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WEAK EXTENDED ORDER ALGEBRAS HAVING ADJOINT TRIPLES

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Abstract. We study the properties of weak extended order algebras having adjoint pairs (triples) or Galois pairs. In particular, we investigate the various laws on weak extended order algebras.

1. Introduction

Wille [9] introduced the structures on lattices which are important mathematical tools for data analysis and knowledge processing. MV-algebra was introduced by Chang [2] to provide algebraic models for many valued propositional logic. Recently, it is developed in many directions (BL-algebra, residuated algebra) [1,3,4,7,9]. Recently, Guido et al. [5] introduced extended order algebras as the generalization of residuated algebras. Kim and Ko [6] introduced the properties of weak extended order algebras. In particular, we investigate the properties of commutative and associative extended-order algebras.

In this paper, we study the properties of weak extended order algebras having adjoint pairs (triples) or Galois pairs. In particular, we investigate the various laws on weak extended order algebras.

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2. Preliminaries

Definition 2.1. [5,6] Let $(L, \wedge, \vee)$ be a lattice. A triple $(L, \Rightarrow, \top)$ is called a *weak extended order algebra* (shortly, w-eo algebra) iff it satisfies the following properties:

(O1) $a \Rightarrow \top = \top$ (upper bounded condition)

(O2) $a \Rightarrow a = \top$ (reflexive condition)

(O3) if $a \Rightarrow b = \top$ and $b \Rightarrow a = \top$, then $a = b$

(O4) if $a \Rightarrow b = \top$ and $b \Rightarrow c = \top$, then $a \Rightarrow c = \top$.

A triple $(L, \Rightarrow, \top)$ is called a *right w-eo algebra* if it satisfies (O1), (O2), (O3) and

(O5) if $a \Rightarrow b = \top$, then $(c \Rightarrow a) \Rightarrow (c \Rightarrow b) = \top$.

A triple $(L, \Rightarrow, \top)$ is called a *left w-eo algebra* if it satisfies (O1), (O2), (O3) and

(O6) if $a \Rightarrow b = \top$, then $(b \Rightarrow c) \Rightarrow (a \Rightarrow c) = \top$.

A w-eo algebra is called a *right distributive w-eo algebra* if

(O7) $a \Rightarrow \bigwedge_i b_i = \bigwedge_i (a \Rightarrow b_i)$.

A w-eo algebra is called a *left distributive w-eo algebra* if

(O8) $\bigvee_i a_i \Rightarrow b = \bigwedge_i (a_i \Rightarrow b)$.

(1) A w-eo algebra has an *adjoint pair* $(\Rightarrow, \odot)$ if there exists a binary operation $\odot$ such that

$$a \odot b \leq c \text{ iff } b \leq a \Rightarrow c.$$

(2) A w-eo algebra has a *Galois pair* $(\Rightarrow, \rightarrow)$ if there exists a binary operation $\rightarrow$ such that

$$b \leq a \Rightarrow c \ \text{ iff } a \leq b \rightarrow c.$$

(3) A w-eo algebra has *symmetrical* if it has a Galois pair $(\Rightarrow, \rightarrow)$ and $(L, \rightarrow, \top)$ is a w-eo algebra.

(4) A w-eo algebra has an *adjoint triple* $(\Rightarrow, \odot, \rightarrow)$ if there exists binary operation $\odot$ and $\rightarrow$ such that $a \odot b \leq c$ iff $b \leq a \Rightarrow c$ iff $a \leq b \rightarrow c$.

(5) A w-eo algebra is called a *w-ceo algebra* if L is complete.

Theorem 2.2. [6] (1) *If $(L, \Rightarrow, \top)$ is a right w-eo algebra, then it is a w-eo algebra.*

(2) *If $(L, \Rightarrow, \top)$ is a left w-eo algebra, then it is a w-eo algebra.*

Theorem 2.3. [5] *Let $(L, \Rightarrow, \top)$ be a right-distributive w-ceo algebra and $\odot$ be defined by*

$$a \odot x = \bigwedge \{ y \in L \mid x \leq a \Rightarrow y \}.$$

Then $\odot$ *and* $\Rightarrow$ *form an adjoint pair, i.e.*

$$x \odot y \leq z \text{ iff } y \leq x \Rightarrow z.$$

THEOREM 2.4. [5] *Let* $(L, \Rightarrow, \top)$ *be a left-distributive w-ceo algebra and* $\to$ *be defined by*

$$g_a(y) = y \to a = \bigvee \{x \in L \mid y \leq x \Rightarrow a\}.$$

Then $(\Rightarrow, \to)$ *forms an adjoint pair, i.e.*

$$y \leq x \Rightarrow a \text{ iff } x \leq y \to a.$$

THEOREM 2.5. [6] *Let* $(L, \Rightarrow, \top)$ *be a w-eo algebra having an adjoint pair* $(\Rightarrow, \odot)$ *and* $\Rightarrow$ *be defined by*

$$a \Rightarrow b = \top \text{ iff } a \leq b$$

For each $a, b, c, a_i, b_i \in L$*, the following properties hold.*

(1) $a \odot b \leq a$ *and* $a \odot (a \Rightarrow b) \leq b \leq a \Rightarrow a \odot b$*;*

(2) $a \odot \top = a$*;*

(3) $a \odot \bot = \bot \odot a = \bot$*;*

(4) *If* $b \leq c$*, then* $a \odot b \leq a \odot c$*.*

(5) $(L, \Rightarrow, \top)$ *be a right w-eo algebra.*

(6) *If* $(L, \Rightarrow, \top)$ *is a complete lattice,* $(L, \Rightarrow, \top)$ *is a right distributive w-eo algebra and* $a \odot (\bigvee_{i\in\Gamma} b_i) = \bigvee_{i\in\Gamma}(a \odot b_i)$*.*

(7) $(L, \Rightarrow, \top)$ *is a left w-eo algebra iff* $a \odot c \leq b \odot c$ *for* $a \leq b$*.*

(8) *If* $(L, \Rightarrow, \top)$ *is a left w-eo algebra, then* $(L, \Rightarrow, \top)$ *be a right w-eo algebra.*

(9) *If* $(L, \Rightarrow, \top)$ *is a left w-eo algebra and* $a \odot (b \odot c) = (a \odot b) \odot c$*, then*

$$(a \Rightarrow b) \Rightarrow ((c \Rightarrow a) \Rightarrow (c \Rightarrow b)\Big) = \top.$$

(10) *If* $(L, \Rightarrow, \top)$ *is a complete and left w-eo algebra, then* $(L, \Rightarrow, \top)$ *is a left distributive w-eo algebra and* $(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma}(a_i \odot b)$*.*

(11) *If* $\top \Rightarrow a = a$*, then* $\top \odot a = a$*,* $a = \bigwedge_b (b \Rightarrow (b \odot a))$ *and* $a = \bigvee_b (b \odot (b \Rightarrow a))$*.*

(12) *If* $(L, \Rightarrow, \top)$ *is a left w-eo algebra and* $\top \Rightarrow a = a$*, then* $a \odot b \leq b$ *and* $a \Rightarrow (b \Rightarrow a) = \top$*.*

THEOREM 2.6. [6] *Let* $(L, \Rightarrow, \top)$ *be a w-eo algebra having a Galois pair* $(\Rightarrow, \to)$*. For each* $a, b, c \in L$*, the following properties hold.*

(1) $a \leq (a \to b) \Rightarrow b$ *and* $a \leq (a \Rightarrow b) \to b$*.*

(2) $\top \to a = a$. *Furthermore, if* $(L, \Rightarrow, \to, \top)$ *is a symmetrical w-eo algebra, then* $\top \Rightarrow a = a$.

(3) If $\top \Rightarrow a = a$, *then* $a \to b = \top$ *iff* $a \le b$.

(4) $(L, \Rightarrow, \top)$ *is a left w-eo algebra. If* $\top \Rightarrow a = a$, $(L, \to, \top)$ *is a left w-eo algebra.*

(5) $(L, \Rightarrow, \top)$ *is a right w-eo algebra iff* $(L, \to, \top)$ *is a right w-eo algebra.*

(6) $a = \bigwedge_b((a \Rightarrow b) \to b)$. *If* $\top \Rightarrow a = a$, *then* $a = \bigwedge_b((a \to b) \Rightarrow b)$.

(7) If $(L, \Rightarrow, \top)$ *is a complete lattice,* $\bigvee_i a_i \Rightarrow b = \bigwedge_i (a_i \Rightarrow b)$ *and* $\bigvee_i a_i \to b = \bigwedge_i (a_i \to b)$.

THEOREM 2.7. [6] *Let* $(L, \Rightarrow, \top)$ *be a w-eo algebra having adjoint triple with* $\odot$ *and* $\to$. *Then the following properties hold.*

(1) $(L, \Rightarrow, \top)$ *is an eo algebra;*

(2) If $\top \Rightarrow a = a$, *then* $(L, \to, \top)$ *is an eo algebra;*

(3) If $a \le b$, *then* $a \odot c \le b \odot c$ *and* $c \odot a \le c \odot b$.

(4) If $\top \Rightarrow a = a$, *then* $\top \odot a = a$.

DEFINITION 2.8. [5] A w-eo algebra $(L, \Rightarrow, \top)$ is commutative iff it satisfies

$$a \Rightarrow (b \Rightarrow c) = \top \text{ iff } b \Rightarrow (a \Rightarrow c) = \top$$

THEOREM 2.9. [6] *Let* $(L, \Rightarrow, \top)$ *be a w-eo algebra having adjoint triple with* $\odot$ *and* $\to$. *Then the following statements are equivalent:*

(1) $(L, \Rightarrow, \top)$ *is commutative;*

(2) $(L, \odot, \top)$ *is commutative;*

(3) $(L, \to, \top)$ *is commutative eo algebra with* $\Rightarrow = \to$.

(4) $a \to b \le a \Rightarrow b$ *for all* $a, b \in L$.

(5) $a \Rightarrow b \le a \to b$ *for all* $a, b \in L$.

(6) $a \le (a \Rightarrow b) \Rightarrow b$ *for all* $a, b \in L$.

(7) $a \le (a \to b) \to b$ *for all* $a, b \in L$.

(8) $(a \Rightarrow b) \odot a \le b$ *for all* $a, b \in L$.

(9) $a \odot (a \to b) \le b$ *for all* $a, b \in L$.

(10) $b \le a \Rightarrow b \odot a$ *for all* $a, b \in L$.

(11) $a \le b \to b \odot a$ *for all* $a, b \in L$.

3. Weak extended order algebras having adjoint triples

THEOREM 3.1. *Let $(L, \Rightarrow, \top)$ be a w-eo algebra having adjoint triple with $\odot$ and $\to$. Then the following statements are equivalent:*
(1) $a \Rightarrow (b \Rightarrow c) = b \Rightarrow (a \Rightarrow c)$ *for all* $a, b, c \in L$.
(2) $b \to c \leq (a \Rightarrow b) \to (a \Rightarrow c)$ *for all* $a, b, c \in L$.
(3) $a \odot (b \odot c) = b \odot (a \odot c)$ *for all* $a, b, c \in L$.
(4) $b \to c \leq (a \odot b) \to (a \odot c)$ *for all* $a, b, c \in L$.
(5) $b \to (a \Rightarrow c) = (a \odot b) \to c$ *for all* $a, b, c \in L$.

Proof. (1)$\Leftrightarrow$ (2) Since $a \Rightarrow (b \Rightarrow c) \leq b \Rightarrow (a \Rightarrow c)$, then $b \leq (a \Rightarrow (b \Rightarrow c)) \to (a \Rightarrow c)$. Put $b = b \to c$. Since $b \leq (b \to c) \Rightarrow c$, then

$$\begin{aligned} b \to c &\leq (a \Rightarrow ((b \to c) \Rightarrow c)) \to (a \Rightarrow c) \\ &\leq (a \Rightarrow b) \to (a \Rightarrow c). \end{aligned}$$

Conversely, since $b \leq (b \Rightarrow c) \to c \leq (a \Rightarrow (b \Rightarrow c)) \to (a \Rightarrow c)$, then $a \Rightarrow (b \Rightarrow c) \leq b \Rightarrow (a \Rightarrow c)$.

(1)$\Leftrightarrow$ (3) Since $a \Rightarrow (b \Rightarrow a \odot (b \odot c)) = b \Rightarrow (a \Rightarrow a \odot (b \odot c)) \geq b \Rightarrow (b \odot c) \geq c$, then

$$\begin{aligned} a \Rightarrow (b \Rightarrow a \odot (b \odot c)) \geq c \ \text{ iff } & b \Rightarrow a \odot (b \odot c) \geq a \odot c \\ \text{iff } & a \odot (b \odot c) \geq b \odot (a \odot c). \end{aligned}$$

Similarly, $a \odot (b \odot c) \leq b \odot (a \odot c)$.

Conversely, since $a \odot (b \odot (a \Rightarrow (b \Rightarrow c))) = b \odot (a \odot (a \Rightarrow (b \Rightarrow c))) \leq b \odot (b \Rightarrow c) \leq c$, we have

$$\begin{aligned} a \odot (b \odot (a \Rightarrow (b \Rightarrow c))) \leq c \ \text{ iff } & b \odot (a \Rightarrow (b \Rightarrow c)) \leq a \Rightarrow c \\ \text{iff } & a \Rightarrow (b \Rightarrow c) \leq b \Rightarrow (a \Rightarrow c) \end{aligned}$$

Similarly, $a \Rightarrow (b \Rightarrow c) \geq b \Rightarrow (a \Rightarrow c)$.

(3)$\Leftrightarrow$ (4)

$$\begin{aligned} &(b \to c) \odot (a \odot b) = a \odot ((b \to c) \odot b\Big) \leq a \odot c \\ &\text{iff } b \to c \leq (a \odot b) \to (a \odot c.) \end{aligned}$$

$$\begin{aligned} &b \leq c \to b \odot c \leq (a \odot c) \to a \odot (b \odot c) \\ &\text{iff } b \odot (a \odot c) \leq a \odot (b \odot c). \end{aligned}$$

(3)$\Leftrightarrow$ (5) By (4), put $c = a \Rightarrow c$.

$$b \to (a \Rightarrow c) \leq (a \odot b) \to a \odot (a \Rightarrow c) \leq a \odot b \to c.$$

$$\begin{aligned} &(a \odot b \to c) \odot (a \odot b) \leq c \\ &\text{iff } a \odot ((a \odot b \to c) \odot b) \leq c \\ &\text{iff } (a \odot b \to c) \odot b \leq a \Rightarrow c \\ &\text{iff } a \odot b \to c \leq b \to (a \Rightarrow c). \end{aligned}$$

Hence $a \odot b \to c \leq b \to (a \Rightarrow c)$.

Conversely, since $b \odot c \to b \odot (a \odot c) = c \to (b \Rightarrow b \odot (a \odot c)) \geq c \to (a \odot c) \geq a$, we have

$$b \odot c \to b \odot (a \odot c) \geq a \text{ iff } a \odot (b \odot c) \leq b \odot (a \odot c).$$

□

THEOREM 3.2. *Let $(L, \Rightarrow, \top)$ be a w-eo algebra having adjoint triple with $\odot$ and $\to$. Then the following statements are equivalent:*

(1) $a \to (b \to c) = b \to (a \to c)$ *for all* $a, b, c \in L$.

(2) $b \Rightarrow c \leq (a \to b) \Rightarrow (a \to c)$ *for all* $a, b, c \in L$.

(3) $(a \odot b) \odot c = (a \odot c) \odot b$ *for all* $a, b, c \in L$.

(4) $b \Rightarrow c \leq (b \odot a) \Rightarrow (c \odot a)$ *for all* $a, b, c \in L$.

(5) $b \Rightarrow (a \to c) = (b \odot a) \Rightarrow c$ *for all* $a, b, c \in L$.

Proof. (1)$\Leftrightarrow$ (2) Since $a \to (b \to c) \leq b \to (a \to c)$, then $b \leq (a \to (b \to c)) \Rightarrow (a \to c)$. Put $b = b \Rightarrow c$. Since $b \leq (b \Rightarrow c) \to c$, then

$$\begin{aligned} b \Rightarrow c &\leq (a \to ((b \Rightarrow c) \to c)) \Rightarrow (a \to c) \\ &\leq (a \to b) \Rightarrow (a \to c) \end{aligned}$$

Since $b \leq (b \to c) \Rightarrow c \leq (a \to (b \to c)) \Rightarrow (a \to c)$, then $a \to (b \to c) \leq b \to (a \to c)$.

(1)$\Leftrightarrow$ (3) Since $c \to (b \to (a \odot b) \odot c) = b \to (c \to (a \odot b) \odot c) \geq b \to (a \odot b) \geq a$, we have

$$\begin{aligned} c \to (b \to (a \odot b) \odot c) \geq a \ &\text{iff } b \to (a \odot b) \odot c \geq a \odot c \\ &\text{iff } (a \odot b) \odot c \geq (a \odot c) \odot b. \end{aligned}$$

Conversely, since $((a \to (b \to c)) \odot b) \odot a = ((a \to (b \to c)) \odot a) \odot b \leq (b \to c) \odot b \leq c$, we have

$$\begin{aligned} ((a \to (b \to c)) \odot b) \odot a \leq c \ &\text{iff } (a \to (b \to c)) \odot b \leq a \to c \\ &\text{iff } a \to (b \to c) \leq b \to (a \to c). \end{aligned}$$

(3)⇔ (4)

$$(b \odot a) \odot (b \Rightarrow c) = (b \odot (b \Rightarrow c)) \odot a \leq c \odot a$$
$$\text{iff } b \Rightarrow c \leq (b \odot a) \Rightarrow (c \odot a)$$

Conversely, since $c \leq a \Rightarrow a \odot c \leq (a \odot b) \Rightarrow (a \odot c) \odot b$, we have

$$c \leq (a \odot b) \Rightarrow (a \odot c) \odot b \text{ iff } (a \odot b) \odot c \leq (a \odot c) \odot b.$$

(3)⇔ (5) By (4), put $c = a \to c$.

$$b \Rightarrow (a \to c) \leq (b \odot a) \Rightarrow (a \to c) \odot a \leq b \odot a \Rightarrow c.$$

Since $(b \odot (b \odot a \Rightarrow c)) \odot (b \Rightarrow b \odot a) = (b \odot (b \Rightarrow b \odot a)) \odot (b \odot a \to c) \leq (b \odot a) \odot (b \odot a \Rightarrow c) \leq c$, then $b \odot (b \odot a \Rightarrow c) \leq (b \Rightarrow b \odot a) \to c \leq a \to c$. So, $b \odot a \Rightarrow c \leq b \Rightarrow (a \to c)$. Hence $b \odot a \Rightarrow c = b \Rightarrow (a \to c)$.

Conversely, since $a \odot b \Rightarrow (a \odot c) \odot b = a \Rightarrow (b \to (a \odot c) \odot b) \geq a \Rightarrow a \odot c \geq c$, we have

$$a \odot b \Rightarrow (a \odot c) \odot b \geq c \text{ iff } (a \odot b) \odot c \leq (a \odot c) \odot b.$$

□

Definition 3.3. [5] Let $(L, \Rightarrow, \top)$ be a w-eo algebra having an adjoint pair $(\Rightarrow, \odot)$. A w-eo algebra $(L, \Rightarrow, \top)$ is associative iff it satisfies

$$a \Rightarrow (b \Rightarrow c) = (b \odot a) \Rightarrow c.$$

Theorem 3.4. *Let $(L, \Rightarrow, \top)$ be a w-eo algebra having an adjoint pair $(\Rightarrow, \odot)$. Then the following statements are equivalent:*
(1) *$(L, \Rightarrow, \top)$ is associative;*
(2) *$(a \odot b) \odot c = a \odot (b \odot c)$ for all $a, b, c \in L$.*

Proof. (1)⇒ (2)

$$(a \odot b) \odot c \leq a \odot (b \odot c)$$
$$\text{iff } c \leq (a \odot b) \Rightarrow (a \odot (b \odot c)) = b \Rightarrow (a \Rightarrow (a \odot (b \odot c)))$$
$$\text{iff } b \odot c \leq a \Rightarrow a \odot (b \odot c)$$
$$\text{iff } a \odot (b \odot c) \leq a \odot (b \odot c).$$

$$\begin{aligned}
&a \odot (b \odot c) \leq (a \odot b) \odot c \\
&\text{iff } b \odot c \leq a \Rightarrow ((a \odot b) \odot c) \\
&\text{iff } c \leq b \Rightarrow (a \Rightarrow ((a \odot b) \odot c)) = a \odot b \Rightarrow ((a \odot b) \odot c) \\
&\text{iff } (a \odot b) \odot c \leq (a \odot b) \odot c.
\end{aligned}$$

(2)$\Rightarrow$ (1) Since $(b \odot a) \odot (a \Rightarrow (b \Rightarrow c)) = b \odot (a \odot (a \Rightarrow (b \Rightarrow c))) \leq b \odot (b \Rightarrow c) \leq c$, we have $a \Rightarrow (b \Rightarrow c) \leq (b \odot a) \Rightarrow c$.

$$\begin{aligned}
&(b \odot a) \Rightarrow c \leq a \Rightarrow (b \Rightarrow c) \\
&\text{iff } a \odot ((b \odot a) \Rightarrow c) \leq b \Rightarrow c \\
&\text{iff } (b \odot a) \odot ((b \odot a) \Rightarrow c) \leq c.
\end{aligned}$$

□

THEOREM 3.5. *Let $(L, \Rightarrow, \top)$ be a right w-eo algebra having a Galois pair $(\Rightarrow, \rightarrow)$. Then the following statements are equivalent:*
(1) $a \Rightarrow (b \rightarrow c) = b \rightarrow (a \Rightarrow c)$.
(2) $b \rightarrow a \leq (a \rightarrow c) \Rightarrow (b \rightarrow c)$ *and* $b \Rightarrow a \leq (a \Rightarrow c) \rightarrow (b \Rightarrow c)$.

Proof. (1)$\Rightarrow$ (2) Since $(a \rightarrow c) \Rightarrow c \geq a$, we have

$$(a \rightarrow c) \Rightarrow (b \rightarrow c) = b \rightarrow ((a \rightarrow c) \Rightarrow c) \geq b \rightarrow a.$$

Since $(a \Rightarrow c) \rightarrow c \geq a$, we have

$$(a \Rightarrow c) \rightarrow (b \Rightarrow c) = b \Rightarrow ((a \Rightarrow c) \rightarrow c) \geq b \Rightarrow a.$$

(2)$\Rightarrow$ (1) Since $a \leq (a \Rightarrow c) \rightarrow c$,

$$\begin{aligned}
a \Rightarrow (b \rightarrow c) &\geq ((a \Rightarrow c) \rightarrow c) \Rightarrow (b \rightarrow c) \\
&\geq b \rightarrow (a \Rightarrow c).
\end{aligned}$$

Since $b \leq (b \rightarrow c) \Rightarrow c$,

$$\begin{aligned}
b \rightarrow (a \Rightarrow c) &\geq ((b \rightarrow c) \Rightarrow c) \rightarrow (a \Rightarrow c) \\
&\geq a \Rightarrow (b \rightarrow c).
\end{aligned}$$

□

THEOREM 3.6. *Let $(L, \Rightarrow, \top)$ be a right w-eo algebra having adjoint triple with $\odot$ and $\rightarrow$. Then the following statements are equivalent:*
(1) $a \Rightarrow (b \rightarrow c) = b \rightarrow (a \Rightarrow c)$.
(2) $(L, \odot, \top)$ *is associative.*
(3) $a \rightarrow (b \rightarrow c) = (a \odot b) \rightarrow c$.

Proof. (1)$\Rightarrow$ (2)

$$\begin{aligned}
&(a \odot b) \odot c \leq a \odot (b \odot c)\\
&\text{iff } a \odot b \leq c \to (a \odot (b \odot c))\\
&\text{iff } b \leq a \Rightarrow (c \to (a \odot (b \odot c))) = c \to (a \Rightarrow (a \odot (b \odot c)))\\
&\text{iff } b \odot c \leq a \Rightarrow (a \odot (b \odot c))\\
&\text{iff } a \odot (b \odot c) \leq a \odot (b \odot c)
\end{aligned}$$

$$\begin{aligned}
&a \odot (b \odot c) \leq (a \odot b) \odot c\\
&\text{iff } b \odot c \leq a \Rightarrow (a \odot b) \odot c\\
&\text{iff } b \leq c \to (a \Rightarrow ((a \odot b) \odot c)) = a \Rightarrow (c \to ((a \odot b) \odot c)))\\
&\text{iff } a \odot b \leq c \to (a \odot b) \odot c\\
&\text{iff } (a \odot b) \odot c \leq (a \odot b) \odot c
\end{aligned}$$

(2)$\Rightarrow$ (1) Since $a \odot ((a \Rightarrow (b \to c)) \odot b) = (a \odot (a \Rightarrow (b \to c))) \odot b \leq (b \to c) \odot b \leq c$, then $(a \Rightarrow (b \to c)) \odot b \leq a \Rightarrow c$. So, $a \Rightarrow (b \to c) \leq b \to (a \Rightarrow c)$.

Since $(a \odot (b \to (a \Rightarrow c))) \odot b = a \odot ((b \to (a \Rightarrow c)) \odot b) \leq a \odot (a \Rightarrow c) \leq c$, then $a \odot (b \to (a \Rightarrow c)) \leq b \to c$. So, $b \to (a \Rightarrow c) \leq a \Rightarrow (b \to c)$.

(2)$\Leftrightarrow$ (3) Since $(a \to (b \to c)) \odot (a \odot b) = ((a \to (b \to c)) \odot a) \odot b \leq c$, then $a \to (b \to c) \leq (a \odot b) \to c$.

Since $(((a \odot b) \to c) \odot a) \odot b = ((a \odot b) \to c) \odot (a \odot b) \leq c$, then $(a \odot b) \to c \leq a \to (b \to c)$.

Conversely, it follows from;

$$\begin{aligned}
&(a \odot b) \odot c \leq (a \odot b) \odot c\\
&\text{iff } a \odot b \leq c \to (a \odot b) \odot c\\
&\text{iff } a \leq b \to (c \to ((a \odot b) \odot c)) = (b \odot c) \to (a \odot b) \odot c\\
&\text{iff } a \odot (b \odot c) \leq (a \odot b) \odot c,
\end{aligned}$$

$$\begin{aligned}
&a \odot (b \odot c) \leq a \odot (b \odot c)\\
&\text{iff } a \leq (b \odot c) \to a \odot (b \odot c)\\
&\text{iff } a \leq b \to (c \to a \odot (b \odot c))\\
&\text{iff } a \odot b \leq c \to a \odot (b \odot c)\\
&\text{iff } (a \odot b) \odot c \leq (a \odot b) \odot c.
\end{aligned}$$

□

THEOREM 3.7. *Let $(L, \Rightarrow, \top)$ be a w-eo algebra having adjoint triple with $\odot$ and $\rightarrow$ satisfying $\top \Rightarrow a = a$ for all $a \in L$. Then the following statements are equivalent:*

(1) $a \Rightarrow (b \Rightarrow c) = b \Rightarrow (a \Rightarrow c)$ *for all* $a, b, c \in L$.

(2) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$ *for all* $a, b, c \in L$.

(3) $a \odot b = b \odot a$ *and* $(a \odot b) \odot c = a \odot (b \odot c)$ *for all* $a, b, c \in L$.

Proof. (1)$\Leftrightarrow$ (2). Since $a \Rightarrow (b \Rightarrow c) = \top$ iff $b \Rightarrow (a \Rightarrow c) = \top$, then $\Rightarrow = \rightarrow$.

Conversely, since $(L, \rightarrow, \top)$ be an eo algebra having adjoint triple with $\odot^*$ and $\Rightarrow$ with $a \odot^* b = b \odot a$ from Theorem 2.7 (2), then $(L, \rightarrow, \top)$ is commutative. Hence $\Rightarrow = \rightarrow$.

(1)$\Leftrightarrow$ (3). By Theorem 2.9, $a \odot b = b \odot a$. By Theorems 3.1 and 3.2, $(a \odot b) \odot c = (a \odot c) \odot b = b \odot (a \odot c) = a \odot (b \odot c)$.

Conversely, by Theorem 3.1, it follows from $a \odot (b \odot c) = b \odot (a \odot c)$. □

THEOREM 3.8. *Let $(L, \Rightarrow, \top)$ be a w-eo algebra having $a \Rightarrow (b \Rightarrow c) = b \Rightarrow (a \Rightarrow c)$ for all $a, b, c \in L$ and $(a \Rightarrow \bot) \Rightarrow \bot = a$. Then $a \Rightarrow b = (b \Rightarrow \bot) \Rightarrow (a \Rightarrow \bot)$.*

Proof. It follows from $a \Rightarrow b = a \Rightarrow ((b \Rightarrow \bot) \Rightarrow \bot) = (b \Rightarrow \bot) \Rightarrow (a \Rightarrow \bot)$. □

EXAMPLE 3.9. (1) Let $([0,1], \Rightarrow)$ be a unit interval defined as

$$a \Rightarrow b = \begin{cases} 1, & \text{if } a \le b, \\ (\frac{1}{2} - a) \vee b, & \text{otherwise.} \end{cases}$$

We easily show that $([0,1], \Rightarrow, 1)$ is an eo algebra having $\bigvee_i x_i \Rightarrow y = \bigwedge_i (x_i \Rightarrow y)$ and $x \Rightarrow \bigwedge_i y_i = \bigwedge_i (x \Rightarrow y_i)$. Put $f_b(x) = x \Rightarrow b$. Define $g_b(y) = y \rightarrow b = \bigvee\{x \in L \mid y \le f_b(x)\}$. Then $y \le f_b(x)$ iff $x \le g_b(y)$. We obtain:

$$a \rightarrow b = \begin{cases} 1, & \text{if } a \le b, \\ (\frac{1}{2} - a) \vee b, & \text{otherwise.} \end{cases}$$

Put $f_b(x) = b \Rightarrow x$. Define $g_b(y) = b \odot y = \bigwedge\{x \in L \mid y \le f_b(x)\}$. Then $g_b(y) \le x$ iff $y \le f_b(x)$. We obtain:

$$a \odot b = \begin{cases} a \wedge b, & \text{if } a + b \ge \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $a \Rightarrow (b \Rightarrow c) = b \Rightarrow (a \Rightarrow c)$ for all $a, b, c \in L$, we have $a \odot b = b \odot a$ and $(a \odot b) \odot c = a \odot (b \odot c)$ for all $a, b, c \in L$. Since $(0.7 \Rightarrow \bot) \Rightarrow \bot = \top$, we have

$$0.7 = 0.8 \Rightarrow 0.7 \neq (0.7 \Rightarrow \bot) \Rightarrow (0.8 \Rightarrow \bot) = \bot \Rightarrow \bot = \bot.$$

EXAMPLE 3.10. Let $K = \{(x, y) \in R^2 \mid x > 0\}$ be a set and we define an operation $\otimes : K \times K \to K$ as follows:

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1x_2, x_1y_2 + y_1).$$

Then $(K, \otimes)$ is a group with $e = (1, 0)$, $(x, y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$.

We have a positive cone $P = \{(a, b) \in R^2 \mid a = 1, b \geq 0 \text{ ,or } a > 1\}$ because $P \cap P^{-1} = \{(1, 0)\}$, $P \otimes P \subset P$, $(a, b)^{-1} \otimes P \otimes (a, b) = P$ and $P \cup P^{-1} = K$. For $(x_1, y_1), (x_2, y_2) \in K$, we define

$$\begin{aligned} &(x_1, y_1) \leq (x_2, y_2) \\ &\Leftrightarrow (x_1, y_1)^{-1} \otimes (x_2, y_2) \in P,\ (x_2, y_2) \otimes (x_1, y_1)^{-1} \in P \\ &\Leftrightarrow x_1 < x_2 \text{ or } x_1 = x_2, y_1 \leq y_2. \end{aligned}$$

Then $(K, \leq \otimes)$ is a lattice-group. (ref. [1])

The structure $(L, \odot, \Rightarrow, \to, (\frac{1}{2}, 1), (1, 0))$ is a generalized residuated lattice with strong negation where $\bot = (\frac{1}{2}, 1)$ is the least element and $\top = (1, 0)$ is the greatest element from the following statements:

$$\begin{aligned} (x_1, y_1) \odot (x_2, y_2) &= (x_1, y_1) \otimes (x_2, y_2) \vee (\frac{1}{2}, 1) \\ &= (x_1x_2, x_1y_2 + y_1) \vee (\frac{1}{2}, 1), \\ (x_1, y_1) \Rightarrow (x_2, y_2) &= ((x_1, y_1)^{-1} \otimes (x_2, y_2)) \wedge (1, 0) \\ &= (\frac{x_2}{x_1}, \frac{y_2 - y_1}{x_1}) \wedge (1, 0), \\ (x_1, y_1) \to (x_2, y_2) &= ((x_2, y_2) \otimes (x_1, y_1)^{-1}) \wedge (1, 0) \\ &= (\frac{x_2}{x_1}, -\frac{x_2y_1}{x_1} + y_2) \wedge (1, 0). \end{aligned}$$

The structure $(L, \odot, \Rightarrow, \to, (\frac{1}{2}, 1), (1, 0))$ is a w-eo algebra having an adjoint triple as follows:

$$\begin{aligned} (x_1, y_1) \odot (x_2, y_2) \leq (x_3, y_3) &\text{ iff } (x_2, y_2) \leq (x_1, y_1) \Rightarrow (x_3, y_3) \\ &\text{ iff } (x_1, y_1) \leq (x_2, y_2) \to (x_3, y_3) \end{aligned}$$

Since

$$(x_1,y_1)\Rightarrow((x_2,y_2)\Rightarrow(x_3,y_3))=(\frac{x_3}{x_1x_2},\frac{y_3-x_2y_1-y_2}{x_1x_2})\wedge(1,0)$$
$$=((x_2,y_2)\odot(x_1,y_1))\Rightarrow(x_3,y_3),$$

$(L,\odot,\Rightarrow,\rightarrow,(\frac{1}{2},1),(1,0))$ is associative. Since $(1,0)\Rightarrow(x,y)=(x,y)$ and

$$(\frac{4}{5},1)\Rightarrow((\frac{5}{6},3)\Rightarrow(\frac{2}{3},-1))=(\frac{4}{5},1)\Rightarrow(\frac{4}{5},-\frac{24}{5})=(1,-\frac{29}{4}),$$

$$(\frac{5}{6},3)\Rightarrow((\frac{4}{5},1)\Rightarrow(\frac{2}{3},-1))=(\frac{5}{6},3)\Rightarrow(\frac{5}{6},-\frac{5}{2})=(1,-\frac{33}{5}),$$

$$(\frac{4}{5},1)\Rightarrow((\frac{5}{6},3)\Rightarrow(\frac{2}{3},-1))\neq(\frac{5}{6},3)\Rightarrow((\frac{4}{5},1)\Rightarrow(\frac{2}{3},-1)),$$

by Theorems 3.1, 3.2 and 3.7,

$$(x_1,y_1)\rightarrow((x_2,y_2)\rightarrow(x_3,y_3))\neq(x_2,y_2)\rightarrow((x_1,y_1)\rightarrow(x_3,y_3))$$
$$(x_1,y_1)\odot((x_2,y_2)\odot(x_3,y_3))\neq(x_2,y_2)\odot((x_1,y_1)\odot(x_3,y_3))$$
$$((x_1,y_1)\odot(x_2,y_2))\odot(x_3,y_3)\neq((x_1,y_1)\odot(x_3,y_3))\odot(x_2,y_2)$$
$$(x_1,y_1)\odot(x_2,y_2)\neq(x_2,y_2)\odot(x_1,y_1).$$

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