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SOME EXAMPLES OF WEAKLY FACTORIAL RINGS

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Abstract. Let D be a principal ideal domain, X be an indeterminate over D, $D[X]$ be the polynomial ring over D, and $R_n = D[X]/(X^n)$ for an integer $n \geq 1$. Clearly, R_n is a commutative Noetherian ring with identity, and hence each nonzero nonunit of R_n can be written as a finite product of irreducible elements. In this paper, we show that every irreducible element of R_n is a primary element, and thus every nonunit element of R_n can be written as a finite product of primary elements.

1. Introduction

Let D be an integral domain, X be an indeterminate over D, $D[X]$ be the polynomial ring over D, and $R_n = D[X]/(X^n)$ for an integer $n \geq 1$. Clearly, R_n is a commutative ring with identity $1 + (X^n)$, and since $(X^n) \cap D = (0)$, D can be considered as a subring of R_n. Note that if $\alpha \in R_n$, then $\alpha = a_0 + a_1X + \cdots + a_{n-1}X^{n-1} + (X^n)$ for some unique $a_i \in D$; so if we let $x = X + (X^n)$, then x is a prime element of R_n, $\alpha = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$, and $\alpha = 0$ if and only if $a_0 = a_1 = \cdots = a_{n-1} = 0$. Also, if $\beta = b_0 + b_1x + \cdots b_{n-1}x^{n-1} \in R_n$, then

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$\alpha + \beta = \sum_{k_0}^{n-1}(a_k + b_k)x^k$ and $\alpha \cdot \beta = a_0b_0 + (a_1b_0 + a_0b_1)x + \cdots + (a_{n-1}b_0 + a_{n-2}b_1 + \cdots + a_1b_{n-2} + a_0b_{n-1})x^{n-1} = \sum_{k=0}^{n-1}(\sum_{i+j=k} a_ib_j)x^k$.

Let R be a commutative ring with identity and $U(R)$ be the set of units of R. An $a \in R$ is said to be *primary* if aR is a primary ideal. An integral domain is called a *weakly factorial domain* if its nonzero nonunit can be written as a finite product of primary elements [1]. For convenience, in this paper, we will say that R is a *weakly factorial ring* if every nonzero nonunit of R can be written as a finite product of primary elements. Hence, weakly factorial domains are weakly factorial rings. Two elements $a, b \in R$ are said to be *associates* if $aR = bR$, i.e., $a = bc_1$ and $b = ac_2$ for some $c_1, c_2 \in R$. An $a \in R$ is said to be *irreducible* if $a = bc$ implies that either b or c is associated with a.

Let D be a principal ideal domain (PID). Clearly, $R_1 = D$, and thus R_1 is a weakly factorial ring. Moreover, in [2, Corollary 11], it was proved that R_2 is a weakly factorial ring. In this short paper, we show that R_n is a weakly factorial ring for all integers $n \geq 1$. Note that R_n is a commutative Noetherian ring with identity, and hence each nonzero nonunit of R_n can be written as a finite product of irreducible elements. Thus, to prove that R_n is a weakly factorial ring, it suffices to show that every irreducible element of R_n is primary. This will be proved by a series of lemmas.

2. Main Results

Let D be an integral domain, $D[X]$ be the polynomial ring over D, and $R_n = D[X]/(X^n)$ for an integer $n \geq 1$. In this section, we show that R_n is a weakly factorial ring by a series of lemmas.

Lemma 1. (cf. [2, Lemma 1]) *Let $\alpha = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \in R_n$. Then α is a unit of R_n if and only if a_0 is a unit of D.*

Proof. If α is a unit, then there is a $\beta = b_0 + b_1x + \cdots + b_{n-1}x^{n-1} \in R_n$ such that $\alpha \cdot \beta = 1$. Thus, $a_0b_0 = 1$. Conversely, assume that a_0 is a unit of D, and let $c \in D$ with $a_0c = 1$. Note that $\alpha R_n = c\alpha R_n$; so replacing α with $\alpha \cdot c$ if necessary, we may assume that $a_0 = 1$. Let $c_0, c_1, \ldots, c_{n-1} \in D$ be such that

$$\begin{cases} c_0 = 1 \\ c_1 + a_1c_0 = 0 \\ c_2 + c_1a_1 + c_0a_2 = 0 \\ \cdots \\ c_{n-1} + c_{n-2}a_1 + \cdots + c_0a_{n-1} \quad = 0 \end{cases}$$

and let $\gamma = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$. Clearly, such c_i's exist and $\alpha\gamma = 1$. □

Lemma 2. (cf. [2, Lemma 2]) *Let $\alpha, \beta \in R_n$. Then α and β are associates if and only if there is a $\theta \in U(R_n)$ such that $\alpha = \theta\beta$. Hence $\alpha \in R_n$ is irreducible if and only if $\alpha = \beta\gamma$ for $\beta, \gamma \in R_n$ implies that either β or γ is a unit.*

Proof. Let $\alpha = a_ix^i + a_{i+1}x^{i+1} + \cdots + a_{n-1}x^{n-1}$ and $\beta = b_jx^j + b_{j+1}x^{j+1} + \cdots + b_{n-1}x^{n-1}$ such that $a_i \neq 0$, $b_j \neq 0$, and $0 \leq i \leq j$. If α and β are associates, then $\alpha = \beta \cdot \theta$ for some $\theta \in R_n$. Note that $i \leq j$; so if we let $\gamma = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$, then $j = i$ and $a_i = c_0b_j$. Similarly, we can find an element $d \in D$ such that $b_j = a_id$. Hence $a_i = c_0da_i$, and since $a_i \neq 0$, we have $c_0d = 1$ or $c_0 \in U(D)$. Thus, θ is a unit of R_n by Lemma 1. The converse is clear. □

Lemma 3. (cf. [2, Theorem 5]) *Let D be a PID and $\alpha = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \in R_n$. If α is irreducible, then either (i) $a_0 = 0$ and $a_1 \in U(D)$ or (ii) $a_0 = up^k$ for some prime $p \in D$, $u \in U(D)$, and integer $k \geq 1$.*

Proof. Assume that $a_0 = 0$. Then $a_1 \neq 0$, because $a_2x^2 + \cdots + a_{n-1}x^{n-1} = x(a_2x + \cdots + a_{n-1}x^{n-2})$ and both x and $a_2x + \cdots + a_{n-1}x^{n-2}$ are not units by Lemma 1. Moreover, if a_1 is a nonzero nonunit, then $a_1x + \cdots + a_{n-1}x^{n-1} = x(a_1 + a_2x \cdots + a_{n-1}x^{n-2})$, and since both x and $a_1 + a_2x \cdots + a_{n-1}x^{n-2}$ are not units by Lemma 1, α is not irreducible by Lemma 2, a contradiction. Thus, a_1 is a unit of D.

Next, assume that $a_0 \neq 0$. If a_0 is not of the form up^k, then there are nonzero $b_0, c_0 \in D$ such that $a_0 = b_0c_0$ and $\gcd(b_0, c_0) = 1$. Since D is a PID, there exist $b_1, c_1 \in D$ so that $b_0c_1 + b_1c_0 = a_1$. Again, D being a PID guarantees that there are $b_2, c_2 \in D$ such that $b_0c_2 + b_2c_0 = a_2 - b_1c_1$. Repeating this process, we can choose $b_2, \ldots, b_{n-1}, c_2, \ldots, c_{n-1} \in D$ so that

$$(b_0 + b_1x + \cdots + b_{n-1}x^{n-1}) \cdot (c_0 + c_1x + \cdots + c_{n-1}x^{n-1}) = \alpha,$$

hence α is not irreducible by Lemmas 1 and 2. Thus, a_0 must be of the form up^k for some prime $p \in D$, $u \in U(D)$, and integer $k \geq 1$. □

We are now ready to prove the main result of this paper.

THEOREM 4. (cf. [2, Corollary 11]) *If D is a PID, then the ring $R_n = D[X]/(X^n)$ is a weakly factorial ring for all integers $n \geq 1$.*

Proof. Note that R_n is a Noetherian ring; hence each element of R_n can be written as a finite product of irreducible elements. Hence if we show that each irreducible element of R_n is primary, then R_n is a weakly factorial ring.

Let $\alpha = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \in R_n$ be irreducible. By Lemma 3, there are only two cases we have to consider. First, assume $a_0 = 0$ and $a_1 \in U(D)$. Then $\alpha R_n = xR_n$ by Lemma 1, and hence α is prime (so primary). Next, assume $a_0 = up^k$ for some $u \in U(D)$, prime $p \in D$ and integer $k \geq 1$. It is known that if $\sqrt{\alpha R_n}$ is a maximal ideal, then αR_n is primary [2, Lemma 10]; so it suffices to show that $\sqrt{\alpha R_n}$ is maximal. Let $\beta \in R_n \setminus \sqrt{\alpha R_n}$, and put $\beta = b_0 + b_1x + \cdots + b_{n-1}x^{n-1}$. Note that if $\delta = c_1x + \cdots + c_{n-1}x^{n-1} \in R_n$, then $\delta^n = 0$, and hence $\delta \in \sqrt{\alpha R_n}$. Hence $b_0 \notin \sqrt{\alpha R_n}$ and $p \in \sqrt{\alpha R_n}$. Note also that if $b_0 \in pD$, then $b_0 = pz$ for some $z \in D$, and so $b_0 = pz \in \sqrt{\alpha R_n}$, a contradiction. So $b_0 \notin pD$, and since D is a PID, we have $b_0z_1 + pz_2 = 1$ for some $z_1, z_2 \in D$. Thus, $1 = \beta z_1 + pz_2 - z_1(b_1x + \cdots + b_{n-1}x^{n-1}) \in \beta R_n + \sqrt{\alpha R_n}$. Therefore, $\sqrt{\alpha R_n}$ is maximal. □

COROLLARY 5. *If $\mathbb{Z}$ is the ring of integers, then $\mathbb{Z}[X]/(X^n)$ is a weakly factorial ring for all integers $n \geq 1$.*

Proof. This follows directly from Theorem 4 because $\mathbb{Z}$ is a PID. □

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