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SOME PROPERTIES OF SCHENSTED ALGORITHM USING VIENNOT'S GEOMETRIC INTERPRETATION

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Abstract. Schensted algorithm was first described in 1938 by Robinson [5], in a paper dealing with an attempt to prove the correctness of the Littlewood-Richardson rule. Schensted [9] rediscovered Schensted algorithm independently in 1961 and Viennot [12] gave a geometric interpretation for Schensted algorithm in 1977. In this paper we describe some properties of Schensted algorithm using Viennot's geometric interpretation.

1. Introduction

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of the nonnegative integer n, denoted $\lambda \vdash n$, so λ is a weakly decreasing sequence of positive integers summing to n. We will also let λ stand for the Ferrers diagram D_λ of λ written in English notation with λ_i nodes or cells in the ith row from the top.

Given $\lambda \vdash n$, a *standard Young tableau T of shape* λ is a filling of the diagram D_λ with positive integers $1, 2, \ldots, n$ such that rows and columns strictly increase. For example,

$$\begin{matrix} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & 7 & \end{matrix}$$

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is a standard Young tableau of the shape $(3, 2, 2)$.

There is a remarkable combinatorial correspondence associated with the theory of symmetric functions, called the Schensted algorithm.

THEOREM 1.1. (**Schensted algorithm**) *Let S_n be the symmetric group of degree n. Then there is a bijection*

$$\pi \mapsto (P, Q)$$

between permutations π of S_n and the set of all pairs $(P_\lambda\ ,\ Q_\lambda)$ of standard Young tableaux of the same shape λ, where $\lambda \vdash n$.

It was first described in 1938 by Robinson [5], in a paper dealing with an attempt to prove the correctness of the Littlewood-Richardson rule. Schensted algorithm was rediscovered independently by Schensted [9] in 1961, whose main objective was counting permutations with given lengths of their longest increasing and decreasing subsequences. Schensted correspondence about increasing and decreasing subsequences is extended by C. Greene [2], to give a direct interpretation of the shape of the standard Young tableaux corresponding to a permutation. The combinatorial significance of Schensted algorithm was indicated by Schützenberger [11], who introduced the evacuation algorithm. Knuth [4] gave a generalization of the Schensted algoritm, where standard Young tableaux are replaced by column strict tableaux, and permutations are replaced by multi-permutations. And he described conditions for two permutation to have the same P-tableaux under Schensted algorithm. In [12] Viennot gave a geometric interpretation for Schensted algorithm.

After Knuth generalized Schensted algorithm to column strict tableaux, various analogs of the Schented algorithm came: versions for rim hook tableaux [10], shifted tableaux ([6]), oscillating tableaux [1], skew tableaux [8], and shifted rim hook tableaux [3].

The bijection in Theorem 1.1 is denoted $\pi \leftrightarrow (P, Q)$ or $\pi \overset{[S]}{\longmapsto} (P(\pi), Q(\pi))$ and $P(\pi), Q(\pi)$ are called the P-tableau and Q-tableau of π, respectively. For example, if $\pi = 3\ a\ 7\ b\ 2\ 4\ d\ 5\ e\ 9\ 1\ 6\ c\ 8\ \in S_{14}$, then the P-tableau and Q-tableau of π are given as

$$\pi \overset{[S]}{\longmapsto} (P(\pi), Q(\pi)) = \begin{pmatrix} \begin{matrix} 1\,4\,5\,6\,8 \\ 2\,7\,9\,c\,e \\ 3\,b\,d \\ a \end{matrix} & , & \begin{matrix} 1\,2\,4\,7\,9 \\ 3\,6\,8\,a\,d \\ 5\,c\,e \\ b \end{matrix} \end{pmatrix}$$

where $a = 10,\ b = 11,\ c = 12,\ d = 13$ and $e = 14$.

In this paper we describe the ways to find P-tableaux and Q-tableaux of permutations π^r, π^* and $\pi^\#$ without using Schützenberger's evacuation algorithm.

Section 2 gives Viennot's geometric interpretation for Schensted algorithm. In Section 3 we describe the ways to find P-tableaux and Q-tableaux of permutations π^r, π^* and $\pi^\#$ using Viennot's geometric interpretation for Schensted algorithm.

2. Geometric interpretation for Schensted algorithm

In this section we describe Viennot's geometric interpretation for Schensted algorithm. See [12] or [7] for further exposition.

Example 2.1. Let

$$\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7\ \in\ S_7$$

- Consider the first quadrant of the Cartesian plane. Given a permutation $\pi = x_1 x_2 \cdots x_n$, represent x_i by a box with coordinates (i, x_i). See the figure 1.
- Imagine a light shining from the origin so that each box casts a shadow with boundaries parallel to the coordinate axes. The shadow cast by the box at $(4, 6)$ looks like the figure 2.
- Consider those points of the permutation that are in the shadow of no other point. In this case $(1, 4)$, $(2, 2)$, and $(6, 1)$. The first shadow line, L_1, is the boundary of the combined shadows of these boxes. In the figure 3, the appropriate line has been thickened. Note that this is a broken line consisting of line segments and exactly one horizontal and one vertical ray. To form the second shadow line, L_2, one removes the boxes on the first shadow line and repeats this procedure.

Given a permutation displayed in the plane, we form its *shadow lines* L_1, L_2, $\ldots$ as follows. Assuming that $L_1, \ldots, L_{i-1}$ have been constructed, remove all boxes on these lines. Then L_i is the boundary of the shadow of the remaining boxes. The x-coordinate of L_i is

$$x_{L_i} = \text{ the } x\text{-coordinate of } L_i\text{'s vertical ray}$$

and the y-coordinate is

$$y_{L_i} = \text{ the } y\text{-coordinate of } L_i\text{'s horizontal ray}$$

The shadow lines make up the *shadow diagram* of π.

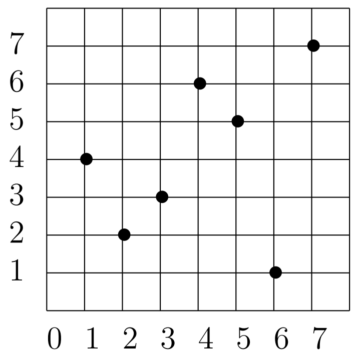


FIGURE 1. The coordinates (i, x_i) of π.

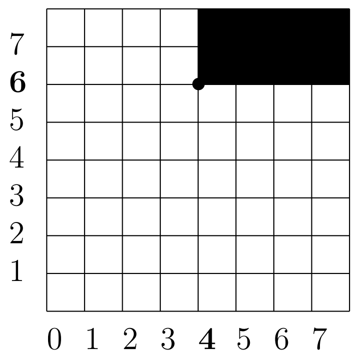


FIGURE 2. The shadow at $(4, 6)$.

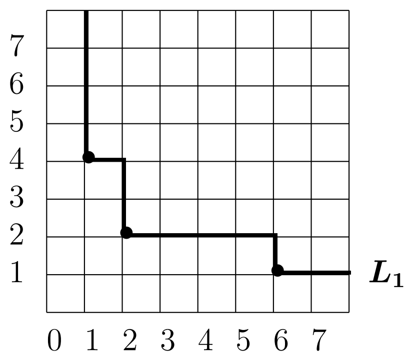


FIGURE 3. The first shadow line, L_1.

EXAMPLE 2.2. In the previous example, there are four shadow lines, and their x- and y-coordinates are shown above and to the left of the figure 4, respectively.

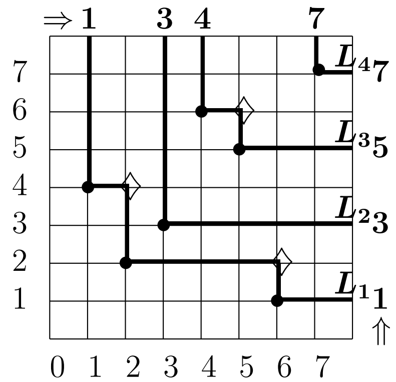


FIGURE 4. Four shadow lines for $\pi = 4236517$

Compare the coordinates of our shadow lines with the first rows of the tableaux

$$P(\pi) = \begin{matrix} \mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{7} \\ 2 & 6 & & \\ 4 & & & \end{matrix} \qquad \text{and} \qquad Q(\pi) = \begin{matrix} \mathbf{1} & \mathbf{3} & \mathbf{4} & \mathbf{7} \\ 2 & 5 & & \\ 6 & & & \end{matrix}$$

computed by Schensted Algorithm. It seems as if

$$P_{1,j} = y_{L_j} \quad \text{and} \quad Q_{1,j} = x_{L_j}$$

for all j.
In fact, even more is true. The boxes on line L_j are precisely those elements passing through the $(1, j)$ cell during the construction of P, as the next result shows.

LEMMA 2.3. *Let the shadow diagram of $\pi = x_1x_2\cdots x_n$ be constructed as before. Suppose the vertical line $x = k$ intersects i of the shadow lines. Let y_j be the y-coordinate of the lowest point of the intersection with L_j. Then the first row of the $P_k = P(x_1 \ldots x_k)$ is*

$$R_1 = y_1y_2\cdots y_i \tag{1}$$

Proof. : See [7]. □

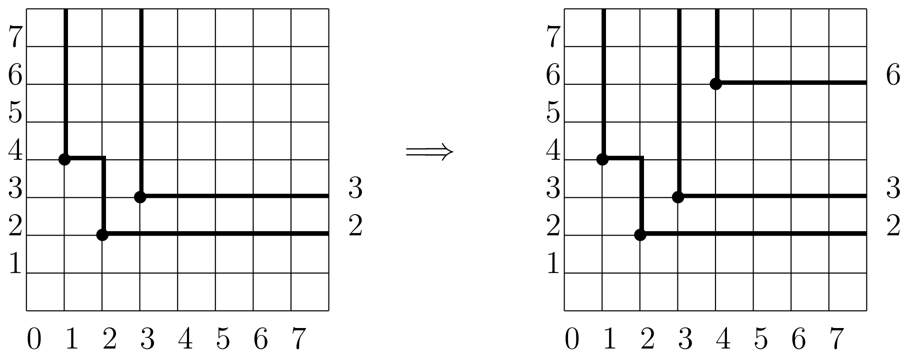
EXAMPLE 2.4. Let $\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$, then

$P_0 = \emptyset \qquad P_1 = 4 \qquad P_2 = \begin{array}{l} 2 \\ 4 \end{array} \qquad P_3 = \begin{array}{ll} 2 & 3 \\ 4 & \end{array}$

$P_4 = \begin{array}{lll} 2 & 3 & 6 \\ 4 & & \end{array} \qquad P_5 = \begin{array}{lll} 2 & 3 & 5 \\ 4 & 6 & \end{array} \qquad P_6 = \begin{array}{lll} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & & \end{array} \qquad P_7 = \begin{array}{llll} 1 & 3 & 5 & 7 \\ 2 & 6 & & \\ 4 & & & \end{array}$

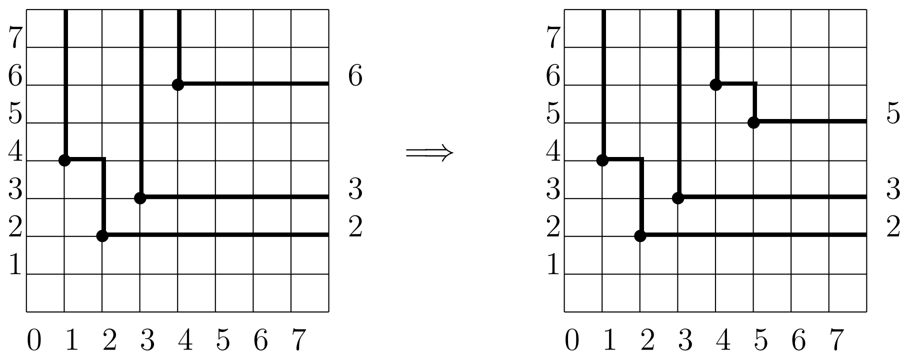
(Case 1) $k = 3$, $P_3 = P(x_1x_2x_3) = P(423) = \begin{array}{ll} 2 & 3 \\ 4 & \end{array}$, $\quad R_1 = y_1y_2$

Then $x_{k+1} = x_4 = 6 > 3 = y_2$ and so $(k+1, x_{k+1}) = (4, 6)$ starts a new shadow line. Hence, $y_3 = x_4 = 6$.



(Case 2) $k = 4$, $P_4 = P(4236) = \begin{array}{lll} 2 & 3 & 6 \\ 4 & & \end{array}$, $R_1 = 236 = y_1y_2y_3$

Then $y_1 < \cdots < y_{j-1} < x_5 = 5 < y_j < \cdots < y_3 = 6$ and so $(k+1, x_{k+1}) = (5, 5)$ is added to line L_j. Hence, $y'_j = x_{k+1} = x_5 = 5$.



It follows from the previous lemma that the shadow diagram of π can be read left to right like a time-line recording the construction of $P(\pi)$. At the k-th stage, the line $x = k$ intersects one shadow line in a ray or line segment and all the rest in single points. In terms of the first row of P_k: a ray corresponds to placing an element at the end, a line segment corresponds to displacing an element, and the points correspond to elements that are unchanged.

COROLLARY 2.5. *If the permutation π has Schensted tableaux (P, Q) and shadow lines L_j, then, for all j,*

$$P_{1,j} = y_{L_j} \qquad \text{and} \qquad Q_{1,j} = x_{L_j}.$$

Proof. See [7]. □

EXAMPLE 2.6. Let

$$\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7\ \in\ S_7$$

Then the first, second and third rows come from the thickened and dashed lines, respectively, of the figure 5, 6 and 7.

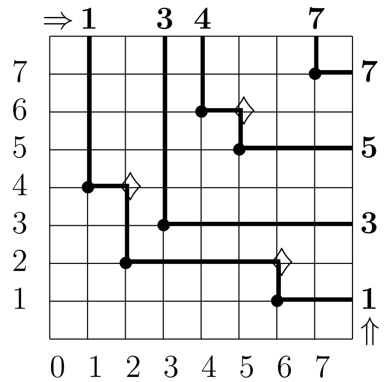


FIGURE 5. The first row of P and Q.

The ***i**-th skeleton* of $\pi \in S_n$, $\boldsymbol{\pi^{(i)}}$, is defined inductively by $\pi^{(1)} = \pi$ and

$$\pi^{(i)} = \begin{matrix} k_1 & k_2 & \cdots & k_m \\ l_1 & l_2 & \cdots & l_m \end{matrix}$$

where $(k_1, l_1), \ldots, (k_m, l_m)$ are the NorthEast corners of the shadow diagram of $\pi^{(i-1)}$ listed in lexicographic order. The shadow lines for $\pi^{(i)}$ are denoted $\boldsymbol{L_j^{(i)}}$.

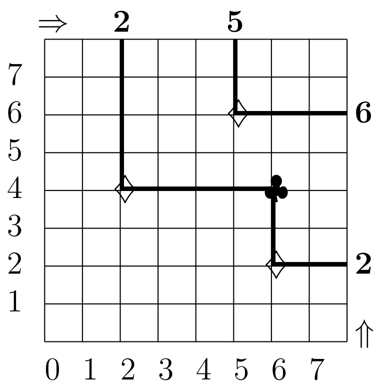


Figure 6. The second row of P and Q.

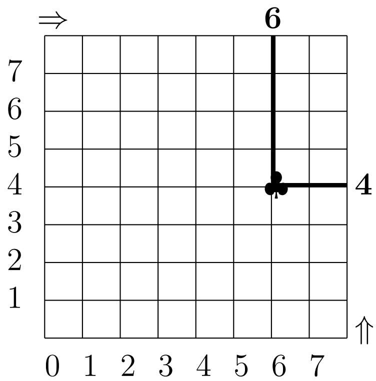


Figure 7. The third row of P and Q.

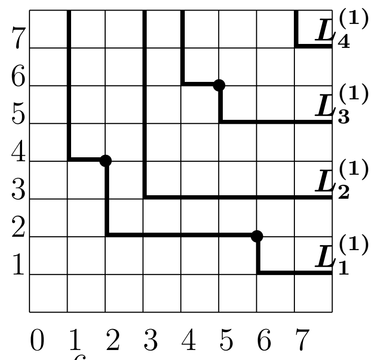
Example 2.7. Let $\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$ and

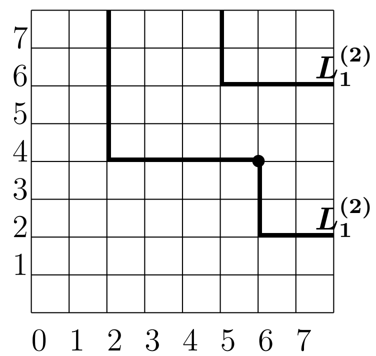
$$(P,Q) = \begin{pmatrix} 1 & 3 & 5 & 7 & & 1 & 3 & 4 & 7 \\ 2 & 6 & & & , & 2 & 5 & & \\ 4 & & & & & 6 & & & \end{pmatrix}.$$

$\pi^{(2)} = \begin{matrix} 2 & 5 & 6 \\ 4 & 6 & 2 \end{matrix}$, where $\{4,6,2\}$ and $\{2,5,6\}$ are the remainder except for the first row of P and Q, respectively.

$$P_{1,j} = y_{L_j^{(1)}} = y_{L_1^{(1)}} y_{L_2^{(1)}} y_{L_3^{(1)}} y_{L_4^{(1)}} = 1\ 3\ 5\ 7$$

$$Q_{1,j} = x_{L_j^{(1)}} = x_{L_1^{(1)}} x_{L_2^{(1)}} x_{L_3^{(1)}} x_{L_4^{(1)}} = 1\ 3\ 4\ 7$$





$\pi^{(3)} = \begin{matrix} 6 \\ 4 \end{matrix}$, where 4 and 6 are the remainder except for the first, second rows of P and Q, respectively.

$$P_{2,j} = y_{L_j^{(2)}} = y_{L_1^{(2)}} y_{L_2^{(2)}} = 2\ 6 \quad , \quad Q_{2,j} = x_{L_j^{(2)}} = x_{L_1^{(2)}} x_{L_2^{(2)}} = 2\ 5$$

Continuing this processing , $P_{3,j} = y_{L_1^{(3)}} = 4 \quad , \quad Q_{3,j} = x_{L_1^{(3)}} = 6$.

THEOREM 2.8. *Suppose $\pi \to (P, Q)$. Then $\pi^{(i)}$ is a partial permutation such that*

$$\pi^{(i)} \to (P^{(i)}, Q^{(i)})$$

where $P^{(i)}$ (respectively, $Q^{(i)}$) consists of the rows i and below of P (respectively, Q). Furthermore,

$$P_{i,j} = y_{L_j^{(i)}} \qquad \text{and} \qquad Q_{i,j} = x_{L_j^{(i)}}$$

for all i, j.

Proof. See [7]. □

3. Main results

Given a permutation $\pi = x_1 x_2 \cdots x_{n-1} x_n \in S_n$, we define new permutations π^r, π^* and $\pi^{\#}$ as follows.

$$\pi^r = x_n x_{n-1} \cdots x_2 x_1$$
$$\pi^* = (n+1-x_1)(n+1-x_2)\cdots(n+1-x_{n-1})(n+1-x_n)$$
$$\pi^{\#} = (n+1-x_n)(n+1-x_{n-1})\cdots(n+1-x_2)(n+1-x_1).$$

Note that $(\pi^*)^r = \pi^{\#}$. Until now we used Schützenberger's evacuation algorithm to compute P-tableaux and Q-tableaux of permutations

π^r, π^* and $\pi^\#$. See [7] for detail. In this section we describe the ways to find P-tableaux and Q-tableaux of permutations π^r, π^* and $\pi^\#$ directly from Viennot's geometric interpretation for Schensted algorithm.

Given a permutation $\pi \in S_n$, let $\pi \overset{[S]}{\longleftrightarrow} (P, Q)$. Then we can find P-tableaux and Q-tableaux of permutations π^r, π^* and $\pi^\#$ as the following propositions.

PROPOSITION 3.1. *Let* $\pi^* \overset{[S]}{\longleftrightarrow} (P^*, Q^*)$. *Then* (P^*, Q^*) *can be obtained as follows:*

1. *Imagine a light shining from* $(0, n+1)$ *and get a new shadow diagram of* π.
2. *Change any coordinate* y_{L_i} *to* $(n+1) - y_{L_i}$.
3. P*-tableaux and* Q*-tableaux are obtained similarly as if we read the original shadow diagram of* π.

EXAMPLE 3.2. Let $\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$. Then,

$$\pi \overset{[S]}{\longleftrightarrow} (P, Q) = \left(\begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 6 & & \\ 4 & & & \end{matrix} \ , \ \begin{matrix} 1 & 3 & 4 & 7 \\ 2 & 5 & & \\ 6 & & & \end{matrix} \right).$$

By the reading coordinate steps of Figure 8 and 9, we obtain that

$$(P^*, Q^*) = \left(\begin{matrix} 1 & 3 & 7 \\ 2 & 5 & \\ 4 & & \\ 6 & & \end{matrix} \ , \ \begin{matrix} 1 & 2 & 6 \\ 3 & 5 & \\ 4 & & \\ 7 & & \end{matrix} \right)$$

Note that $P^* = (\text{ev }(P))^t$ and $Q^* = Q^t$.

PROPOSITION 3.3. *Let* $\pi^r \overset{[S]}{\longleftrightarrow} (P^r, Q^r)$. *Then* (P^r, Q^r) *can be obtained as follows:*

1. *Imagine a light shining from* $(n+1, 0)$ *and get a new shadow diagram of* π.
2. *Change any coordinate* x_{L_i} *to* $(n+1) - x_{L_i}$.
3. P*-tableaux and* Q*-tableaux are obtained similarly as if we read the original shadow diagram of* π.

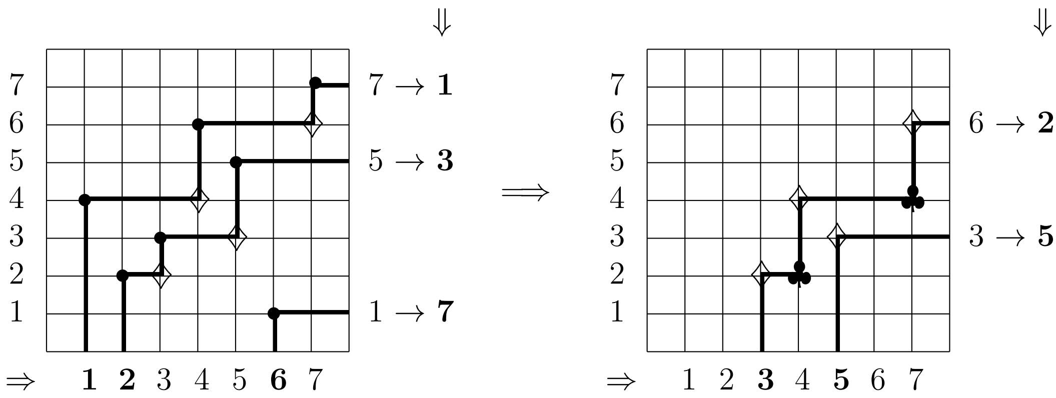


FIGURE 8. First and Second rows for π^*

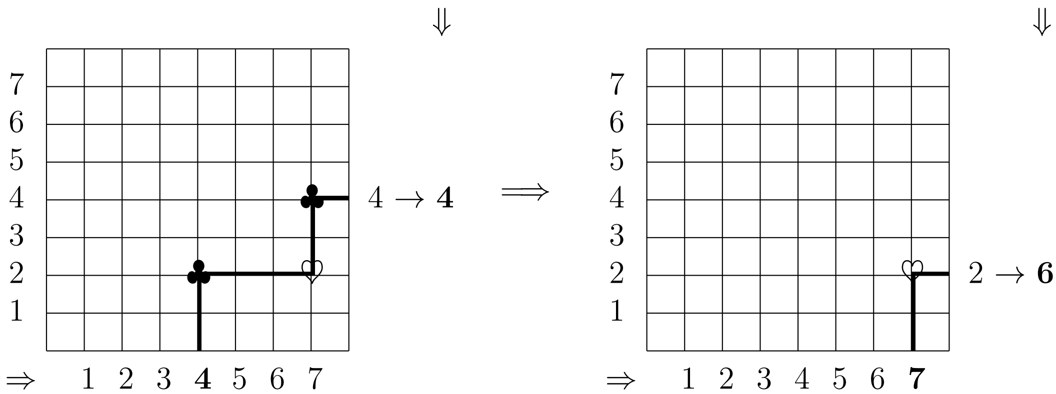


FIGURE 9. Third and Fourth rows for π^*

EXAMPLE 3.4. Let $\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$. By the reading coordinate steps of Figure 10 and 11, we obtain that

$$(P^r, Q^r) = \left(\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 6 & \\ 5 & & \\ 7 & & \end{array} , \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 7 & \\ 5 & & \\ 6 & & \end{array} \right)$$

Note that $P^r = P^t$ and $Q^r = (\text{ev}\ (Q))^t$.

PROPOSITION 3.5. *Let* $\pi^\sharp \xleftrightarrow{[\mathrm{S}]} (P^\sharp, Q^\sharp)$. *Then* $(P^\sharp, Q^\sharp)$ *can be obtained as follows:*

1. *Imagine a light shining from* $(n+1, n+1)$ *and get a new shadow diagram of* π.

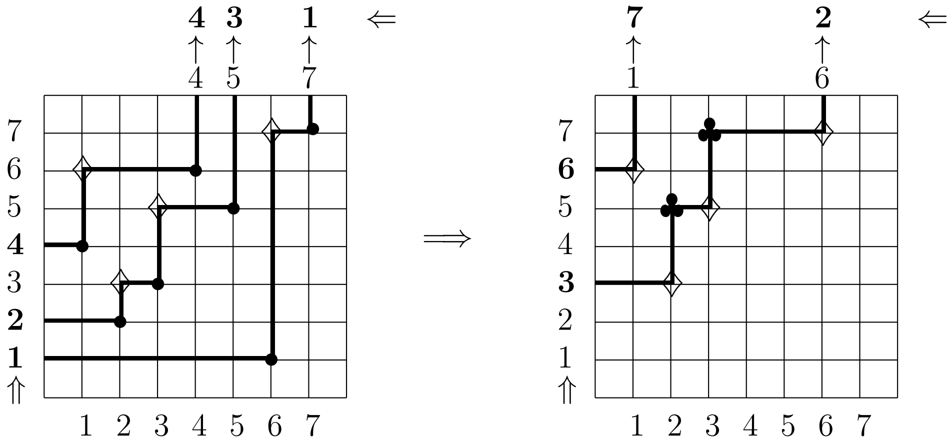


FIGURE 10. First and Second rows for π^r

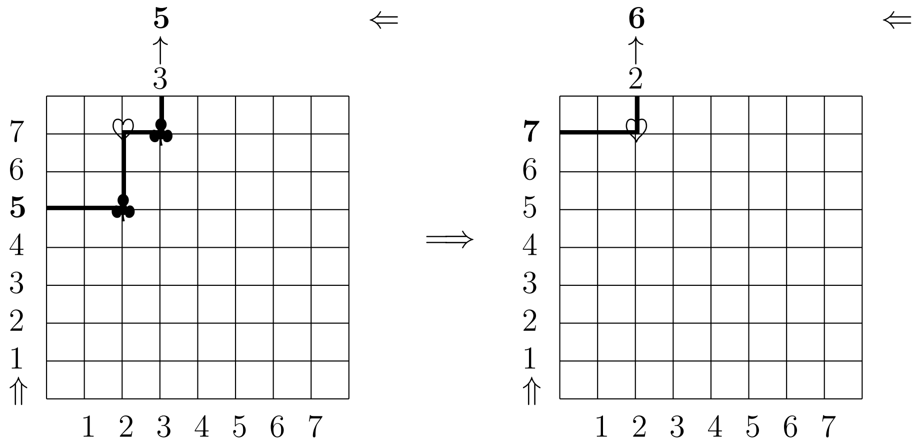


FIGURE 11. Third and Fourth rows for π^r

2. *Change any coordinate* x_{L_j} *and* y_{L_i} *to* $(n+1) - x_{L_j}$ *and* $(n+1) - y_{L_i}$, *respectively.*
3. P*-tableaux and* Q*-tableaux are obtained similarly as if we read the original shadow diagram of* π.

EXAMPLE 3.6. Let $\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$. By the reading coordinate steps of Figure 12 and 13, we obtain that

$$(P^\sharp, Q^\sharp) = \left(\begin{matrix} 1 & 2 & 4 & 6 \\ 3 & 5 & & \\ 7 & & & \end{matrix} \;,\; \begin{matrix} 1 & 2 & 5 & 6 \\ 3 & 7 & & \\ 4 & & & \end{matrix} \right)$$

Note that $P^\sharp = \text{ev}\,(P)$ and $Q^\sharp = \text{ev}\,(Q)$.

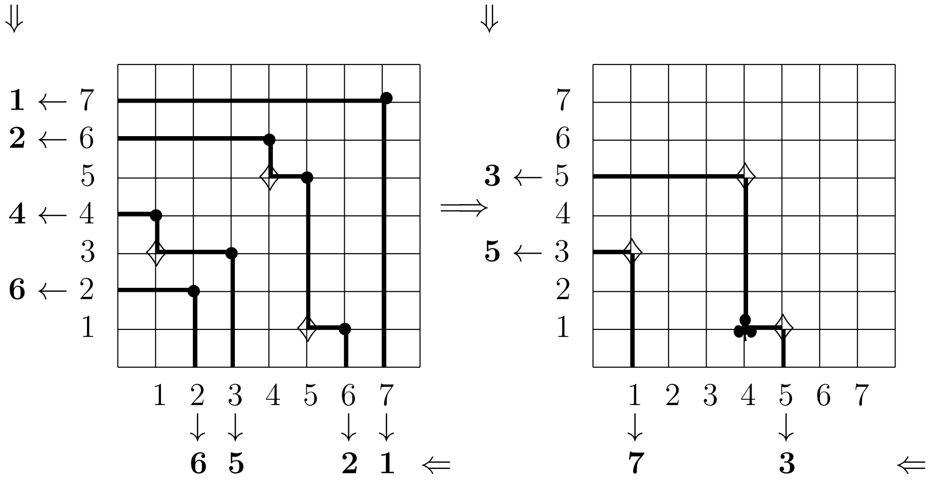


FIGURE 12. First and Second rows for $\pi^\sharp$

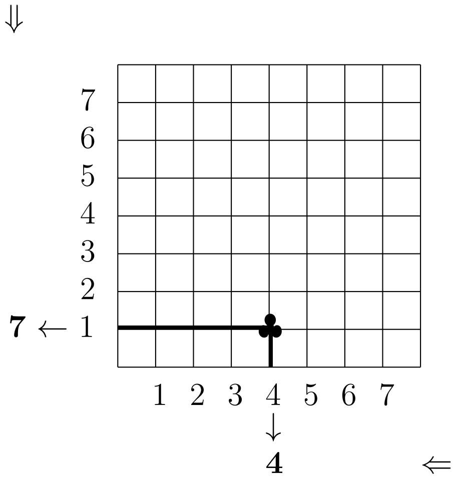


FIGURE 13. Third row for $\pi^\sharp$

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