Korean J. Math. **21** (2013), No. 3, pp. 311–318
http://dx.doi.org/10.11568/kjm.2013.21.3.311

A REMARK ON IFP RINGS

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Abstract. We continue the study of power-Armendariz rings over IFP rings, introducing k-power Armendariz rings as a generalization of power-Armendariz rings. Han et al. showed that IFP rings are 1-power Armendariz. We prove that IFP rings are 2-power Armendariz. We moreover study a relationship between IFP rings and k-power Armendariz rings under a condition related to nilpotency of coefficients.

1. IFP rings and 2-power Armendariz rings

Throughout this note every ring is associative with identity unless otherwise stated. Let R be a ring (possibly without identity). $R[x]$ denotes the polynomial ring with an indeterminate x over R. For $f(x) \in R[x]$, let $C_{f(x)}$ denote the set of all coefficients of $f(x)$. We use $deg\, f(x)$ to denote the degree of $f(x)$. Denote the n by n full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Use e_{ij} for the matrix with (i, j)-entry 1 and elsewhere 0. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). Let $J(R)$, $N_*(R)$, $N^*(R)$, and $N(R)$ denote the Jacobson radical, the prime radical, the upper nilradical (i.e., sum of all

Received May 28, 2013. Revised July 4, 2013. Accepted July 4, 2013.
2010 Mathematics Subject Classification: 16U80, 16N40.
Key words and phrases: IFP ring, k-power-Armendariz ring.
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nil ideals), and the set of all nilpotent elements in R, respectively. It is well-known that $N^*(R) \subseteq J(R)$ and $N_*(R) \subseteq N^*(R) \subseteq N(R)$.

Following Bell [3], a ring R is called to satisfy the *Insertion-of-Factors-Property* (simply, an *IFP* ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. A ring is usually called *Abelian* if every idempotent is central. It is easily checked that IFP rings are Abelian. It is well-known that $N_*(R) = N^*(R) = N(R)$ for an IFP ring R.

A ring (possibly without identity) is usually called *reduced* if it has no nonzero nilpotent elements. For a reduced ring R and $f(x), g(x) \in R[x]$, Armendariz [2, Lemma 1] proved that

$$ab = 0 \text{ for all } a \in C_{f(x)}, b \in C_{g(x)} \text{ whenever } f(x)g(x) = 0.$$

Rege-Chhawchharia [9] called a ring (possibly without identity) *Armendariz* if it satisfies this property. So reduced rings are clearly Armendariz. Armendariz rings are also Abelian by the proof of [1, Theorem 6] (or [7, Lemma 7]).

Note that IFP rings and Armendariz rings are independent of each other by [9, Example 3.2] and [5, Example 14]. However for a semiprime right Goldie ring R, R is Armendariz if and only if R is IFP by [5, Corollary 13].

Given a ring R and $n \geq 2$, we usually write

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in U_n(R) \mid a, a_{ij} \in R \right\},$$

$$N_n(R) = \{(a_{ij}) \in D_n(R) \mid a_{ii} = 0 \text{ for all } i\}, \text{ and}$$

$$V_n(R) = \{(a_{ij}) \in D_n(R) \mid a_{ij} = a_{(i+1)(j+1)} \text{ for } i = 1, \ldots, n-2 \text{ and } j = 2, \ldots, n-1\}.$$

Note that $V_n(R) \cong \frac{R[x]}{R[x]x^nR[x]}$ by [8].

A ring R is reduced if and only if $D_3(R)$ is Armendariz by [6, Proposition 2.8], but $D_n(A)$ cannot be Armendariz for any ring A when $n \geq 4$ by [7, Example 3]. Let R be a division ring and consider $f(x) = \sum_{i=0}^{s} A_i x^i, g(x) = \sum_{j=0}^{t} B_j x^j \in D_n(R)[x]$ with $f(x)g(x) = 0$. Since $J(D_n(R)) = N_n(R)$ and $\frac{D_n(R)}{N_n(R)} \cong R$, $f(x)g(x) = 0$ implies that $A_i, B_j \in N_n(R)$ for all i, j. This yields $A_i^n = 0$, $B_j^n = 0$, and $A_i^n B_j^n = 0$.

Following Han et al. [4], a ring R (possibly without identity) is called *power-Armendariz* if whenever $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$, there exist $m, n \geq 1$ such that

$$a^m b^n = 0 \text{ for all } a \in C_{f(x)}, b \in C_{g(x)}.$$

It is obvious that $a^m b^n = 0$ for some $m, n \geq 1$ if and only if $a^\ell b^\ell = 0$ for some $\ell \geq 1$, in the preceding definition. Armendariz rings are clearly power-Armendariz, but the converse need not be true. Consider a non-reduced, IFP and Armendariz ring A (e.g., $D_2(\mathbb{Z})$). Then $D_3(A)$ is power-Armendariz by [4, Theorem 1.4(1)]; but $D_3(A)$ is not Armendariz by [6, Proposition 2.8]. Power-Armendariz rings are also Abelian by [4, Proposition 1.1(5)].

In this note, a ring R (possibly without identity) will be called *k-power-Armendariz* if whenever $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$ with $deg\, f(x), deg\, g(x) \leq k$, there exist $m, n \geq 1$ such that

$$a^m b^n = 0 \text{ for all } a \in C_{f(x)}, b \in C_{g(x)}.$$

It is obvious that a ring R is k-power-Armendariz if and only if there exist $m, n \geq 1$ such that $a^m b^n = 0$ for any pair $(a, b) \in C_{f(x)} \times C_{g(x)}$, whenever $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$ with $deg\, f(x), deg\, g(x) \leq k$. Note that a ring is power-Armendariz if and only if it is k-power-Armendariz for all $k \geq 0$. k-power-Armendariz rings are Abelian by the proof of [4, Proposition 1.1(5)] for any $k \geq 1$.

IFP rings are 1-power-Armendariz by [4, Proposition 1,6]. We continue this study by investigating more properties of IFP rings which are related to k-power-Armendariz rings.

Proposition 1.1. *IFP rings are 2-power-Armendariz.*

Proof. Let R be an IFP ring, and suppose that $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{2} a_i x^i, g(x) = \sum_{j=0}^{2} b_j x^j$ in $R[x]$. Then we have

$$a_0 b_0 = 0, \tag{1}$$

$$a_0 b_1 + a_1 b_0 = 0, \tag{2}$$

$$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0, \tag{3}$$

$$a_2 b_1 + a_1 b_2 = 0, \tag{4}$$

$$a_2 b_2 = 0. \tag{5}$$

We will use the IFP property of R freely. Multiplying the equality (2) by b_0 on the right (resp., by a_0 on the left), we have

$$(6) \qquad 0 = (a_0b_1 + a_1b_0)b_0 = a_1b_0^2 \text{ (resp., } 0 = a_0(a_0b_1 + a_1b_0) = a_0^2b_1)$$

by the equality (1).

Multiplying the equality (3) by b_0^2 on the right (resp., by a_0^2 on the left), we have

$$(7) \qquad 0 = (a_0b_2 + a_1b_1 + a_2b_0)b_0^2 = a_2b_0^3$$
$$\text{(resp., } 0 = a_0^2(a_0b_2 + a_1b_1 + a_2b_0) = a_0^3b_2)$$

by the equalities (1) and (6).

Multiplying the equality (4) by b_2 on the right (resp., by a_2 on the left), we have

$$(8) \qquad 0 = (a_2b_1 + a_1b_2)b_2 = a_1b_2^2 \text{ (resp., } 0 = a_2(a_2b_1 + a_1b_2) = a_2^2b_1)$$

by the equality (5).

Multiplying the equality (3) by b_2^2 on the right (resp., by a_2^2 on the left), we have

$$(9) \qquad 0 = (a_0b_2 + a_1b_1 + a_2b_0)b_2^2 = a_0b_2^3$$
$$\text{(resp., } 0 = a_2^2(a_0b_2 + a_1b_1 + a_2b_0) = a_2^3b_0)$$

by the equalities (5) and (7).

Lastly we will find $s, t \geq 1$ such that $a_1^sb_1^t = 0$. From the equalities (1) $\sim$ (5), we have

$$(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) = 0.$$

This equality yields that

$$(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2) = 0 \text{ for all } r \in R.$$

So we have

$$(10) \qquad a_1rb_1 = -a_0rb_1 - a_1rb_0 - a_0rb_2 - a_2rb_0 - a_1rb_2 - a_2rb_1$$

for all $r \in R$. Taking $r = a_1a_1b_1$ in the equality (10), we get

$$\begin{aligned}
a_1^3b_1^2 =&\ a_1(a_1a_1b_1)b_1\\
=&- a_0(a_1a_1b_1)b_1 - a_1(a_1a_1b_1)b_0 - a_0(a_1a_1b_1)b_2\\
&- a_2(a_1a_1b_1)b_0 - a_1(a_1a_1b_1)b_2 - a_2(a_1a_1b_1)b_1\\
=&\ a_0a_1(-a_1b_1)b_1 + a_1a_1(-a_1b_1)b_0 + a_0a_1(-a_1b_1)b_2\\
&+ a_2a_1(-a_1b_1)b_0 + a_1a_1(-a_1b_1)b_2 + a_2a_1(-a_1b_1)b_1\\
=&\ a_0a_1(a_0b_2 + a_2b_0)b_1 + a_1a_1(a_0b_2 + a_2b_0)b_0\\
&+ a_0a_1(a_0b_2 + a_2b_0)b_2 + a_2a_1(a_0b_2 + a_2b_0)b_0\\
&+ a_1a_1(a_0b_2 + a_2b_0)b_2 + a_2a_1(a_0b_2 + a_2b_0)b_1 = \ 0
\end{aligned}$$

by help of the equalities (1) $\sim$ (9). Thus R is 2-power-Armendariz. $\square$

2. IFP rings and k-power Armendariz rings

In this section we study a relationship between IFP rings and k-power Armendariz rings under a condition.

Proposition 2.1. *Let R be an IFP ring and $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^k a_ix^i, g(x) = \sum_{j=0}^k b_jx^j \in R[x]$ $(k \geq 3)$ such that $a_1, \ldots, a_{k-1} \in N(R)$ or $b_1, \ldots, b_{k-1} \in N(R)$. Then there exist $m, n \geq 1$ such that $a^mb^n = 0$ for all $a \in C_{f(x)}, b \in C_{g(x)}$.*

Proof. Let R be an IFP ring, and assume that $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^k a_ix^i, g(x) = \sum_{j=0}^k b_jx^j \in R[x]$ $(k \geq 3)$ with $a_1, \ldots, a_{k-1} \in N(R)$ or $b_1, \ldots, b_{k-1} \in N(R)$. Then we first have

$$a_0b_0 = 0, \tag{11}$$
$$a_0b_1 + a_1b_0 = 0, \tag{12}$$
$$a_0b_2 + a_1b_1 + a_2b_0 = 0, \tag{13}$$
$$\cdots \tag{14}$$
$$a_0b_k + a_1b_{k-1} + \cdots + a_{k-1}b_1 + a_kb_0 = 0, \tag{15}$$
$$\cdots \tag{16}$$
$$a_{k-2}b_k + a_{k-1}b_{k-1} + a_kb_{k-2} = 0, \tag{17}$$
$$a_{k-1}b_k + a_kb_{k-1} = 0, \tag{18}$$
$$a_kb_k = 0. \tag{19}$$

We will use the IFP property of R freely. Multiplying the equality (12) by b_0 on the right (resp., by a_0 on the left), we have

$$0 = (a_0b_1 + a_1b_0)b_0 = a_1b_0^2 \text{ (resp., } 0 = a_0(a_0b_1 + a_1b_0) = a_0^2b_1) \tag{20}$$

by the equality (11).

Multiplying the equality (13) by b_0^2 on the right (resp., by a_0^2 on the left), we have

$$0 = (a_0b_2 + a_1b_1 + a_2b_0)b_0^2 = a_2b_0^3$$
$$\text{(resp., } 0 = a_0^2(a_0b_2 + a_1b_1 + a_2b_0) = a_0^3b_2)$$

by the equalities (11) and (20).

We proceed by induction and assume that $a_ib_0^{i+1} = 0$ for $i = 0, 1, \ldots, k-1$.

Multiplying the equality (15) by b_0^k on the right (resp., by a_0^k on the left), we have

$$0 = (a_0b_k + a_1b_{k-1} + \cdots + a_{k-1}b_1 + a_kb_0)b_0^k = a_kb_0^{k+1}$$
$$\text{(resp., } 0 = a_0^k(a_0b_k + a_1b_{k-1} + \cdots + a_{k-1}b_1 + a_kb_0) = a_0^{k+1}b_k)$$

by assumption.

From the equalities (15) $\sim$ (19), we can obtain

$$a_k^{k+1}b_0 = a_k^kb_1 = \cdots = a_k^2b_{k-1} = 0 \quad \text{and}$$
$$a_0b_k^{k+1} = a_1b_k^k = \cdots = a_{k-1}b_k^2 = 0$$

similarly.

We already have $a_1, \ldots, a_{k-1} \in N(R)$ or $b_1, \ldots, b_{k-1} \in N(R)$ by assumption, so there exist $s, t \geq 1$ such that $a_i^sb_j^t = 0$ for $i = 1, \ldots, k-1$ and $j = 1, \ldots, k-1$.

Therefore there exist $m, n \geq 1$ such that $a^mb^n = 0$ for all $a \in C_{f(x)}$, $b \in C_{g(x)}$. $\square$

We do not answer whether IFP rings are k-power Armendariz for $k \geq 3$. So we end this note by raising the following question.

Question. Are IFP rings k-power Armendariz for $k \geq 3$?

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