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CYCLIC SUBGROUP SEPARABILITY OF CERTAIN GRAPH PRODUCTS OF SUBGROUP SEPARABLE GROUPS

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Abstract. In this paper, we show that tree products of certain subgroup separable groups amalgamating normal subgroups are cyclic subgroup separable. We then extend this result to certain graph product of certain subgroup separable groups amalgamating normal subgroups, that is we show that if the graph has exactly one cycle and the cycle is of length at least four, then the graph product is cyclic subgroup separable.

1. Introduction

Cyclic subgroup separability or π_c was introduced by Stebe [22]. Kim [12, 13] had given useful criteria for certain generalized free products and HNN extensions to be cyclic subgroup separable. By using Kim's criterion for HNN extensions, Wong and Wong [28] had given a characterization for certain HNN extensions with central associated subgroups to be cyclic subgroup separable. Kim and Tang [17] had given a sufficient and necessary condition for HNN extensions of cyclic subgroup separable groups with cyclic associated subgroups to be cyclic subgroup separable.

Cyclic subgroup separability is used to show that certain generalized free products are conjugacy separable (see [15, 16, 23, 24]). Conjugacy separability is used by Grossman [8] to show that certain outer automorphism groups are residually finite. In fact, she showed that if all the class-preserving automorphisms of a finitely generated conjugacy separable group G are inner, then the outer automorphism group of G is residually finite. This criterion has been used by many authors to show that certain outer automorphism groups are residually finite (see [2, 3, 6, 14, 18, 26, 29]). Recently, Zhou and Kim [31, 32] had studied the class-preserving automorphisms of certain groups. Raptis, Talelli and Varsos [21] showed that conjugacy separability and residually finiteness are equivalent in certain HNN extensions (see also [26, 27] for similar results).

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Subgroup separability is a property stronger than cyclic subgroup separability. It is well known that polycyclic groups and free groups are subgroup separable (Hall [9], Mal'cev [19]). Since a finite extension of a subgroup separable group is again subgroup separable, polycyclic-by-finite groups and free-by-finite groups are subgroup separable. Metaftsis and Raptis [20] gave a sufficient and necessary condition for certain HNN extensions to be subgroup separable. By applying their result, Wong and Wong [30] showed that subgroup separability and conjugacy separability are equivalent in certain HNN extensions.

In this paper, we will study cyclic subgroup separability of certain graph products. This paper is motivated by the works of Kim [11], Allenby [1], and Wong and Wong [25]. We will give a generalization of the Allenby's Theorem [1, Theorem C], which is a generalization of the Kim's Theorem [11, Theorem 2.11]. In Section 2, we will discuss the generalization of Allenby's Theorem (see Theorem 2.5). In Sections 3, 4 and 5, we will provide the details of the proofs.

2. Generalizing Allenby's theorem

The notation used here is standard. In addition, the following will be used for any group G, $N \lhd_f G$ means N is a normal subgroup of finite index in G. We denote by $A \underset{H}{*} B$ the generalised free product of A and B with the subgroup H amalgamated. If $G = A \underset{H}{*} B$ and $x \in G$, then $\|x\|$ denotes the free product length of x in G. If $\overline{G}$ is a homomorphic image of G, then we use $\overline{x}$ to denote the image of x in $\overline{G}$.

Definition 2.1. A group G is called *H-separable* for the subgroup H if for each $x \in G \backslash H$, there exists $N \lhd_f G$ such that $x \notin HN$.

G is called HK-separable for the subgroups H, K if for each $x \in G \backslash HK$, there exists $N \lhd_f G$ such that $x \notin HKN$.

A group G is termed *subgroup separable* if G is H-separable for every finitely generated subgroup H. A group G is termed *cyclic subgroup separable* (or π_c for short) if G is H-separable for every cyclic subgroup H. A group G is termed *residually finite* if G is 1-separable.

Definition 2.2. Let Q be a simple graph (without loops and multiple edges) with vertex set $V(Q)$ and edge set $E(Q)$. To each vertex v of Q assign a vertex group A_v, and to each edge e of Q assign an edge group H_e together with monomorphisms α_e and β_e embedding H_e into the two vertex groups at the end of e. The *graph product* $G(Q)$ of the vertex groups amalgamating the edge groups is defined to be the group generated by all the generators of the vertex groups with defining relations given by the defining relations of all the vertex groups together with the relations $\alpha_e(g_e) = \beta_e(g_e)$ for each g_e in H_e.

Roughly speaking, if ij is an edge in $E(Q)$, then H_{ij} is a subgroup of A_i and A_j. Since ij and ji represent the same edge in Q, we have $H_{ij} = H_{ji}$.

If Q is a tree, then $G(Q)$ is called the *tree product*, whereas if Q is a cycle (polygon), then $G(Q)$ is called the *polygonal product*.

The polygonal products of groups were introduced by Karrass, Pietrowski and Solitar [10] in their study of the subgroup structure of the Picard group PSL(2, $\mathbb{Z}[i]$). By using their results, Brunner, Frame, Lee and Wielenberg [5] characterized all the torson-free subgroups of finite index in the Picard group. Polygonal products also form a large subclass in the class of one-relator products of cyclic groups. For certain one-relator products, Fine, Howie and Rosenberger [7] had proved a Freiheitssatz but the word problem and residual finiteness are still unknown.

Definition 2.3. Let $u, v \in V(Q)$. If u is adjacent to v in Q, i.e., $uv \in E(Q)$, then we shall write $u \sim v$.

Suppose that for each $u \in V(Q)$, the edge group H_{uv} is normal in the vertex group A_u for all $v \in V(Q)$ with $v \sim u$. We say that the edge groups satisfy the *intersection property* if for each edge $uv \in E(Q)$,

$$H_{uv} \cap \prod_{\substack{w\sim u,\\ w\neq v}} H_{uw} = 1.$$

We remark here that, since all the edge groups are normal in its vertex group, the product $\prod_{w\sim u, w\neq v} H_{uw}$ is the subgroup generated by all the edge groups $\langle H_{uw} \ : \ w \sim u, w \neq v\rangle$ in A_u. In fact, if the edge groups satisfy the intersection property, the subgroup generated by all the edge groups in A_u is the direct product

$$\bigotimes_{w\sim u} H_{uw}.$$

Note that if Q is a cycle of length 4 and the intersection property holds, then we have vertex groups A_1, A_2, A_3, A_4 and edge groups H_1, H_2, H_3, H_4 such that $A_i \cap A_{i+1} = H_i$ and $H_i \cap H_{i+1} = 1$ for $i = 1, 2, 3, 4$ where the subscripts are taken modulo 4. What Allenby [1, Theorem C] has proved is the following theorem (see also [25, Theorem 4.6]):

Theorem 2.4 (Allenby's Theorem)**.** *If Q is a cycle of length at least 4 and the intersection property holds, then the polygonal product $G(Q)$ of polycyclic-by-finite groups amalgamating finitely generated normal subgroups is π_c.*

The objective of this paper is to prove the following theorem, which is a generalization of Theorem 2.4.

Theorem 2.5. *Suppose Q is a simple graph that has exactly one cycle. If the length of the cycle is at least 4 and the intersection property holds, then the graph product $G(Q)$ of subgroup separable groups amalgamating finitely generated normal subgroups is π_c.*

Since polycyclic-by-finite is subgroup separable, the following corollary is a consequence of Theorem 2.5.

Corollary 2.6. *Suppose Q is a simple graph that has exactly one cycle. If the length of the cycle is at least 4 and the intersection property holds, then the* *graph product $G(Q)$ of polycyclic-by-finite groups amalgamating normal subgroups is π_c.* □

We shall need the following result of Kim [12, Proposition 1.2].

Theorem 2.7. *Let $G = A \ast_H B$. Suppose that*

(a) *A and B are π_c and H-separable,*
(b) *for each $N \lhd_f H$, there exist $N_A \lhd_f A$ and $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subset N$.*

Then G is π_c.

Note that to prove Theorem 2.5, we may assume that Q is connected. Let C be the cycle of length at least 4 in Q. Then C contains at least 4 vertices, say u_1, u_2, v_1, v_2, such that $u_1 \sim v_1$ and $u_2 \sim v_2$. Now if we remove the edges u_1v_1 and u_2v_2 from Q, then the resulting graph $Q - \{u_1v_1, u_2v_2\}$, is the union of two trees, say T_1 and T_2. Since the intersection property holds, we conclude that the subgroup generated by $H_{u_1v_1}$ and $H_{u_2v_2}$ in $G(T_1)$ is the free product $H_{u_1v_1} \ast H_{u_2v_2}$. Similarly, the free product $H_{u_1v_1} \ast H_{u_2v_2}$ is the subgroup generated by $H_{u_1v_1}$ and $H_{u_2v_2}$ in $G(T_2)$. Therefore

$$G(Q) = G(T_1) \ast_{(H_{u_1v_1} \ast H_{u_2v_2})} G(T_2).$$

If we could show that $G(T_1)$ and $G(T_2)$ satisfy the conditions (a) and (b) of Theorem 2.7, then Theorem 2.5 follows.

From now onwards throughout the paper, we shall assume the following:

(a) T is a tree;
(b) the intersection property holds;
(c) $G(T)$ is the tree product of subgroup separable groups amalgamating finitely generated normal subgroups;
(d) the vertex groups are denoted by A_u, $u \in V(T)$, and the edge groups are denoted by H_{uv}, $uv \in E(T)$.

Now, Theorem 2.5 follows from Theorem 2.7 applying Theorem 2.8, Lemma 2.9 and Lemma 2.10.

Theorem 2.8. *$G(T)$ is π_c.*

Lemma 2.9. *Let M and K be finitely generated normal subgroups of A_r and A_k respectively where $M \cap \prod_{j\sim r} H_{rj} = 1 = K \cap \prod_{j\sim k} H_{kj}$ and $r \neq k$. Then $G(T)$ is $M \ast K$-separable.*

Let G be a finitely generated group and $S \lhd_f G$. If S is a characteristic subgroup of G, then we set $f_G(S) = S$. Suppose S is not a characteristic subgroup of G. Let $[G : S] = m$ where m is a positive integer. Since G is finitely generated, the number of subgroups of index m in G is finite. Let N be the intersection of all these subgroups. Then N is a characteristic subgroup of finite index in G and $N \subseteq S$. We set $f_G(S) = N$ (see [25, Lemma 3.1]).

Lemma 2.10. *Let M and K be finitely generated normal subgroups of A_r and A_k respectively where $M \cap \prod_{j\sim r} H_{rj} = 1 = K \cap \prod_{j\sim k} H_{kj}$ and $r \neq k$. Then for each $S \lhd_f (M * K)$, there exists $N \lhd_f G(T)$ such that $N \cap (M * K) = f_{M*K}(S)$.*

3. Proof of Theorem 2.8

Lemma 3.1. *Let A be a subgroup separable group and $H_1, H_2, \ldots, H_n$ be finitely generated normal subgroups of A such that $H_i \cap \prod_{j\neq i} H_j = 1$ for $i = 1, 2, \ldots, n$. If $S_i \lhd_f H_i$, then there exists $N \lhd_f A$ such that $N \cap H_i = f_{H_i}(S_i)$, and $NH_i \cap NH_j = N$, $j \neq i$, $1 \leq i, j \leq n$.*

Proof. Let $S = \Pi_{i=1}^{n} f_{H_i}(S_i)$ and $H = \Pi_{i=1}^{n} H_i$. Since H_i is finitely generated and $f_{H_i}(S_i) \lhd_f H_i$, we have $f_{H_i}(S_i)$ is finitely generated and thus S is finitely generated. Note that S is a finitely generated normal subgroup in A. Therefore $\overline{A} = A/S$ is residually finite and $\overline{H} = H/S$ is finite. So, there exists $\overline{N} \lhd_f \overline{A}$ such that $\overline{N} \cap \overline{H} = 1$. Let N be the preimage of $\overline{N}$. Now, we show that $N \cap H_i = f_{H_i}(S_i)$. Clearly $f_{H_i}(S_i) \subseteq N \cap H_i$. Let $y \in N \cap H_i$. This implies that $\overline{y} = 1$, and thus $y \in S$. So $y = a_1 a_2 \cdots a_n$ where $a_k \in f_{H_k}(S_k)$, and $y a_i^{-1} \in H_i \cap \Pi_{k\neq i} H_k = 1$. Hence $y = a_i \in f_{H_i}(S_i)$ and $N \cap H_i = f_{H_i}(S_i)$.

Next, we show that $NH_i \cap NH_j = N$. Clearly $N \subseteq NH_i \cap NH_j$. Let $x \in NH_i \cap NH_j$ where $i \neq j$. Then $x = n_1 h_i = n_2 h_j$ where $n_1, n_2 \in N$, $h_i \in H_i$ and $h_j \in H_j$. This implies that $h_i h_j^{-1} \in N \cap H$. Therefore $\overline{h_i h_j^{-1}} = 1$, and thus $h_i h_j^{-1} \in S$. Let $h_i h_j^{-1} = b_1 b_2 \cdots b_n$ where $b_k \in f_{H_k}(S_k)$. Then $h_i b_i^{-1} \in H_i \cap \Pi_{k\neq i} H_k = 1$ and $h_i = b_i \in f_{H_i}(S_i) \subseteq N$. Hence $x \in N$ and $NH_i \cap NH_j = N$. □

Lemma 3.2. *Let M and K be finitely generated normal subgroups of A_r and A_k respectively where $M \cap \prod_{j\sim r} H_{rj} = 1 = K \cap \prod_{j\sim k} H_{kj}$ and $r \neq k$. Then for any $S_r \lhd_f M$ and $S_k \lhd_f K$, there exists $N \lhd_f G(T)$ such that $N \cap M = f_M(S_r)$, $N \cap K = f_K(S_k)$ and $NM \cap NK = N$.*

Proof. By Lemma 3.1, there exists $N_r \lhd_f A_r$ such that $N_r \cap M = f_M(S_r)$ and $N_r \cap H_{rj} = f_{H_{rj}}(H_{rj}) = H_{rj}$ for all j with $j \sim r$. Similarly, there exists $N_k \lhd_f A_k$ such that $N_k \cap K = f_K(S_k)$ and $N_k \cap H_{kj} = H_{kj}$ for all j with $j \sim k$.

Let $N_i = \prod_{j\sim i} H_{ij}$ for $i \neq r, k$. Let ϕ_i be the natural epimorphism from A_i onto A_i/N_i for all $i \in V(T) = \{1, 2, \ldots, n\}$. Then these epimorphisms can be extended to an epimorphism ϕ from $G(T)$ onto $\overline{G(T)}$ where $\overline{G(T)} = A_1/N_1 * A_2/N_2 * \cdots * A_n/N_n$.

Since $\overline{G}$ is a free product and $r \neq k$, $\overline{M} \cap \overline{K} = 1$. Since $\overline{G}$ is residually finite by [4, Theorem 2] and $\overline{MK}$ is finite, there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{N} \cap \overline{MK} = 1$. Let N be the preimage of $\overline{N}$. We shall show that N is the required subgroup.

First we note that $N \cap M = N \cap A_r \cap M = N_r \cap M = f_M(S_r)$. Similarly, $N \cap K = f_K(S_k)$. So, it remains to show that $NM \cap NK = N$. Clearly $N \subseteq NM \cap NK$. Now let $y \in NM \cap NK$. Then $y = n_1 m_1 = n_2 m_2$ where $n_1, n_2 \in N$, $m_1 \in M$ and $m_2 \in K$. It is sufficient to show $m_1 \in N$. Now

$m_1 m_2^{-1} \in N$ implies $\overline{m}_1 \overline{m}_2^{-1} \in \overline{N} \cap \overline{MK} = 1$. Since $\overline{M} \cap \overline{K} = 1$, we have $\overline{m}_1 = 1$ and $m_1 \in N$. □

We shall need the following lemma of Kim [11, Theorem 2.3].

Lemma 3.3. *Let $G = A \ast_H B$ where A, B are H-separable. Suppose for each $N_H \lhd_f H$, there exist $N_A \lhd_f A, N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$. Let S be any subgroup of B. If B is S-separable, then G is S-separable.* □

Lemma 3.4. *Let M be a finitely generated subgroup of A_r. Then $G(T)$ is M-separable.*

Proof. Let $V(T) = \{1, 2, \dots, n\}$. We shall use induction on n. The case $n = 2$ follows from Lemma 3.1 and Lemma 3.3. Suppose $n \geq 3$. Note that the tree T has a vertex of degree one, say n, which is joined to a unique vertex, say $n-1$. Let T_1 be the tree obtained by removing the vertex n and the edge $n(n-1)$ from T. Then $G(T) = G(T_1) \ast_H A_n$ where $H = H_{(n-1)n} = H_{n(n-1)}$. By induction, $G(T_1)$ is H-separable. By Lemma 3.2, for any $N_H \lhd_f H$, there exists $N_1 \lhd_f G(T_1)$ such that $N_1 \cap H = f_H(N_H)$. Since A_n is subgroup separable, it is H-separable. By Lemma 3.1, there exists $N_2 \lhd_f A_n$ such that $N_2 \cap H = f_H(N_H)$.

Suppose $r = n$. Since A_n is M-separable, by Lemma 3.3, $G(T)$ is M-separable.

Suppose $r \neq n$. Then by induction, $G(T_1)$ is M-separable and thus by Lemma 3.3 again, $G(T)$ is M-separable. □

We are now ready to prove Theorem 2.8.

Proof of Theorem 2.8. We use induction on n. The case $n = 2$ follows from Lemma 3.1 and Theorem 2.7. Suppose $n \geq 3$. As in Lemma 3.4, we may assume that n is a vertex of degree one and is joined to a unique vertex, say $n-1$. Then $G(T) = G(T_1) \ast_H A_n$ where T_1 is the tree obtained by removing the vertex n and the edge $n(n-1)$ from T and $H = H_{(n-1)n} = H_{n(n-1)}$. By Lemma 3.4, $G(T_1)$ is H-separable, and by Lemma 3.2, for any $N_H \lhd_f H$, there exists $N_1 \lhd_f G(T_1)$ such that $N_1 \cap H = f_H(N_H)$. Furthermore, A_n is H-separable, and by Lemma 3.1, there exists $N_2 \lhd_f A_n$ such that $N_2 \cap H = f_H(N_H)$. Since $G(T_1)$ is π_c by the induction hypothesis and A_n is π_c, it follows from Theorem 2.7 that $G(T)$ is π_c. □

4. Proof of Lemma 2.9

We shall need the following two lemmas from [25, Lemmas 4.1 and 4.2]

Lemma 4.1. *Let $G = A \ast_H B$ and M, K be subgroups of A, B respectively with $M \cap H = 1 = K \cap H$. Suppose for each $N_H \lhd_f H$ there exist $N_A \lhd_f A, N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$ and $N_A H \cap N_A M = N_A$, $N_B H \cap N_B K = N_B$. If A is H, M, HM, MH-separable and B is H, K, HK, KH-separable, then G is MK-separable.*

Lemma 4.2. *Let $G = A \ast_H B$ and M, K be subgroups of A, B respectively with $M \cap H = 1 = K \cap H$. Suppose A, B, M, K satisfy the hypothesis of Lemma 4.1. Then G is $M \ast K$-separable.*

Lemma 4.3. *Let $G = A \ast_H B$ and M, K be subgroups of B. Suppose for each $N_H \lhd_f H$ there exist $N_A \lhd_f A, N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$. If A is H-separable and B is H, MK-separable, then G is MK-separable.*

Proof. Let $g \in G - MK$.

Case 1. $g \in B$. Since B is MK-separable, there exists $P \lhd_f B$ such that $g \notin PMK$. Let $N_H = P \cap H$. By assumption, there exist $N_A \lhd_f A, N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$. Let $R_A = N_A, R_B = N_B \cap P$. Then $R_A \cap H = R_B \cap H$. Let $\overline{G} = A/R_A \ast_{\overline{H}} B/R_B$ where $\overline{H} = HR_A/R_A = HR_B/R_B$. Clearly, $\overline{G}$ is a homomorphic image of G, and $\overline{g} \notin \overline{MK}$. Since $\overline{G}$ is residually finite by [4, Theorem 2] and $\overline{MK}$ is finite, there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{g} \notin \overline{NMK}$. Let N be the preimage of $\overline{N}$. Then $g \notin NMK$.

Case 2. $g \in A - H$ or $g \notin A \cup B$. We will only consider the case $g = a_1 b_1 a_2 b_2 \cdots a_n b_n$ where $a_i \in A - H$, $b_i \in B - H$, $i = 1, 2, \ldots, n$. The other cases are similar. Since A is H-separable, B is H-separable, there exist $M_A \lhd_f A$, $M_B \lhd_f B$ such that $a_i \notin M_A H$, $b_i \notin M_B H$, $i = 1, 2, \ldots, n$. Let $N_H = M_A \cap M_B$. By assumption, there exist $N_A \lhd_f A$, $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq N_H$. Let $R_A = N_A \cap M_A, R_B = N_B \cap M_B$. Then $R_A \cap H = R_B \cap H$. Let $\overline{G} = A/R_A \ast_{\overline{H}} B/R_B$ where $\overline{H} = HR_A/R_A = HR_B/R_B$. Clearly, $\|\overline{g}\| = \|g\|$. So $\overline{g} \notin \overline{MK}$. Furthermore, $\overline{MK}$ is finite. We can now proceed as in Case 1 to complete the proof. □

Lemma 4.4. *Let M and K be finitely generated normal subgroups of A_r and A_k respectively where $M \cap \prod_{j \sim r} H_{rj} = 1 = K \cap \prod_{j \sim k} H_{kj}$ (here we allow $r = k$). Then $G(T)$ is MK-separable.*

Proof. Let $V(T) = \{1, 2, \ldots, n\}$. We shall use induction on n. The case $n = 2$ follows from Lemma 3.1, Lemma 4.1 and Lemma 4.3. Suppose $n \geq 3$. As in Lemma 3.4, we may assume that n is a vertex of degree one and is joined to a unique vertex, say $n - 1$. Then $G(T) = G(T_1) \ast_H A_n$ where T_1 is the tree obtained by removing the vertex n and the edge $n(n-1)$ from T and $H = H_{(n-1)n} = H_{n(n-1)}$.

Case 1. $M \leq G(T_1)$ and $K \leq A_n$. By the induction, $G(T_1)$ is MH, HM-separable. By Lemma 3.4, $G(T_1)$ is M, H-separable. By Lemma 3.2, for any $S_H \lhd_f H$, $S_M \lhd_f M$ and $S_K \lhd_f K$, there exists $N_1 \lhd_f G(T_1)$ such that $N_1 \cap M = f_M(S_M)$, $N_1 \cap H = f_H(S_H)$ and $N_1 M \cap N_1 H = N_1$. By Lemma 3.1, there exists $N_2 \lhd_f A_n$ such that $N_2 \cap K = f_K(S_K)$, $N_2 \cap H = f_H(S_H)$ and $N_2 K \cap N_2 H = N_2$.

Since A_n is subgroup separable, it is H, K, HK, KH-separable. Hence by Lemma 4.1, $G(T)$ is MK-separable.

Case 2. $K \leq G(T_1)$ and $M \leq A_n$. The argument is similar to Case 1.

Case 3. $K, M \leq G(T_1)$. By induction, $G(T_1)$ is MK-separable. By Lemma 3.4, $G(T_1)$ is H-separable, and by Lemma 3.2, for any $S_H \lhd_f H$ there exists

$N_1 \lhd_f G(T_1)$ such that $N_1 \cap H = f_H(S_H)$. Furthermore, A_n is H-separable, and by Lemma 3.1, for any $S_H \lhd_f H$ there exists $N_2 \lhd_f A_n$ such that $N_2 \cap H = f_H(S_H)$. Hence by Lemma 4.3, $G(T)$ is MK-separable.

Case 4. $K, M \leq A_n$. The argument is similar to Case 3. □

We are now ready to prove Lemma 2.9.

Proof of Lemma 2.9. Let $V(T) = \{1, 2, \ldots, n\}$. We shall use induction on n. The case $n = 2$ follows from Lemma 3.1 and Lemma 4.2. Suppose $n \geq 3$. As in Lemma 3.4, we may assume that n is a vertex of degree one and is joined to a unique vertex, say $n-1$. Then $G(T) = G(T_1) \underset{H}{*} A_n$ where T_1 is the tree obtained by removing the vertex n and the edge $n(n-1)$ from T and $H = H_{(n-1)n} = H_{n(n-1)}$.

Case 1. $M \leq G(T_1)$ and $K \leq A_n$. By Lemma 4.4, $G(T_1)$ is MH-separable and HM-separable. By Lemma 3.4, $G(T_1)$ is M, H-separable. By Lemma 3.1 and Lemma 3.2, $G(T_1)$ and A_n satisfy the hypothesis of Lemma 4.2. Hence $G(T)$ is $M * K$-separable.

Case 2. $K \leq G(T_1)$ and $M \leq A_n$. The argument is similar to Case 1.

Case 3. $K, M \leq G(T_1)$. By induction, $G(T_1)$ is $M * K$-separable. By Lemma 3.4, $G(T_1)$ is H-separable. It then follows from Lemma 3.1, Lemma 3.2 and Lemma 3.3 that $G(T)$ is $M * K$-separable. □

5. Proof of Lemma 2.10

Lemma 5.1. *Let $G = A \underset{H}{*} B$ and M, K be subgroups of A, B respectively with $M \cap H = 1 = K \cap H$. Let $S \lhd_f (M * K)$ and suppose there exist $N_A \lhd_f A$, $N_B \lhd_f B$ such that $N_A \cap H = H = N_B \cap H$ and $N_A \cap M = S \cap M$, $N_B \cap K = S \cap K$. Then there exists $N \lhd_f G$ such that $N \cap (M * K) = S$.*

Proof. Let $\overline{G} = A/N_A * B/N_B$. Since $N_A \cap H = H = N_B \cap H$, the natural epimorphisms from A onto A/N_A and from B onto B/N_B can be extended to an epimorphism ψ from G onto $\overline{G}$.

First we show that $\text{Ker } \psi \cap (M * K) \subseteq S$. Let g be any element with the smallest length such that $g \in \text{Ker } \psi \cap (M * K)$ but $g \notin S$. Without loss of generality we may assume $g = m_1 k_1 \cdots m_n k_n$ where $m_i \in M, k_i \in K$. Then $\overline{g} = \overline{m_1 k_1 \cdots m_n k_n} = 1$. Hence $\overline{m_i} = 1$ or $\overline{k_i} = 1$ for some i. Suppose $\overline{m_i} = 1$ (the case $\overline{k_i} = 1$ is similar). Then $m_i \in N_A$ for $\overline{M} = MN_A/N_A$. Hence $m_i \in N_A \cap M = S \cap M \subseteq S$. Now

$$g = (m_1 k_1 \cdots m_{i-1} k_{i-1}) m_i (m_1 k_1 \cdots m_{i-1} k_{i-1})^{-1} (m_1 k_1 \cdots m_{i-1} k_{i-1}) k_i \cdots m_n k_n.$$

But $(m_1 k_1 \cdots m_{i-1} k_{i-1}) m_i (m_1 k_1 \cdots m_{i-1} k_{i-1})^{-1} \in S$ since $m_i \in S \lhd_f (M * K)$. This implies that $g_1 = (m_1 k_1 \cdots m_{i-1} k_{i-1}) k_i \cdots m_n k_n \notin S$. Furthermore $\overline{g_1} = 1$ for $\overline{m_i} = 1$. Therefore $g_1 \in \text{Ker } \psi \cap (M * K)$, but $\|g_1\| < \|g\|$, a contradiction. Thus $\text{Ker } \psi \cap (M * K) \subseteq S$.

Next we show that $\overline{S} \cap \overline{M} = 1$ and $\overline{S} \cap \overline{K} = 1$. Let $\overline{y} \in \overline{S} \cap \overline{M}$. Then $\overline{y} = \overline{s} = \overline{m}$ where $s \in S$ and $m \in M$. So $\overline{sm^{-1}} = 1$, and this implies that

$sm^{-1} \in \text{Ker } \psi \cap (M * K) \subseteq S$. Therefore $m \in S \cap M \subseteq N_A$. This implies that $\overline{m} = 1$, and thus $\overline{y} = 1$. Hence $\overline{S} \cap \overline{M} = 1$. Similarly, $\overline{S} \cap \overline{K} = 1$.

Now $\overline{G} = A/N_A * B/N_B = A/N_A \underset{\overline{M}}{*} (\overline{M} * \overline{K}) \underset{\overline{K}}{*} B/N_B$. Since $\overline{S} \cap \overline{M} = 1$, $\overline{\overline{M}} = \overline{MS}/\overline{S} \simeq \overline{M}/(\overline{S} \cap \overline{M}) \simeq \overline{M}$. Similarly, $\overline{S} \cap \overline{K} = 1$ implies that $\overline{\overline{K}} = \overline{KS}/\overline{S} \simeq \overline{K}/(\overline{S} \cap \overline{K}) \simeq \overline{K}$. So, we can form $\overline{\overline{G}} = A/N_A \underset{\overline{M}}{*} ((\overline{M} * \overline{K})/\overline{S}) \underset{\overline{K}}{*} B/N_B$. Clearly, $\overline{\overline{G}}$ is a homomorphic image of $\overline{G}$, and thus $\overline{\overline{G}}$ is a homomorphic image of G.

Since $(\overline{M} * \overline{K})/\overline{S}$ is finite and $\overline{\overline{G}}$ is residually finite by [4, Theorem 2], there exists $\overline{\overline{N}} \lhd_f \overline{\overline{G}}$ such that $\overline{\overline{N}} \cap ((\overline{M} * \overline{K})/\overline{S}) = 1$. Let N be the preimage of $\overline{\overline{N}}$ in G. We shall show that $N \cap (M * K) = S$. Clearly $S \subseteq N \cap (M * K)$. Now let $g_2 \in N \cap (M * K)$. Since $\overline{\overline{g_2}} = 1$, we have $\overline{g_2} \in \overline{S}$. Let $\overline{g_2} = \overline{t}$ where $t \in S$. Then $\overline{g_2 t^{-1}} = 1$, and this implies that $g_2 t^{-1} \in \text{Ker } \psi \cap (M * K) \subseteq S$. Hence $g_2 \in S$ and $N \cap (M * K) = S$. □

We are now ready to prove Lemma 2.10.

Proof of Lemma 2.10. Let $S \lhd_f (M * K)$ be given. Let $S_M = f_{M*K}(S) \cap M$ and $S_K = f_{M*K}(S) \cap K$. Then S_M and S_K are characteristic subgroups of M and K, respectively.

Since T is a tree, there exists a unique path from the vertex r to the vertex k. Let $i_0 j_0$ be an edge on this path. Let T_1 and T_2 be the two disjoint trees obtained by removing the edge $i_0 j_0$ from T. Then $G(T) = G(T_1) \underset{H}{*} G(T_2)$ where $H = H_{i_0 j_0} = H_{j_0 i_0}$ and $A_r \subseteq G(T_1)$ and $A_k \subseteq G(T_2)$. For all j with $j \sim r$, let $S_{rj} = H_{rj}$, and for all j with $j \sim k$, let $S_{kj} = H_{kj}$.

By Lemma 3.1, there exists $N_r \lhd_f A_r$ such that $N_r \cap H_{rj} = f_{H_{rj}}(S_{rj}) = H_{rj}$ and $N_r \cap M = f_M(S_M) = S_M$. Let ψ_1 be the epimorphism from $G(T_1)$ onto A_r/N_r where $\psi_1(a) = 1$, $\forall a \in A_i \subseteq G(T_1)$, $i \neq r$ and $\psi_1(a) = aN_r$, $\forall a \in A_r$. Let $N_1 = Ker\ \psi_1$. Then $N_1 \lhd_f G(T_1)$, $N_1 \cap M = f_M(S_M) = S_M$ and $N_1 \cap H = H$. Similarly there is a $N_2 \lhd_f G(T_2)$ with $N_2 \cap K = f_K(S_K) = S_K$ and $N_2 \cap H = H$. Therefore by Lemma 5.1, there exists $N \lhd_f G(T)$ such that $N \cap (M * K) = f_{M*K}(S)$. □

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