}} g^n(U)$ admits a dominated splitting

$$T_{\bigcap_{n\in\mathbb{Z}} g^n(U)} M = E'(g) \oplus F'(g)$$

with $\dim E'(g) = \dim E$.

We say that $f$ is *Kupka-Smale* if every periodic point is hyperbolic and for any $p, q \in P(f)$, $W^s(p)$ and $W^u(q)$ are transverse and $W^u(p)$ and $W^s(q)$ intersect transversally. Note that the Kupka-Smale diffeomorphism is a residual subset of $\text{Diff}(M)$.

*Proof of Proposition 2.6.* Suppose $f$ has the $C^1$-stably limit shadowing property on $\Lambda$. Then there is a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ such that for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_g(U)}$ has the limit shadowing property. Assume that Proposition 2.6 is not true. Then there is $p, q \in \Lambda \cap P(f)$ such that $\text{index}(p) \neq \text{index}(q)$. Thus we know that $\dim W^u(q) + \dim W^s(p) < \dim M$ or $\dim W^s(q) + \dim W^u(p) < \dim M$. Without loss of generality, we may assume that $\dim W^s(p) + \dim W^u(q) < \dim M$. Since $f$ has the $C^1$-stably limit shadowing property on $\Lambda$, we take a Kupka-Smale diffeomorphism $g \in \mathcal{U}(f)$. Then $g$ has the limit shadowing property on $\Lambda_g(U)$ and $p_g, q_g \in \Lambda_g(U)$, where $p_g$ and $q_g$ are the continuation of $p$ and $q$, respectively. One can see that $\dim W^s(p_g) = \dim W^s(p)$ and $\dim W^u(q_g) = \dim W^u(q)$. Since $g$ is Kupka-Smale, $W^s(p_g) \cap W^u(q_g) = \emptyset$.

This is a contradiction by Lemma 2.7. □

Let us recall Mañé's ergodic closing lemma in [3]. For any $\epsilon > 0$, let $B_\epsilon(f, x)$ be an $\epsilon$-tubular neighborhood of $f$-orbit of $x$, that is, $B_\epsilon(f, x) = \{y \in M : d(f^n(x), y) < \epsilon$ for some $n \in \mathbb{Z}\}$. Let $\Sigma_f$ be the set of points $x \in M$ such that for any $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ and $\epsilon > 0$, there are $g \in \mathcal{U}(f)$ and $y \in P(g)$ satisfying $g = f$ on $M \setminus B_\epsilon(f, x)$ and $d(f^i(x), g^i(y)) \leq \epsilon$ for $0 \leq i \leq \pi(y)$.

Remark 2.8. ([3, Theorem A]). For any $f$-invariant probability measure $\mu$, we have $\mu(\Sigma_f) = 1$.

By Lemma 2.4 and Pugh's closing Lemma, we see that $\overline{P(f) \cap \Lambda} = \Lambda$.

*Proof of Theorem 1.1.* Let $\mathcal{U}_0(f)$ be the $C^1$-neighborhood of $f$ given by Proposition 2.5. To get the conclusion, it is sufficient to show that $\Lambda_i(f)$ is hyperbolic, where $\Lambda_i(f) = \overline{P_i(f|_\Lambda)}$, and $i$ is an index of $\Lambda$. Fix any neighborhood $U_i \subset U$ of $\Lambda_i(f)$. Note that by Proposition 2.6, $\Lambda_j(f) = \overline{P_j(f|_\Lambda)} = \emptyset$ if $i \neq j$. Thus we show that the following: let $\mathcal{V}(f) \subset \mathcal{U}_0(f)$ be a small connected $C^1$-neighborhood of $f$. If any $g \in \mathcal{V}(f)$ satisfies $q = f$ on $M \setminus U_i$, then index $(p)$ = index $(q)$ for any $p, q \in \Lambda_g(U) \cap P(g)$. Indeed, suppose not, then there are $g_1 \in \mathcal{V}(f)$ and $q \in \Lambda_g(U) \cap P(g_1)$ such that $g_1 = f$ on $M \setminus U_i$ and index $(p) \neq$ index $(q)$. Suppose that $g_1^n(q) = q, k = \text{index}(q)$, and define $\gamma : \mathcal{V}(f) \to \mathbb{Z}$ by

$$\gamma(g) = \sharp\{y \in \Lambda_g(U) \cap P(g) : g^n(y) = y \quad \text{and} \quad \text{index}(y) = k\}.$$

By Lemma 2.4, the function $\gamma$ is continuous, and since $\mathcal{V}(f)$ is connected, it is constant. But the property of $g_1$ implies $\gamma(g_1) > \gamma(f)$. This is a contradiction. We will finish the proof of Theorem 1.1. Then we use the proof of Theorem B in [3]. Thus we show that

$$\liminf_{n\to\infty} \|D_x f^n|_{E_x}\| = 0 \text{ and } \liminf_{n\to\infty} \|D_x f^{-n}|_{F_x}\| = 0$$

for all $x \in \Lambda$, and thus the splitting is hyperbolic. More precisely, we will prove the case of $\liminf_{n\to\infty} \|D_x f^n_{|E}\| = 0$ (the other case is similar). We will derive a contraction. If it is not true, then there is $x \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \|Df^m|_{E_{f^{mj}(x)}}\| \geq 1$$

for all $n \geq 0$. Thus

$$\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df^m|_{E_{f^{mj}(x)}}\| \geq 0$$

for all $n \geq 0$. The proof is similar to end of the proof of Theorem 1.3 (see [2]). Then we know that $\liminf_{n\to\infty}\|D_x f^n|_{E_x}\| = 0$ for all $x \in \Lambda$. Thus $\Lambda$ is hyperbolic. This completes the proof of the "only if" part of Theorem 1.1. □

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