Korean J. Math. **21** (2013), No. 3, pp. 247–263
http://dx.doi.org/10.11568/kjm.2013.21.3.247

INFINITUDE OF MINIMALLY SUPPORTED TOTALLY INTERPOLATING BIORTHOGONAL MULTIWAVELET SYSTEMS WITH LOW APPROXIMATION ORDERS

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Abstract. By analyzing one-parameter families of totally interpolating multiwavelet systems of minimal total length with low approximation orders, whose explicit formulas were obtained with the aid of well-known relations of filters, we demonstrate the infinitude of such systems.

1. Introduction

Wavelet theory has been a popular tool in signal/image processing, computer graphics, and many other applied mathematics. A wavelet is based on multiresolution analysis(MRA) derived from one scaling function [6]. The Haar scaling function is the only orthogonal scaling function of compact support having symmetry and Shannon-like sampling properties [16]. Therefore, multiwavelet theory or biorthogonal wavelets have been studied to achieve many desirable properties such as orthogonality, symmetry, short support, and approximation order, simultaneously.

Totally interpolating multiwavelet systems have many advantages. The interpolating condition provides Shannon-like sampling property [12, 13] and perfect and fast reconstruction/decomposition algorithm in signal processing [17]. Moreover, under the interpolating condition

Received May 15, 2013. Revised July 30, 2013. Accepted August 7, 2013.
2010 Mathematics Subject Classification: Primary 42C40; Secondary 65T60.
Key words and phrases: biorthogonal multiwavelets, stability, totally interpolating.

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the equivalence of approximation and balancing orders was proved in terms of orthogonal/biorthogonal multiwavelet systems [1, 7–9]. Therefore, a prefiltering can be avoided with a totally interpolating orthogonal/biorthogonal multiwavelet system satisfying a suitable approximation order condition [14, 15]. It is well-known that both approximation order and short support conditions are important properties for applications such as denoising and compression. In this article, we are mainly concerned with totally interpolating biorthogonal multiwavelet systems having minimal support and approximation order property.

One can find sufficiently many refinable function vectors. For example, a one-parameter family of interpolating refinable function vectors with a given approximation order is constructed in [7]. Nevertheless, it is difficult to know how many interpolating multiwavelet systems there are that satisfy suitable regularity and stability. There are many totally interpolating orthogonal/biorthogonal multiwavelet systems of minimal total length for some given approximation orders [2, 3, 7, 12, 17]. Naturally, we are concerned about a question how many such systems there are. In this article, we investigate the cardinality of L^2-stable totally interpolating biorthogonal multiwavelet systems of minimal total length. Under FIR condition a degree of freedom provides the possibility of infinitely many totally interpolating biorthogonal multiwavelet systems from a suitable perturbation without lengthening their supports.

This paper is organized as follows. The second section introduces elementary notions of biorthogonal multiwavelet systems, interpolating condition, approximation order, and regularity. In third section, we prove that such systems should have even total length and there are infinitely many L^2-stable systems with minimal total length for approximation orders 1, 2, and 3.

2. Preliminaries

In this section, we introduce basic notions on biorthogonal multiwavelet systems and interpolating properties. A vector function $\mathbf{f} \in L^2(\mathbb{R})^2$ is said to be L^2-*stable* if there are constants $0 < A \leq B < \infty$ such that

$$A \sum_{k=-\infty}^{\infty} \mathbf{b}_k^* \mathbf{b}_k \leq \left\| \sum_{k=-\infty}^{\infty} \mathbf{b}_k^* \mathbf{f}(\cdot - k) \right\|_{L^2}^2 \leq B \sum_{k=-\infty}^{\infty} \mathbf{b}_k^* \mathbf{b}_k$$

holds for any vector sequence $\{\mathbf{b}_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})^2$ [11].

A vector function $\mathbf{\Phi} = (\phi_1, \phi_2)^T$ is said to be a *multiscaling function of multiplicity* 2 if $\mathbf{\Phi}$ satisfies a matrix refinement equation

$$\mathbf{\Phi}(t) = 2\sum_{\ell\in\mathbb{Z}} \mathbf{P}_\ell \mathbf{\Phi}(2t-\ell) \tag{2.1}$$

for some 2×2 real matrices $\{\mathbf{P}_\ell\}$. The Fourier transform is defined by $\hat{\mathbf{\Phi}} := \left(\hat{\phi}_1, \hat{\phi}_2\right)^T$, where $\hat{\phi}_j(\omega) := \int_{-\infty}^{\infty} \phi_j(t)\, e^{-i\omega t} dt$ with $i = \sqrt{-1}$ for $j = 1, 2$. By taking Fourier transform, (2.1) leads to $\hat{\mathbf{\Phi}}(\omega) = \mathbf{P}\left(\frac{\omega}{2}\right)\hat{\mathbf{\Phi}}\left(\frac{\omega}{2}\right)$. Here, $\mathbf{P}(\omega) := \sum_{k\in\mathbb{Z}} \mathbf{P}_k e^{-i\omega k}$ is called the *two-scale matrix symbol* or the *refinement mask* corresponding to $\mathbf{\Phi}$.

Let $\tilde{\mathbf{\Phi}}$ be a multiscaling function such that for some 2×2 real matrices $\left\{\tilde{\mathbf{P}}_\ell\right\}$

$$\tilde{\mathbf{\Phi}}(t) = 2\sum_{\ell\in\mathbb{Z}} \tilde{\mathbf{P}}_\ell \tilde{\mathbf{\Phi}}(2t-\ell).$$

A pair of multiscaling functions $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ is said to be *biorthogonal* if

$$\left\langle \mathbf{\Phi}(\cdot), \tilde{\mathbf{\Phi}}(\cdot - k)\right\rangle = \delta_{k,0}\mathbf{I}_2,$$

where $\langle \mathbf{f}, \mathbf{g}\rangle := \int_{-\infty}^{\infty} \mathbf{f}(t)\,\mathbf{g}^*(t)\,dt$ and $\delta_{k,\ell}$ denotes the *Kronecker δ-symbol.*

Here and in what follows, $\mathbf{I}_2$ and $\mathbf{0}$ denote the 2×2 identity and zero matrices, respectively. Consider multiwavelets $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$ associated with $\{\mathbf{\Phi}, \tilde{\mathbf{\Phi}}\}$ given by

$$\mathbf{\Psi}(t) = 2\sum_{\ell\in\mathbb{Z}} \mathbf{Q}_\ell \mathbf{\Phi}(2t-\ell) \quad \text{and} \quad \tilde{\mathbf{\Psi}}(t) = 2\sum_{\ell\in\mathbb{Z}} \tilde{\mathbf{Q}}_\ell \tilde{\mathbf{\Phi}}(2t-\ell)$$

with 2×2 real matrices $\{\mathbf{Q}_\ell\}$ and $\left\{\tilde{\mathbf{Q}}_\ell\right\}$. A pair of multiwavelets $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$ is said to be *biorthogonal* if for all $k\in\mathbb{Z}$

$$\begin{aligned} \left\langle \mathbf{\Psi}(\cdot), \tilde{\mathbf{\Psi}}(\cdot - k)\right\rangle &= \delta_{k,0}\mathbf{I}_2 \quad \text{and} \\ \left\langle \mathbf{\Phi}(\cdot), \tilde{\mathbf{\Psi}}(\cdot - k)\right\rangle = \left\langle \tilde{\mathbf{\Phi}}(\cdot), \mathbf{\Psi}(\cdot - k)\right\rangle &= \mathbf{0}. \end{aligned}$$

For a given mask $\mathbf{P}(\omega)$, the *transition operator* $T : (L^2_{2\pi})^{2\times 2} \longrightarrow (L^2_{2\pi})^{2\times 2}$ acting on 2×2 matrix $\mathbf{H}(\omega)$ with 2π-periodic square integrable entries in [5] is defined by

$$T\mathbf{H}(2\omega) := \mathbf{P}(\omega)\,\mathbf{H}(\omega)\,\mathbf{P}^*(\omega) + \mathbf{P}(\omega+\pi)\,\mathbf{H}(\omega+\pi)\,\mathbf{P}^*(\omega+\pi).$$

Let $\mathbf{H}_L$ be the space of 2×2 matrices of trigonometric polynomials with degree at most L. A matrix is said to satisfy *Condition E* if it has a simple eigenvalue 1 and the moduli of all other eigenvalues are less than 1. The *Kronecker product* $\mathbf{X} \otimes \mathbf{Y}$ of $\mathbf{X} = (x_{jk})_{j,k=1}^2 \in \mathbb{C}^{2\times 2}$ and $\mathbf{Y} \in \mathbb{C}^{2\times 2}$ [4] is defined by

$$\mathbf{X} \otimes \mathbf{Y} = \begin{pmatrix} x_{11}\mathbf{Y} & x_{12}\mathbf{Y} \\ x_{21}\mathbf{Y} & x_{22}\mathbf{Y} \end{pmatrix}.$$

Since we assume that $\mathbf{\Phi}$ is compactly supported, we can write $\mathbf{\Phi}(t) = 2\sum_{\ell=L_1}^{L_2} \mathbf{P}_\ell \mathbf{\Phi}(2t-\ell)$ for some integer $L_1 < L_2$. Let $\mathbf{B}_n$ be the 4×4 matrix given by $\mathbf{B}_n := \sum_{\ell=0}^{L} \mathbf{P}_{\ell-n} \otimes \mathbf{P}_\ell$ for $L \geq L_2 - L_1$. Define the $4(2L+1) \times 4(2L+1)$ matrix

$$\mathbf{T} := (2\mathbf{B}_{2j-k})_{j,k=-L}^{L}. \tag{2.2}$$

Then $T : \mathbf{H}_L \longrightarrow \mathbf{H}_L$ can be represented by $\mathbf{T}$. Recall the following criterion for the L^2-stability.

Theorem 2.1. *(Plonka and Strela, [11]) The refinable function vector $\mathbf{\Phi}$ is L^2-stable if and only if its symbol $\mathbf{P}(0)$ satisfies Condition E, and the corresponding transition operator T (or $\mathbf{T}$) restricted to $\mathbf{H}_L$ satisfies Condition E, where the eigenmatrix corresponding to the eigenvalue 1 is positive definite for all $\omega \in \mathbb{R}$.*

A vector function $\mathbf{f}(t) = [f_1(t), f_2(t)]^T$ is said to be *interpolating* if it satisfies the condition $\left[\mathbf{f}(n), \mathbf{f}\left(n+\frac{1}{2}\right)\right] = \sqrt{2}\delta(n)\mathbf{I}_2$ for $n \in \mathbb{Z}$. If all of $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$, $\mathbf{\Psi}$, and $\tilde{\mathbf{\Psi}}$ in a biorthogonal multiwavelet system are interpolating, then this system is said to be *totally interpolating*. Recall the following lemma: Lemma 2.2 (Zhang et al. [17]).

Lemma 2.2. *If $\mathbf{\Phi}$ is an interpolating multiscaling function, then*

$$\mathbf{P}(\omega) = \begin{pmatrix} \frac{1}{2} & p_1(\omega) \\ \frac{1}{2}e^{-i\omega} & p_2(\omega) \end{pmatrix} \tag{2.3}$$

with $p_j(\omega) := \sum_{k\in\mathbb{Z}} c_k^{j,2} e^{-i\omega k}$ for some $c_k^{j,2} \in \mathbb{R}$ and $j = 1, 2$.

A filter is said to be a *finite impulse response(FIR) filter* or have *FIR property* if it has finite duration. The following theorem for FIR property and the biorthogonality was shown in [2, 17]:

Theorem 2.3. *Let $\{\mathbf{\Phi}, \tilde{\mathbf{\Phi}}\}$ be a biorthogonal pair of interpolating multiscaling functions with two-scale matrix symbols $\mathbf{P}(\omega)$ and $\tilde{\mathbf{P}}(\omega)$*

as in (2.3). If both $p_1(\omega)$ and $p_2(\omega)$ are FIR filters, then $\tilde{\mathbf{P}}(\omega)$ is an FIR filter if and only if

$$\sum_{k\in\mathbb{Z}} c^1_{2k+1} c^2_{2\ell-2k} - \sum_{k\in\mathbb{Z}} c^1_{2k} c^2_{2\ell-2k+1} = \frac{C}{2}\delta_{2\ell+1,-m'} \tag{2.4}$$

for some real constant $C \neq 0$ and some odd integer m'.

Using biorthogonality and FIR property, one can find $\tilde{\mathbf{P}}(\omega)$, $\mathbf{Q}(\omega)$, and $\tilde{\mathbf{Q}}(\omega)$ from a given $\mathbf{P}(\omega)$.

THEOREM 2.4. *Let $\left\{\mathbf{\Phi}, \tilde{\mathbf{\Phi}}, \mathbf{\Psi}, \tilde{\mathbf{\Psi}}\right\}$ be a totally interpolating biorthogonal multiwavelet system with FIR property. The refinement mask $\tilde{\mathbf{P}}(\omega)$ dual to $\mathbf{P}(\omega)$ can be obtained by*

$$\tilde{p}_1(\omega) = \frac{1}{2C} e^{im'\omega} p_2(-\omega+\pi) \quad \textit{and} \quad \tilde{p}_2(\omega) = -\frac{1}{2C} e^{im'\omega} p_1(-\omega+\pi).$$

The corresponding high-pass filters are

$$\mathbf{Q}(\omega) = \begin{pmatrix} \frac{1}{2} & -p_1(\omega) \\ \frac{1}{2}e^{-i\omega} & -p_2(\omega) \end{pmatrix} \quad \textit{and} \quad \tilde{\mathbf{Q}}(\omega) = \begin{pmatrix} \frac{1}{2} & -\tilde{p}_1(\omega) \\ \frac{1}{2}e^{-i\omega} & -\tilde{p}_2(\omega) \end{pmatrix}.$$

We say that a multiscaling function $\mathbf{\Phi}$ *provides approximation order* $M \geq 1$ if there exist vectors $\mathbf{y}^m_\ell \in \mathbb{R}^2$ such that $\sum_{\ell\in\mathbb{Z}} (\mathbf{y}^m_\ell)^T \mathbf{\Phi}(t-\ell) = t^m$ for all $t \in \mathbb{R}$ and $m = 0, \ldots, M-1$. The following theorem of approximation property based on biorthogonality and FIR property is known in [2, 10, 17].

THEOREM 2.5. *Let $\mathbf{\Phi}(t)$ and $\tilde{\mathbf{\Phi}}(t)$ be L^2-stable interpolating biorthogonal multiscaling functions. Assume that $\mathbf{P}(\omega)$ and $\tilde{\mathbf{P}}(\omega)$ are both FIR filters. Then both $\mathbf{\Phi}(t)$ and $\tilde{\mathbf{\Phi}}(t)$ provide approximation order M if and only if*

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} k^n c^1_{2k} &= \frac{(-1)^n\left(1 - 2C(2m'+1)^n\right)}{2^{2n+2}}, \\
\sum_{k\in\mathbb{Z}} k^n c^1_{2k+1} &= \frac{(-1)^n\left(3^n + 2C(2m'+3)^n\right)}{2^{2n+2}}, \\
\sum_{k\in\mathbb{Z}} k^n c^2_{2k} &= \frac{1 + 2C(1-2m')^n}{2^{2n+2}}, \quad \textit{and} \\
\sum_{k\in\mathbb{Z}} k^n c^2_{2k+1} &= \frac{(-1)^n\left(1 - 2C(2m'+1)^n\right)}{2^{2n+2}}
\end{aligned}$$

for some nonzero real constant C, some odd integer m', and $n = 0, \ldots, M-1$.

The *length* of a refinement mask $\mathbf{P}(\omega) = \sum_{\ell=L_1}^{L_2} \mathbf{P}_\ell e^{-i\omega\ell}$ (or multiscaling function $\mathbf{\Phi}(t) = 2\sum_{\ell=L_1}^{L_2} \mathbf{P}_\ell \mathbf{\Phi}(2t-\ell)$) means the number $L_2 - L_1 + 1$ with $L_1 \le L_2$ and $\mathbf{P}_{L_1} \neq \mathbf{0}, \mathbf{P}_{L_2} \neq \mathbf{0}$. Similarly, we say that a biorthogonal multiwavelet system $\{\mathbf{\Phi}, \tilde{\mathbf{\Phi}}, \mathbf{\Psi}, \tilde{\mathbf{\Psi}}\}$ has the *total length* $TL_2 - TL_1 + 1$ if TL_1 is the minimum value of their first indexes and TL_2 is the maximum value of their last indexes of all refinement masks in this system.

3. Existence of L^2-stable totally interpolating biorthogonal multiwavelet systems with minimal total length

The primary concern of this section is how many L^2-stable totally interpolating biorthogonal multiwavelet systems with minimal total length exist. We begin with a simple property on the total length of a totally interpolating biorthogonal multiwavelet systems.

LEMMA 3.1. *The total length of a totally interpolating FIR biorthogonal multiwavelet system $\{\mathbf{\Phi}, \tilde{\mathbf{\Phi}}, \mathbf{\Psi}, \tilde{\mathbf{\Psi}}\}$ is even.*

Proof. Since $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$ have the same support as $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$, respectively, we have only to consider the total length of $\{\mathbf{\Phi}, \tilde{\mathbf{\Phi}}\}$. By Lemma 2.2, we can write $\mathbf{P}(\omega) = \sum_{\ell=L_1}^{L_2} \mathbf{P}_\ell e^{-i\omega\ell}$ with $L_1 \le 0$ and $L_2 \ge 1$.

Since $\tilde{p}_1(\omega) = \frac{1}{2C} e^{im'\omega} p_2(-\omega+\pi)$ and $\tilde{p}_2(\omega) = -\frac{1}{2C} e^{im'\omega} p_1(-\omega+\pi)$ by Theorem 2.4, we have

$$\tilde{\mathbf{P}}(\omega) = \sum_{\ell=-L_2-m'}^{-L_1-m'} \tilde{\mathbf{P}}_\ell e^{-i\omega\ell}$$

for some odd integer m' and 2×2 real non-zero matrices $\tilde{\mathbf{P}}_\ell$. Therefore, we have $TL_1 = \min(L_1, -L_2 - m')$ and $TL_2 = \min(L_2, -L_1 - m')$. The total length is either $2L_2 + m' + 1$ or $-2L_1 - m' + 1$, which finishes the proof. $\square$

There is a family of such systems of total length 2 with approximation order 1 whose the associated filters are follows:

$$\begin{aligned} p_1(\omega) &= \left(\frac{1}{4}-\frac{1}{2}C\right)+\left(\frac{1}{4}+\frac{1}{2}C\right)e^{-i\omega} \quad \text{and} \\ p_2(\omega) &= \left(\frac{1}{4}+\frac{1}{2}C\right)+\left(\frac{1}{4}-\frac{1}{2}C\right)e^{-i\omega}. \end{aligned}$$

However, one can check that each system is discontinuous.

THEOREM 3.2. *There are infinitely many L^2-stable totally interpolating biorthogonal multiwavelet systems with the minimal total length* 4 *and the approximation order* 1.

Proof. We can find a totally interpolating multiwavelet system with the following filters: $p_1(\omega)=\sum_{k=-1}^{2} c_k^1 e^{-i\omega k}$ and $p_2(\omega)=\sum_{k=-1}^{2} c_k^2 e^{-i\omega k}$, where

$$\begin{aligned} c_{-1}^1 &= \frac{1}{16}\left(\frac{16C^2+18C-9}{2C-3}\right), & c_0^1 &= \frac{3}{16}-\frac{1}{2}C, \\ c_1^1 &= -\frac{1}{16}\left(\frac{3+34C}{2C-3}\right), & c_2^1 &= \frac{1}{16}, \\ c_{-1}^2 &= -\frac{1}{8}\left(\frac{8C^2+18C+9}{2C-3}\right), & c_0^2 &= \frac{3}{8}+\frac{1}{2}C, \\ c_1^2 &= \frac{1}{8}\left(\frac{3+34C}{2C-3}\right), & c_2^2 &= -\frac{1}{8} \end{aligned}$$

for some nonzero real constant C. When $C=-\frac{1}{2}$, we have

$$\begin{aligned} p_1(\omega) &= \frac{7}{32}e^{i\omega}+\frac{7}{16}-\frac{7}{32}e^{-i\omega}+\frac{1}{16}e^{-2i\omega}, \\ p_2(\omega) &= \frac{1}{16}e^{i\omega}+\frac{1}{8}+\frac{7}{16}e^{-i\omega}-\frac{1}{8}e^{-2i\omega}, \\ \tilde{p}_1(\omega) &= \frac{1}{8}e^{i\omega}+\frac{7}{16}-\frac{1}{8}e^{-i\omega}+\frac{1}{16}e^{-2i\omega}, \quad \text{and} \\ \tilde{p}_2(\omega) &= \frac{1}{16}e^{i\omega}+\frac{7}{32}+\frac{7}{16}e^{-i\omega}-\frac{7}{32}e^{-2i\omega}. \end{aligned}$$

Let $\mathbf{T}_{\mathbf{\Phi},C}$ be the set of transition matrix as in (2.2) and $\mathcal{E}_{\mathbf{\Phi},C}$ be the set of absolute values of eigenvalues of $\mathbf{T}_{\mathbf{\Phi},C}$ associated with $\mathbf{\Phi}$ and some nonzero real constant C. The sets $\mathcal{E}_{\mathbf{\Phi},-\frac{1}{2}}$ and $\mathcal{E}_{\tilde{\mathbf{\Phi}},-\frac{1}{2}}$ are given by

$$\begin{aligned}\mathcal{E}_{\mathbf{\Phi},-\frac{1}{2}} &\approx \Big\{1, \frac{1}{2}, 0.4206, 0.1866(2), 0.1601(2), 0.1253(2), 0.1169(2),\\ &\quad 0.1026, 0.0881, 0.0646, 0.0489, \frac{1}{64}(2), 0.0110, 0(10)\Big\},\\ \mathcal{E}_{\tilde{\mathbf{\Phi}},-\frac{1}{2}} &\approx \Big\{1, \frac{1}{2}, 0.3312, 0.1497(2), 0.1234(2), 0.1209, 0.1084, 0.0830(2),\\ &\quad 0.0657(2), 0.0517, 0.0413, 0.0273(2), 0.0055, 0(10)\Big\},\end{aligned}$$

where $\lambda(n)$ means that the multiplicity of λ is n. One can easily verify Condition E. If we let $\mathbf{H}_C(\omega)$ be suitably chosen eigenmatrix of $\mathbf{T}_{\mathbf{\Phi},C}$ corresponding to simple eigenvalue 1, then eigenmatrices $\mathbf{H}_{-\frac{1}{2}}(\omega)$ of $\mathbf{T}_{\mathbf{\Phi},-\frac{1}{2}}$ and $\tilde{\mathbf{H}}_{-\frac{1}{2}}(\omega)$ of $\mathbf{T}_{\tilde{\mathbf{\Phi}},-\frac{1}{2}}$ can be chosen as in TABLE 1, where $[\mathbf{H}(\omega)]_{j,k} := \sum_{\ell\in\mathbb{Z}} h_\ell^{j,k} e^{-i\omega\ell}$ with some $h_\ell^{j,k} \in \mathbb{R}$ for $j,k = 1,2$. One can check that the minima of $[\mathbf{H}_{-\frac{1}{2}}(\omega))]_{1,1}$, $\det(\mathbf{H}_{-\frac{1}{2}}(\omega))$, $[\tilde{\mathbf{H}}_{-\frac{1}{2}}(\omega)]_{1,1}$, and $\det(\tilde{\mathbf{H}}_{-\frac{1}{2}}(\omega))$ are all positive. Furthermore, one can choose eigenmatrices that depend smoothly on C, with the aid of CAS packages such as Maple. Together with smooth dependence of eigenvalues on C, we can conclude L^2-stability of the system for C sufficiently close to $-\frac{1}{2}$. Matrix representations of $\mathbf{T}_{\mathbf{\Phi},C}|_{\mathbb{H}_L^0}$ and $\mathbf{T}_{\tilde{\mathbf{\Phi}},C}\big|_{\mathbb{H}_L^0}$ that depend smoothly on C can be obtained. Sobolev exponents of $\mathbf{\Phi}_C$ and $\tilde{\mathbf{\Phi}}_C$ corresponding to $C = -\frac{1}{2}$ are 0.6248 and 0.7971, respectively. This implies the continuity of $\mathbf{\Phi}_C$ and $\tilde{\mathbf{\Phi}}_C$ for C sufficiently close to $-\frac{1}{2}$. $\square$

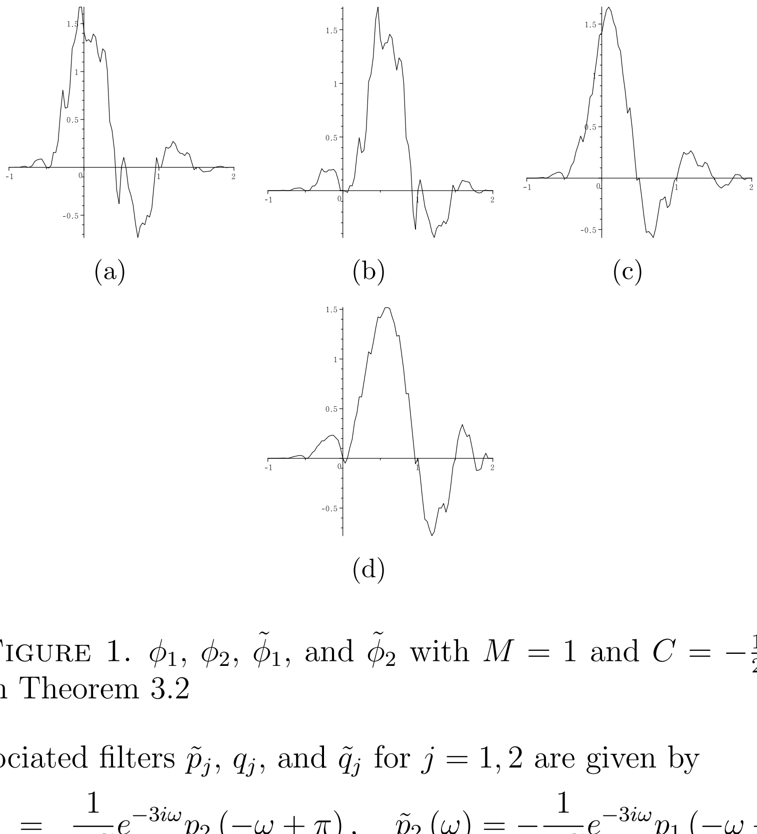
Graphs of the system $\{\mathbf{\Phi}_{-\frac{1}{2}}, \tilde{\mathbf{\Phi}}_{-\frac{1}{2}}, \mathbf{\Psi}_{-\frac{1}{2}}, \tilde{\mathbf{\Psi}}_{-\frac{1}{2}}\}$ are FIGURES 1 and 2.

There are only two candidates of totally interpolating biorthogonal multiwavelet systems of approximation order 2 with total length 4. Such systems are obtained with $m' = -3$ and $C = -\frac{7}{4} \pm \frac{3}{4}\sqrt{5}$ and they are dual to each other. The filters are given by $p_j(\omega) = \sum_{k=0}^{3} c_k^j e^{-i\omega k}$ for $j = 1,2$, where

$$\begin{aligned}&c_0^1 = \frac{5}{16} + \frac{1}{8}C,\quad c_1^1 = \frac{7}{16} + \frac{1}{8}C,\quad c_2^1 = -\frac{1}{16} - \frac{5}{8}C,\quad c_3^1 = -\frac{3}{16} + \frac{3}{8}C,\\ &c_0^2 = \frac{3}{16} - \frac{3}{8}C,\quad c_1^2 = \frac{5}{16} + \frac{1}{8}C,\quad c_2^2 = \frac{1}{16} + \frac{7}{8}C,\quad \text{and}\quad c_3^2 = -\frac{1}{16} - \frac{5}{8}C.\end{aligned}$$

TABLE 1. $\mathbf{H}-\frac{1}{2}(\omega)$ and $\tilde{\mathbf{H}}_{-\frac{1}{2}}(\omega)$ for $M = 1$ and $m' = -1$ in Theorem 3.2

ℓ	$h_\ell^{1,1}$	$h_\ell^{1,2}$	$h_\ell^{2,1}$	$h_\ell^{2,2}$
-2	$\frac{1799}{1116}$	$-\frac{28}{9}$	$\frac{16}{31}$	-1
-1	$-\frac{977179}{2108}$	$\frac{11277083}{18972}$	$-\frac{24411797}{531216}$	$\frac{4003}{62}$
0	$\frac{275949344}{33201}$	$-\frac{7520461}{18972}$	$-\frac{7520461}{18972}$	$\frac{241054435}{33201}$
1	$-\frac{977179}{2108}$	$-\frac{24411797}{531216}$	$\frac{11277083}{18972}$	$\frac{4003}{62}$
2	$\frac{1799}{1116}$	$\frac{16}{31}$	$-\frac{28}{9}$	-1
ℓ	$\tilde{h}_\ell^{1,1}$	$\tilde{h}_\ell^{1,2}$	$\tilde{h}_\ell^{2,1}$	$\tilde{h}_\ell^{2,2}$
-2	$-\frac{1028}{2145}$	$\frac{64}{39}$	$-\frac{16}{55}$	1
-1	$\frac{27456747}{204490}$	$-\frac{241894723}{613470}$	$\frac{3644534}{306735}$	$-\frac{4261}{110}$
0	$\frac{899952974}{306735}$	$\frac{28889194}{306735}$	$\frac{28889194}{306735}$	$\frac{335059778}{102245}$
1	$\frac{27456747}{204490}$	$\frac{3644534}{306735}$	$-\frac{241894723}{613470}$	$-\frac{4261}{110}$
2	$-\frac{1028}{2145}$	$-\frac{16}{55}$	$\frac{64}{39}$	1



(a) (b) (c)

(d)

FIGURE 1. ϕ_1, ϕ_2, $\tilde{\phi}_1$, and $\tilde{\phi}_2$ with $M = 1$ and $C = -\frac{1}{2}$ in Theorem 3.2

The associated filters $\tilde{p}_j$, q_j, and $\tilde{q}_j$ for $j = 1, 2$ are given by

$$\begin{aligned} \tilde{p}_1(\omega) &= \frac{1}{2C} e^{-3i\omega} p_2(-\omega + \pi), \quad \tilde{p}_2(\omega) = -\frac{1}{2C} e^{-3i\omega} p_1(-\omega + \pi), \\ q_j(\omega) &= -p_j(\omega), \quad \text{and} \quad \tilde{q}_j(\omega) = -\tilde{p}_j(\omega). \end{aligned}$$

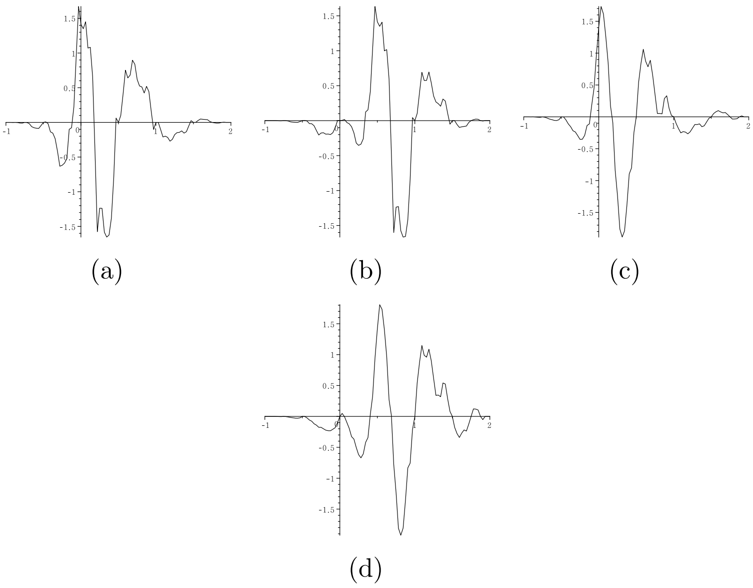


FIGURE 2. ψ_1, ψ_2, $\tilde{\psi}_1$, and $\tilde{\psi}_2$ with $M = 1$ and $C = -\frac{1}{2}$ in Theorem 3.2

Although the transition matrix $\mathbf{T}_{\mathbf{\Phi},C}$ for $C = -\frac{7}{4} + \frac{3}{4}\sqrt{5}$ satisfies condition E, that of $\tilde{\mathbf{\Phi}}_C$ has spectral radius approximately 36.74. Therefore, the system is not L^2-stable.

In view of Lemma 3.1, one can deduce that the total length of a totally interpolating L^2-stable biorthogonal multiwavelet system is at least 6. On the other hand, various examples of L^2-stable totally interpolating biorthogonal multiwavelet systems of approximation order 2 with the minimal total length 6 have been known in [2, 17] so far.

THEOREM 3.3. *There are infinitely many L^2-stable totally interpolating biorthogonal multiwavelet systems with the minimal total length* 6 *and the approximation order* 2.

Proof. For $j = 1, 2$, one can find a one-parameter family of totally interpolating biorthogonal multiwavelet systems based on filter components

$$p_j(\omega) = \sum_{k=-2}^{2} c_k^j e^{-i\omega k} \quad \text{and} \quad \tilde{p}_j(\omega) = \sum_{k=-1}^{3} \tilde{c}_k^j e^{-i\omega k},$$

where

$$
\begin{aligned}
c^1_{-2} &= -\frac{\alpha}{32\gamma}, \quad c^1_{-1} = \left(\frac{3}{16} + \frac{1}{8}C\right), \quad c^1_0 = \left(\frac{\alpha}{16\gamma} + \frac{5}{16} - \frac{3}{8}C\right),\\
c^1_1 &= \left(\frac{1}{16} + \frac{3}{8}C\right), \quad c^1_2 = -\left(\frac{\alpha}{32\gamma} + \frac{1}{16} + \frac{1}{8}C\right),\\
c^2_{-2} &= -\frac{\beta}{32\gamma}, \quad c^2_{-1} = \left(\frac{1}{16} + \frac{1}{8}C\right), \quad c^2_0 = \left(\frac{\beta}{16\gamma} + \frac{3}{16} + \frac{1}{8}C\right),\\
c^2_1 &= \left(\frac{3}{16} - \frac{5}{8}C\right), \quad c^2_2 = -\left(\frac{\beta}{32\gamma} - \frac{1}{16} - \frac{3}{8}C\right),\\
\tilde{c}^1_\ell &= \frac{1}{2C}(-1)^{\ell-1}c^2_{1-\ell}, \quad \text{and} \quad \tilde{c}^2_\ell = -\frac{1}{2C}(-1)^{\ell-1}c^1_{1-\ell}
\end{aligned}
$$

with $\alpha := 8C^3 + 16C^2 + 8C + 3$, $\beta := 8C^3 + 8C^2 + 4C + 1$, and $\gamma := 4C^2 + 4C - 1$ for a nonzero real constant C and $\ell = -1, \dots, 3$. When $C = -\frac{2}{5}$, we get

$$
\begin{aligned}
p_1(\omega) &= \tfrac{33}{1120}e^{2i\omega} + \tfrac{11}{80}e^{i\omega} + \tfrac{113}{280} - \tfrac{7}{80}e^{-i\omega} + \tfrac{19}{1120}e^{-2i\omega},\\
p_2(\omega) &= \tfrac{3}{1120}e^{2i\omega} + \tfrac{1}{80}e^{i\omega} + \tfrac{37}{280} + \tfrac{7}{16}e^{-i\omega} - \tfrac{19}{224}e^{-2i\omega},\\
\tilde{p}_1(\omega) &= \tfrac{95}{896}e^{i\omega} + \tfrac{35}{64} - \tfrac{37}{224}e^{-i\omega} + \tfrac{1}{64}e^{-2i\omega} - \tfrac{3}{896}e^{-3\omega},\\
\tilde{p}_2(\omega) &= \tfrac{19}{896}e^{i\omega} + \tfrac{7}{64} + \tfrac{113}{224}e^{-i\omega} - \tfrac{11}{64}e^{-2i\omega} + \tfrac{33}{896}e^{-3\omega}.
\end{aligned}
$$

The sets $\mathcal{E}_{\mathbf{\Phi},-\frac{2}{5}}$ and $\mathcal{E}_{\tilde{\mathbf{\Phi}},-\frac{2}{5}}$ of absolute values of eigenvalues of $\mathbf{T}_{\mathbf{\Phi},-\frac{2}{5}}$ and $\mathbf{T}_{\tilde{\mathbf{\Phi}},-\frac{2}{5}}$, respectively are given by

$$
\begin{aligned}
\mathcal{E}_{\mathbf{\Phi}} \approx &\Big\{1, \frac{1}{2}, \frac{1}{4}, 0.13603, \frac{1}{8}, 0.07229, 0.07048, 0.06604, 0.05612, 0.03022,\\
&0.02932, 0.01892, 0.01530, 0.01310, 0.01236, 0.01143, 0.00068,\\
&0.00045(2), 0.00038, 0(16)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_{\tilde{\mathbf{\Phi}}} \approx &\Big\{1, \frac{1}{2}, \frac{1}{4}, 0.12799, \frac{1}{8}, 0.05918, 0.05333(2), 0.05076(2), 0.04213,\\
&0.02347, 0.02251(2), 0.019266, 0.01844, 0.00938, 0.00314,\\
&0.00156(2), 0(16)\Big\},
\end{aligned}
$$

from which one can easily verify Condition E. One can check that the minima of $[\mathbf{H}_{-\frac{2}{5}}(\omega))]_{1,1}$, $\det(\mathbf{H}_{-\frac{2}{5}}(\omega))$, $[\tilde{\mathbf{H}}_{-\frac{2}{5}}(\omega)]_{1,1}$, and $\det(\tilde{\mathbf{H}}_{-\frac{2}{5}}(\omega))$ are all positive. Furthermore, one can choose eigenmatrices that depend smoothly on C. Together with smooth dependence of eigenvalues on

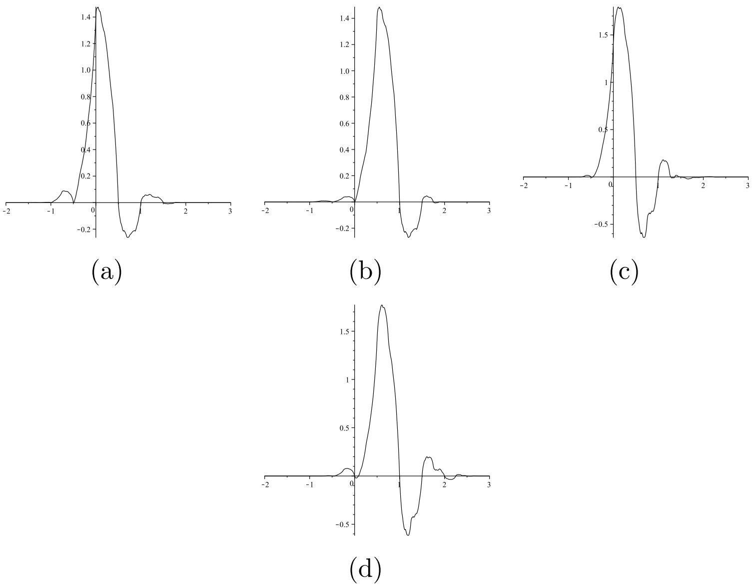


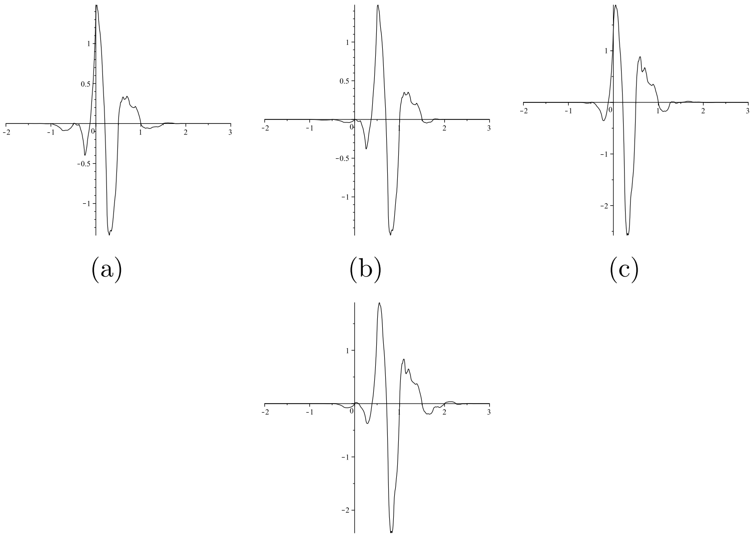
FIGURE 3. ϕ_1, ϕ_2, $\tilde{\phi}_1$, and $\tilde{\phi}_2$ with $M = 2$ and $C = -\frac{2}{5}$ in Theorem 3.3

C, we can conclude L^2-stability of the system for C sufficiently close to $-\frac{2}{5}$. Matrix representations of $\mathbf{T}_{\mathbf{\Phi},C}|_{\mathbb{H}_L^0}$ and $\mathbf{T}_{\tilde{\mathbf{\Phi}},C}\big|_{\mathbb{H}_L^0}$ that depend smoothly on C can be obtained. Sobolev exponents of $\mathbf{\Phi}_{-\frac{2}{5}}$ and $\tilde{\mathbf{\Phi}}_{-\frac{2}{5}}$ are approximately are 0.93898 and 0.98294, respectively. This implies the continuity of $\mathbf{\Phi}_C$ and $\tilde{\mathbf{\Phi}}_C$ for C sufficiently close to $-\frac{2}{5}$. $\square$

Graphs of the system $\{\mathbf{\Phi}_{-\frac{2}{5}}, \tilde{\mathbf{\Phi}}_{-\frac{2}{5}}, \mathbf{\Psi}_{-\frac{2}{5}}, \tilde{\mathbf{\Psi}}_{-\frac{2}{5}}\}$ are FIGURES 3 and 4.

THEOREM 3.4. *There are infinitely many L^2-stable totally interpolating biorthogonal multiwavelet systems with the minimal total length* 8 *and the approximation order* 3.

Proof. A one-parameter family of totally interpolating biorthogonal multiwavelet systems with approximation order 3 of total length 8 can be constructed with $m' = -1$. If we let κ be a root of $Z^2 + (384C^3 - 264C^2 - 156C)Z + 16128C^6 + 28800C^5 - 23616C^4 - 37152C^3 + 3312C^2 + 4968C - 324 = 0$ and $\mu := 4C^2 - 8C + 1$, then the systems are determined



(a) (b) (c)

(d)

FIGURE 4. ψ_1, ψ_2, $\tilde{\psi}_1$, and $\tilde{\psi}_2$ with $M = 2$ and $C = -\frac{2}{5}$ in Theorem 3.3

by the filter components $p_j(\omega) = \sum_{k=-3}^{4} c_k^j e^{-i\omega k}$ for $j = 1, 2$, where

$$\begin{aligned}
c_{-3}^1 &= \frac{\kappa}{1024\mu}, \\
c_{-2}^1 &= \frac{5}{128} + \frac{3}{64}C - \frac{24C^3 - 76C^2 - 64C - 7}{512\mu}, \\
c_{-1}^1 &= \frac{21}{128} + \frac{5}{64}C - \frac{3\kappa}{1024\mu}, \\
c_0^1 &= \frac{15}{64} - \frac{15}{32}C + \frac{3\,(24C^3 - 76C^2 - 64C - 7)}{512\mu}, \\
c_1^1 &= \frac{7}{64} + \frac{15}{32}C + \frac{3\kappa}{1024\mu}, \\
c_2^1 &= -\frac{3}{128} - \frac{5}{64}C - \frac{3\,(24C^3 - 76C^2 - 64C - 7)}{512\mu}, \\
c_3^1 &= -\frac{3}{128} - \frac{3}{64}C - \frac{\kappa}{1024\mu}, \\
c_4^1 &= \frac{(24C^3 - 76C^2 - 64C - 7)}{512\mu},
\end{aligned}$$

and

$$\begin{aligned}
c^2_{-3} &= \frac{56C^3+128C^2+38C-3}{512\mu},\\
c^2_{-2} &= -\frac{3}{128}-\frac{3}{64}C-\frac{192C^3+24C^2-12C-24+\kappa}{1024\mu},\\
c^2_{-1} &= \frac{5}{128}+\frac{3}{64}C-\frac{3\,(56C^3+128C^2+38C-3)}{512\mu},\\
c^2_{0} &= \frac{15}{64}+\frac{7}{32}C+\frac{3\,(192C^3+24C^2-12C-24+\kappa)}{1024\mu},\\
c^2_{1} &= \frac{15}{64}-\frac{15}{32}C+\frac{3\,(56C^3+128C^2+38C-3)}{512\mu},\\
c^2_{2} &= \frac{5}{128}+\frac{21}{64}C-\frac{3\,(192C^3+24C^2-12C-24+\kappa)}{1024\mu},\\
c^2_{3} &= -\frac{3}{128}-\frac{5}{64}C-\frac{56C^3+128C^2+38C-3}{512\mu},\\
c^2_{4} &= \frac{192C^3+24C^2-12C-24+\kappa}{1024\mu}.
\end{aligned}$$

Recall that the filter components $\tilde{p}_1(\omega)$ and $\tilde{p}_2(\omega)$ are determined by $\tilde{p}_1(\omega)=\frac{1}{2C}e^{-i\omega}p_2(-\omega+\pi)$ and $\tilde{p}_2(\omega)=-\frac{1}{2C}e^{-i\omega}p_1(-\omega+\pi)$.

We will analyze the filters corresponding to $C=-\frac{3}{4}$ and $\alpha=\frac{45}{4}$. We get

$$p_1(\omega)=\sum_{k=-3}^{4}c^1_k e^{-i\omega k}\quad\text{and}\quad p_2(\omega)=\sum_{k=-3}^{4}c^2_k e^{-i\omega k},$$

where

$$\begin{aligned}
c^1_{-3} &= \frac{45}{37888},\quad c^1_{-2}=\frac{243}{37888},\quad c^1_{-1}=\frac{3861}{37888},\quad c^1_{0}=\frac{21915}{37888},\\
c^1_{1} &= -\frac{9041}{37888},\quad c^1_{2}=\frac{1617}{37888},\quad c^1_{3}=\frac{399}{37888},\quad c^1_{4}=-\frac{95}{37888},\\
c^2_{-3} &= \frac{135}{37888},\quad c^2_{-2}=\frac{729}{37888},\quad c^2_{-1}=-\frac{257}{37888},\quad c^2_{0}=\frac{1809}{37888},\\
c^2_{1} &= \frac{22605}{37888},\quad c^2_{2}=-\frac{6989}{37888},\quad c^2_{3}=\frac{1197}{37888},\quad\text{and}\quad c^2_{4}=-\frac{285}{37888}.
\end{aligned}$$

The sets $\mathcal{E}_{\mathbf{\Phi},-\frac{3}{4}}$ and $\mathcal{E}_{\tilde{\mathbf{\Phi}},-\frac{3}{4}}$ of absolute values of eigenvalues of $\mathbf{T}_{\mathbf{\Phi},-\frac{3}{4}}$ and $\mathbf{T}_{\tilde{\mathbf{\Phi}},-\frac{3}{4}}$, respectively are given by

$$\begin{aligned}
\mathcal{E}_{\mathbf{\Phi},-\frac{3}{4}} &\approx \Big\{1, \frac{1}{2}, 0.324059, \frac{1}{4}, 0.226937, 0.199128, \frac{1}{8}, 0.069142(2),\\
&\quad 0.065385(2), \frac{1}{16}, 0.047648, 0.032906(2), \frac{1}{32}, 0.028756(2),\\
&\quad 0.019578(2), 0.004469(2), 0.003704(2), 0.002739(2),\\
&\quad 0.002077(2), 0.001718, 0.001697, 0.000260(2), 0.000257(2),\\
&\quad 0.000075(2), 0(24)\Big\},\\
\mathcal{E}_{\tilde{\mathbf{\Phi}},-\frac{3}{4}} &\approx \Big\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 0.096571, \frac{1}{16}, 0.049996(2), 0.048366, 0.034076(2),\\
&\quad \frac{1}{32}, 0.022093, 0.020055, 0.019107, 0.015869, 0.011817(2),\\
&\quad 0.008387(2), 0.005798, 0.004106, 0.003879, 0.003698,\\
&\quad 0.003698, 0.002911(2), 0.002187, 0.000430, 0.000423,\\
&\quad 0.000312(2), 0.000293(2), 0.000043(2), 0(24)\Big\},
\end{aligned}$$

which implies Condition E for $\mathbf{T}_{\mathbf{\Phi},-\frac{3}{4}}$ and $\mathbf{T}_{\tilde{\mathbf{\Phi}},-\frac{3}{4}}$. One can check that $\mathbf{H}_{-\frac{3}{4}}(\omega)$ and $\tilde{\mathbf{H}}_{-\frac{3}{4}}(\omega)$ are positive definite. Sobolev exponents of $\mathbf{\Phi}_{-\frac{3}{4}}$ and $\tilde{\mathbf{\Phi}}_{-\frac{3}{4}}$ are 1.2639 and 1.3791, respectively. As in the proofs of previous theorems, one can verify the L^2-stability. □

Graphs of the system $\{\mathbf{\Phi}_{-\frac{3}{4}}, \tilde{\mathbf{\Phi}}_{-\frac{3}{4}}, \mathbf{\Psi}_{-\frac{3}{4}}, \tilde{\mathbf{\Psi}}_{-\frac{3}{4}}\}$ are FIGURES 5 and 6.

Acknowledgements

This paper was completed with Ajou University research fellowship of 2011.

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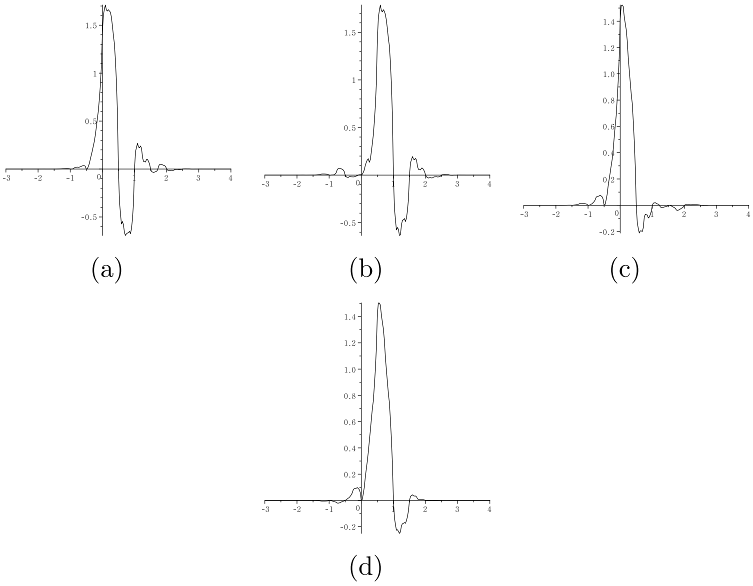

FIGURE 5. ϕ_1, ϕ_2, $\tilde{\phi}_1$, and $\tilde{\phi}_2$ with $M = 3$ and $C = -\frac{3}{4}$ in Theorem 3.4

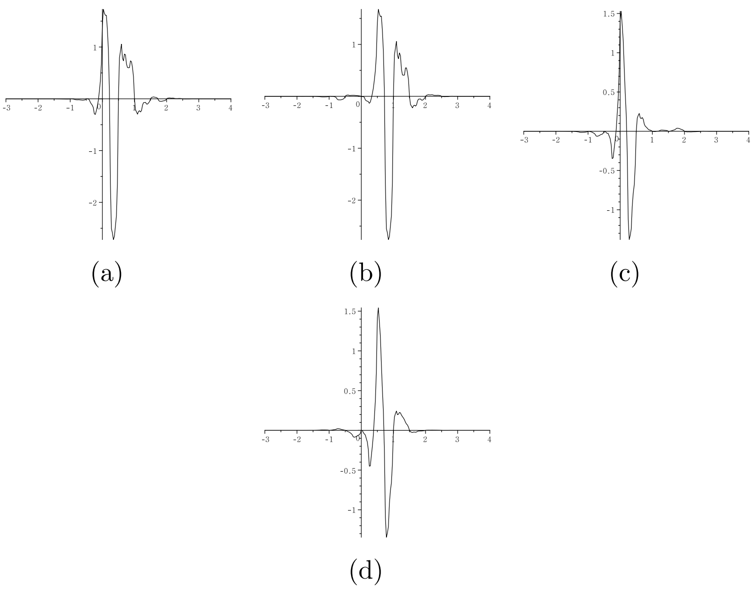

FIGURE 6. ψ_1, ψ_2, $\tilde{\psi}_1$, and $\tilde{\psi}_2$ with $M = 3$ and $C = -\frac{3}{4}$ in Theorem 3.4

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