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# Remark for Certain Elliptic PDE with Exponential Nonlinearity in a Bounded Domain

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Abstract

In this note, we are concerned with a class of semi-linear elliptic pdes with exponential nonlinearity in a bounded domain. Here, the nonlinearity is more or less growing exponentially with power $p$. We consider the problem under two types of Dirichlet boundary condition. We give existence and non-existence of solutions for those problems and some asymptotics.

**Key words**: Semi-linear PDE, Exponential nonlinearity, Existence

## 1. Introduction

Nonlinear elliptic pdes with exponential nonlinearity arises frequently in physics and mathematics such as mean field approximations in statistical physics, static solitons in (2+1)D gauge models, prescirbed Gaussian curvature problems in 2 dimensional surface, etc. We here consider elliptic pdes having exponential nonlinearity in a bounded domain $\Omega$ with Dirichlet type boundary conditions. Specifically, we consider

$$\Delta u = f_1(x, e^u) + f_0(x) + 4\pi \sum_i n_i \delta_{x_i}, \quad x \in \Omega \tag{1}$$

Here, $\Omega$ is a smooth bounded domain in the two dimensional Euclidean space, $f_1(x, t)$ is a measurable function continuous at 0 with respoect to $t$, $f_1(x, 0)=0$, $f_0 \in L^q$, $q > 1$, $n_i$ is positive, and $x_1 \in \Omega$.

Motivated from the Chern-Simons equations[1,2],

$$\Delta u = e^{2u} - e^u + 4\pi \sum_i n_i \delta_{x_i},$$

we restrict ourself $f_1$ satisfying

$$A(x)(t^p+1) > f_1(x,t) > -A(x) \tag{2}$$

for some positive $A \in L^q$, $q > 1$ and $p > 1$.

Examples of such $f$ are $f(t) = t^p - t$, $t \ln t$.

We consider two types of Dirichlet boundary conditions for (1) again motivated from the Chern-Simons model;

$u \to B$ as $x \to \partial\Omega$ (Topological)
$u \to \infty$ as $x \to \partial\Omega$ (Nontopological).

In this note, we shall show that (1) has a solution in the Sobolev space $H^1$ under the topological boundary condition and does not have a solution under the non-topological condition.

## 2. Existence and Nonexistence

Theorem 1
Let $f$ satisfy (2). Then, there exists a solution in $H^1$ of (1) under topological boundary condition.

proof)
We first consider a function satisfying

$$\Delta K = f_0 + \sum_i 4\pi n_i \delta_{x_i}, \quad K|_{\partial\Omega} = B.$$

Existence of such $K$ is guaranteed since we can decompose the above equation as follows:

$$K = K_1 + K_2, \; K_1 = \sum_i n_i \ln|x - x_i|^2,$$

$$\Delta K_2 = f_0, \; K_2|_{\partial\Omega} = -K_1 + B$$

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Since $K_1$ is Holder continuous away from $x_1$, $K_1$ becomes a Holder continuous function near the boundary of Ω. Hence, a Holder continuous $K_2$ exists by the standard elliptic theory.

It is noted that $e^K < C$ and $K \in L^q$, $\forall q > 1$ since $K = \sum_i n_i \ln|x - x_i|^2 + K_2$.

Let $v = u - K$, then it is equivalent to show the existence of a solution of the following equation.

$$\Delta v = f_1(x, e^K e^v),\ v = 0,\ x \in \partial\Omega \tag{3}$$

Let $f_2(x, t) = \frac{f_1(x,t)}{t}$ and $F(x,t) = \int_1^t f_2(x,t)dt$. The variational formulation of (3) becomes then $E(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 + \int_\Omega F(x, e^K e^v)$. The first condition of (2) implies that $F \le \int_1^{e^K e^v} A(t^{p-1} + t^{-1}) \le CA(e^{p(K+v)} + K + v + C)$. Therefore, $E$ is well defined in $H^1$ since $v.\ e^v \in L^q$, $\forall q > 1$ Next, by the second of (2),

$$F(x, e^K e^v) \ge -\int_1^{e^K e^v} \frac{A}{t} dt \ge -A(K+v) \tag{4}$$

$$\int_\Omega F \ge -\|A\|_{L^q}(\|v\|_{L^{q/(q-1)}} + \|K\|_{L^{q/(q-1)}}) \ge C\|\nabla v\|_{L^2} - C \tag{5}$$

Therefore, $E$ becomes strongly coercive and a minimizing sequence converges weakly in $H^1$. By the first of (2) and the Trudinger embedding, $\int_\Omega F$ is compact in $H^1$ and we have a minimizer.

Theorem 2.

Let $f$ satisfy (2). Then, there does not exist a solution in $H^1$ of (1) under the nontopological boundary condition.

proof)

If $v$ is a solution of (1) in $H^1$, by standard bootstrap argument and (2), we get $v$ is Holder continuous.

Therefore, by the nontopological boundary condition, $v$ is bounded from above and thus $e^v$ is bounded. This in turn implies that

$$|\Delta v| = |f_1(x, e^u)| \le A(x)(1 + e^{pu}) \le CA \in L^q.$$

Let $v = v_1 + v_2$, $v_1 = N * \Delta v$, $\Delta v_2 = 0$. Here, $N$ is the Newtonian potential. Then, $v_1$ is uniformly bounded and thus $v_2$ must be a harmonic function which tends to negative infinity near the boundary of Ω. This is impossible due to the maximum principle and we finish the proof.

If we assume differentiability of $f_1$ near $t = 0$, we can get certain compact of solutions under the topological condition. Concretely, we have the following theorem.

Theorem 3.

If the latter condition in (2) is replaced with $f_1 \ge -\frac{At}{1+t}$, the $H^1$ norm of $v$ under topological condition is uniformly bounded. And, $u \to -\infty$ uniformly as $B \to -\infty$.

proof)

Clearly, $\frac{At}{1+t} \le A$. Thus the proof of theorem 1, holds true. Further, (4) becomes $F(x, e^K e^v) \ge -\int_1^{e^K e^v} \frac{A}{1+t} dt \ge -A(\ln(1 + e^{K+v}) + 1) \ge -A(v^+ + \ln(2 + e^K))$. Here, we use the inequality $\ln(1 + \varepsilon e^v) \le v^+ + \ln(1 + \varepsilon)$. Then, (5) becomes $\int_\Omega F \ge -C(\||v| + \ln(2 + e^K)\|_{L^{q/(q-1)}}) \ge -C\|v\|_{L^{q/(q-1)}} - C \ge -C\|\nabla v\|_{L^2} - C$ unoformly if $B \le 0$. Thus, we have a uniform bound of $\|\nabla v\|_{L^2}$ as $B \to -\infty$.

This leads us that $u = v + K \equiv v + \tilde{K} + B$ diverges uniformly.

## References

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