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EVALUATION SUBGROUPS OF HOMOGENEOUS SPACES OF COMPACT LIE GROUPS

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Abstract. In this paper, we compute the images of homotopy groups of various classical Lie groups under the homomorphisms induced by the natural projections from those groups to irreducible symmetric spaces of classical type. We identify that those computations are certain lower bounds of Gottlieb groups of irreducible symmetric spaces. We use the lower bounds to compute some Gottlieb groups.

1. Introduction

The evaluation subgroups of homotopy groups of a space were first introduced by D. Gottlieb in [1] and [2] where their salient properties were described, including its intimate connection with the Euler characteristic. Specifically, it was shown that the non-triviality of the evaluation subgroup of the fundamental group for a finite polyhedron suffices to ensure the vanishing of the Euler characteristic, whereas the triviality of the evaluation subgroups of homotopy groups for a space suffices to ensure the existence of the cross section of a fibration with the space as the fiber and a sphere as the base space. After the original work, many authors referred to the evaluation subgroups of homotopy groups of a space as Gottlieb group of a space, and have made an attempt to compute the evaluation subgroups of homotopy groups for various spaces; for instance, J. Siegel, G. Lang, Oprea, Golasinski and Mukai, etc. For this reason, we will use those terms together in this paper. While many results are already known, explicit computations of Gottlieb groups appear difficult. One reason that accounts in part for this difficulty is the fact that a map between spaces does not necessarily induce a corresponding homomorphism between Gottlieb groups. Woo and the second author ([10], [16]) attempt to circumvent this problem by introducing the so-called G-sequence and by providing the lower bounds or upper bounds of Gottlieb groups. In [14], J. Siegel gave a connection

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between Gottlieb groups and image groups of homomorphism induced by the natural projection map from Lie groups to their homogeneous spaces. In [9], Mimura, Woo and the second author provided some lower bounds of Gottlieb groups of quaternionic projective spaces and Stiefel manifolds, and used them to compute some Gottlieb groups of their homogeneous spaces. Golasinski and Mukai [4] fully determine the groups $G_{n+k}(S_n)$ for $k \neq 13$, except for the 2-primary components in the cases: $k = 9$, $n = 53$; $k = 11$, $n = 115$ by means of the classical homotopy theory.

In this paper, we offer more computations or lower bounds of Gottlieb groups of homogeneous spaces by computing the image of homomorphisms induced by the natural projections from classical Lie groups to their homogeneous spaces; in particular, the symmetric spaces of classical type classified by Cartan. In order to solve those problems, we first apply Lundell's concise tables, which provide various homotopy groups of compact Lie groups and their homogeneous spaces to compute those images. As a result, we obtain Table 1.

Table 1

Type, $p_*\pi_k \backslash$ k_i (mod 8)	0	1	2	3	4	5	6	7
(1) $p_*\pi_{k_1}(SU(n))$	0	$\mathbb{Z}$	0	=	0	$\mathbb{Z}$	0	=
(2) $p_*\pi_{k_2}(SU(2n))$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
(3) $p_*\pi_{k_3}(U(n+m))$	0	=	0	=	0	=	0	=
(4) $p_*\pi_{k_4}(SO(2n))$	0	0	0	0	0	0	0	0
(5) $p_*\pi_{k_5}(SO(n+1))$	0	0	0	0	0	0	0	0
(6) $p_*\pi_{k_6}(Sp(n))$	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0
(7) $p_*\pi_{k_7}(Sp(n+m))$	0	0	0	=	0	0	0	=
(8) $p_*\pi_{k_8}(SO(2n))$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	=	0	0	0	=

where k_i's are integers such that $1 < k_1 \leq n-2$, $1 < k_2 \leq 4n-1$, $1 < k_3 \leq \min\{2n-1, 2m-1\}$, $1 < k_4 \leq n-2$, $1 < k_5 \leq n-1$, $1 < k_6 \leq 2n-1$, $1 < k_7 \leq \min\{4n+1, 4m+1\}$ and $1 < k_8 \leq 2n-2$. Moreover, '=' means that $p_*\pi_{k_i}(G) = \pi_{k_i}(G/H)$ for compact Lie group G , its closed subgroup H and the natural projection $p : G \to G/H$.

Next, we use Table 1 to find computations or ranges of some Gottlieb groups of symmetric spaces of classical type. As a results, we obtain Table 2, where k_i's are integers such that $1 < k_1 \leq n-2$, $1 < k_2 \leq 4n-1$, $1 < k_3 \leq \min\{2n-1, 2m-1\}$, $1 < k_4 \leq n-2$, $1 < k_5 \leq 4n+1$, $1 < k_6 \leq 2n-1$ and $1 < k_7 \leq 2n-2$. Moreover, '=' means that $G_{k_i}(G/H) = \pi_{k_i}(G/H)$ for compact Lie group G and its closed subgroup H and '2 $\subseteq$' means that $\mathbb{Z}_2 \subseteq G_k$.

2. Preliminaries

Let $f : X \to Y$ be a base point preserving map between two based topological spaces. Then, f induces a homomorphism $f_* : \pi_k(X, x_0) \to \pi_k(Y, y_0)$. Then the image $f_*\pi_k(X, x_0)$ is a subgroup of $\pi_k(Y, y_0)$. We call the subgroup

Table 2

$G_{k_i} \backslash\ k_i \pmod 8$	0	1	2	3	4	5	6	7
$G_{k_1}(SU(n)/SO(n))$	0	$\mathbb{Z}$		$\mathbb{Z}_2$	0	∞	0	0
$G_{k_2}(SU(2n)/Sp(n))$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$		$\mathbb{Z}_2$
$G_{k_3}(U(n+m)/U(n)\times U(m))$		$=$		$=$		$=$		$=$
$G_{k_4}(SO(2n)/SO(n)\times SO(n))$				0		0	0	0
$G_{k_5}(Sp(2n)/Sp(n)\times Sp(n))$		0	0	0				0
$G_{k_6}(Sp(n)/U(n))$	0	0		$\mathbb{Z}_2$	$\mathbb{Z}_2$	0		0
$G_{k_7}(SO(2n)/Sp(n))$	$2\subseteq$	$2\subseteq$	$=$					$=$

$f_*\pi_k(X,x_0)$ of $\pi_k(Y,y_0)$ *the image subgroup of* $\pi_k(Y,y_0)$ *induced by* f. If a map $g:(X,x_0)\to(Y,y_0)$ is homotopic to f, then $f_*=g_*$. Hence the image subgroups are determined by the homotopy classes of continuous maps.

Here, we recall the evaluation subgroup defined by D. H. Gottlieb. Let X be any topological space and X^X function space of maps from X to itself with compact open topology. Then the evaluation subgroup (or Gottlieb group) $G_k(X,x_0)$ is defined by the image subgroup of $\pi_k(X,x_0)$ induced by the evaluation map $\omega:(X^X,id)\to(X,x_0)$ given by $\omega(f)=f(x_0)$, where x_0 is a base point of X and $id:X\to X$ is the identity map.

There are other equivalent definitions. Let S^k be the k-sphere. Consider a continuous function $F:X\times S^k\to X$ such that $F(x,s_0)=x$, where $x\in X$ and s_0 is a base point of S^k. Then the map $f:S^k\to X$ defined by $f(s)=F(x_0,s)$ represents an element $\alpha=[f]\in\pi_k(X,x_0)$. Then the evaluation subgroup $G_k(X,x_0)$ is the set of all elements $\alpha\in\pi_k(X,x_0)$ obtained in the above manner from some F. If an element $\alpha\in G_k(X,x_0)$ is associated to a map F then we call F an *affiliated map of* α.

Equivalently, we have

$$G_n(X,x_0)=\{\partial(\iota_{n+1})\ \mid\ \exists \text{ fibration } X\hookrightarrow E\xrightarrow{p}S^{n+1}\},$$

where ι_{n+1} is the homotopy class induced by the identity from S^{n+1} to itself and ∂ is a connecting homomorphism in the homotopy exact sequence induced by p.

Thus, if the Gottlieb group $G_n(X,x_0)$ is trivial, then any fibration with S^{n+1} as a base space and X as a fiber has a cross section.

Let Y be a Lie group and G be any closed subgroup of Y. Then it is well-known that the homogeneous space Y/G is also a Lie group and the natural quotient map $p:Y\to Y/G$ is a fibration. Thus we have the homotopy sequence of the fibration $p:Y\to Y/G$:

$$\cdots\to\pi_k(G)\xrightarrow{i_*}\pi_k(Y)\xrightarrow{p_*}\pi_k(Y/G)\xrightarrow{\partial_k}\pi_{k-1}(G)\to\cdots.$$

Moreover, there exists the natural action $\mu : (Y/G) \times Y \to Y/G$. For any map $f : S^i \to Y$, the composition: $\phi{:}(Y/G) \times S^i \xrightarrow{1\times f} (Y/G) \times Y \xrightarrow{\mu} Y/G$ is an affiliated map of $[p \circ f]$. Thus we have $p_*\pi_i(Y) \leq G_i(Y/G)$ for all $i \geq 1$ [14].

From the fact, we conclude that the image subgroup $p_*\pi_i(Y)$ is a lower bound of the Gottlieb group $G_i(Y/G)$. Thus the image subgroups induced by natural projection maps are very important objects to compute or to guess the Gottlieb groups of homogenous spaces of compact Lie groups.

3. Image subgroups and evaluation groups of irreducible symmetric spaces

Let $O(n)$ be an orthogonal group of order n, $U(n)$ a unitary group of order n, $Sp(n)$ a symplectic group of order n, $SO(n)$ a special orthogonal group of order n and $SU(n)$ a special unitary group of order n. The irreducible symmetric spaces of classical type are classified into the following 8 types by E. Cartan:

(1) $SU(n)/SO(n)$;
(2) $SU(2n)/Sp(n)$;
(3) $U(n \times m)/U(n) \times U(m)$;
(4) $SO(n+m)/SO(n) \times SO(m)$;
(5) $SO(n+1)/SO(n)$;
(6) $Sp(n)/U(n)$;
(7) $Sp(n+m)/Sp(n) \times Sp(m)$;
(8) $SO(2n)/Sp(n)$.

We categorize the computations of the image subgroups and Gottlieb groups by the above types. In order to compute image subgroups of homotopy groups induced by natural projections from compact Lie groups to their homogeneous spaces, we first introduce the well-known stable homotopy groups [11];

Table 3

$\pi_k \backslash\ k \pmod 8$	0	1	2	3	4	5	6	7
$\pi_{k_1}(U(n))$	0	∞	0	∞	0	∞	0	∞
$\pi_{k_2}(SO(n))$	2	2	0	∞	0	0	0	∞
$\pi_{k_3}(SO(2n)/U(n))$	2	0	∞	0	0	0	∞	2
$\pi_{k_4}(U(2n)/Sp(n))$	0	∞	0	0	0	∞	2	2
$\pi_{k_5}(Sp(2n)/Sp(n) \times Sp(n))$	∞	0	0	0	∞	2	2	0
$\pi_{k_6}(Sp(n))$	0	0	0	∞	2	2	0	∞
$\pi_{k_7}(Sp(n)/U(n))$	0	0	∞	2	2	0	∞	0
$\pi_{k_8}(U(n)/SO(n))$	0	∞	2	2	0	∞	0	0
$\pi_{k_9}(SO(2n)/SO(n) \times SO(n))$	∞	2	2	0	∞	0	0	0

where $\infty \equiv \mathbb{Z}$, $2 \equiv \mathbb{Z}_2$, 0 is the trivial group and k_i's are positive integers such that $1 < k_1 \leq 2n-1$, $0 < k_2 \leq n-2$, $1 < k_3 \leq 2n-2$, $k_4 \leq 4n-1$, $1 < k_5 \leq 4n+2$, $1 < k_6 \leq 4n+1$, $k_7 \leq 2n$, $k_8 \leq n-1$ and $0 < k_9 \leq n-1$.

For Type (1), we obtain the following computations of the image subgroups.

Theorem 3.1. *Let $p : SU(n) \to SU(n)/SO(n)$ be the natural projection and $1 < k \leq n-2$. Then the image subgroups of $\pi_k(SU(n)/SO(n))$ induced by p are as follows.*

$$p_*\pi_k(SU(n)) = \begin{cases} 0 & \textit{for } k \equiv 0,2,4,6 \pmod 8, \\ \mathbb{Z} & \textit{for } k \equiv 1,5 \pmod 8, \\ \pi_k(SU(n)/SO(n)) & \textit{for } k \equiv 3,7 \pmod 8. \end{cases}$$

Proof. Consider the exact sequence

$$\cdots \to \pi_k(SO(n)) \xrightarrow{i_*} \pi_k(SU(n)) \xrightarrow{p_*} \pi_k(SU(n)/SO(n)) \to \cdots$$

induced by a fiber bundle $SO(n) \hookrightarrow SU(n) \xrightarrow{p} SU(n)/SO(n)$.

For $k \equiv 0,2,4,6$, $\pi_k(SU(n)) = 0$. Thus we have $p_*\pi_k(SU(n)) = 0$.

For $k \equiv 1$, we have the sequence

$$\mathbb{Z}_2 \xrightarrow{i_*} \mathbb{Z} \xrightarrow{p_*} \pi_k(SU(n)/SO(n)).$$

Thus i_* is trivial, p_* is injective and therefore, $p_*\pi_k(SU(n)) = \mathbb{Z}$.

Since $\pi_k(SO(n)) = 0$ for $k \equiv 5 \pmod 8$, p_* is injective. Thus we have $p_*\pi_k(SU(n)) \cong \pi_k(SU(n)) = \mathbb{Z}$ for $k \equiv 5 \pmod 8$.

Since $\pi_k(SO(n)) = 0$ for $k \equiv 2,6 \pmod 8$, we have the following exact sequence for $k \equiv 3,7$:

$$\cdots \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \xrightarrow{p_*} \pi_k(SU(n)/SO(n)) \to 0 \to \cdots$$

Thus p_* is surjective. □

From Theorem 3.1 and the fact that the image $p_*\pi_k(SU(n))$ is a lower bound and $\pi_k(SU(n)/SO(n))$ is an upper bound of the Gottlieb group

$$G_k(SU(n)/SO(n)),$$

we obtain the following corollary.

Corollary 3.2. *For $1 < k \leq n-2$,*

$$G_k(SU(n)/SO(n)) = \begin{cases} 0 & \textit{for } k \equiv 0,4,6,7 \pmod 8, \\ \mathbb{Z} & \textit{for } k \equiv 1,5 \pmod 8, \\ \mathbb{Z}_2 & \textit{for } k \equiv 3 \pmod 8. \end{cases}$$

For Type (1), we conclude all Gottlieb groups are the same as the image subgroups except the case $k \equiv 2 \pmod 8$. In this case, Gottlieb groups are trivial or $\mathbb{Z}_2$. However, they have not yet been determined.

For Type (2), we have the following theorem.

Theorem 3.3. *Let $p : SU(2n) \to SU(2n)/Sp(n)$ be the natural projection and $1 < k \leq 4n-1$. Then the image subgroups of $\pi_k(SU(2n)/Sp(n))$ induced by p*

are as follows.

$$p_*\pi_k(SU(2n)) = \begin{cases} 0 & \textit{for } k \equiv 0,2,3,4,6 \pmod 8, \\ \mathbb{Z} & \textit{for } k \equiv 1 \pmod 8, \\ 2\mathbb{Z} & \textit{for } k \equiv 5 \pmod 8, \\ \mathbb{Z}_2 & \textit{for } k \equiv 7 \pmod 8. \end{cases}$$

Proof. Consider the exact homotopy sequence induced by a fiber bundle

$$Sp(n) \hookrightarrow SU(2n) \xrightarrow{p} SU(2n)/Sp(n):$$

$$\cdots \to \pi_k(Sp(n)) \xrightarrow{i_*} \pi_k(SU(2n)) \xrightarrow{p_*} \pi_k(SU(2n)/Sp(n)) \to \cdots$$

Since $\pi_k(SU(2n)/Sp(n)) = 0$ for $k \equiv 0,2,3,4 \pmod 8$, $p_*\pi_k(SU(2n)) = 0$. Moreover, since $\pi_k(SU(2n)) = 0$ for $k \equiv 6 \pmod 8$, $p_*\pi_k(SU(2n)) = 0$.

For $k \equiv 1, 0 \pmod 8$, $\pi_k(Sp(n)) = 0$. Thus we have the following exact sequence for $k \equiv 1$:

$$\cdots \to 0 \xrightarrow{i_*} \mathbb{Z} \xrightarrow{p_*} \pi_k(SU(2n)/Sp(n)) \to 0 \to \cdots.$$

Therefore p_* is an isomorphism and $p_*\pi_k(SU(2n)) = \mathbb{Z}$.

For $k \equiv 5 \pmod 8$, we obtain an exact sequence

$$\cdots \to \mathbb{Z}_2 \xrightarrow{i_*} \mathbb{Z} \xrightarrow{p_*} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}_2 \to 0 \to \cdots.$$

Therefore, i_* is trivial and p_* is a monomorphism. Thus

$$p_*\pi_k(SU(2n)) = \text{Ker } \partial = 2\mathbb{Z}.$$

For $k \equiv 7 \pmod 8$, we obtain an exact sequence

$$\cdots \to \mathbb{Z} \xrightarrow{p_*} \mathbb{Z}_2 \xrightarrow{\partial} 0 \to \cdots.$$

Therefore, p_* is an epimorphism and $p_*\pi_k(SU(2n)) = \mathbb{Z}_2$. □

From the above theorem and the fact that the image $p_*\pi_k(SU(n))$ is a lower bound and $\pi_k(SU(2n)/Sp(n))$ is an upper bound of the Gottlieb group $G_k(SU(2n)/Sp(n))$, we obtain the following corollary.

Corollary 3.4. *For* $1 < k \leq n-2$,

$$G_k(SU(2n)/SP(n)) = \begin{cases} 0 & \textit{for } k \equiv 0,2,3,4 \pmod 8, \\ \mathbb{Z} & \textit{for } k \equiv 5 \pmod 8, \\ \mathbb{Z}_2 & \textit{for } k \equiv 7 \pmod 8. \end{cases}$$

For Type (3), we have the following theorem.

Theorem 3.5. *Let* $p : U(n+m) \to U(n+m)/U(n) \times U(m)$ *be the natural projection and* k *be an integer such that* $1 \leq k \leq \min\{2n-1, 2m-1\}$. *Then the image subgroup of* $\pi_k(U(n+m))$ *induced by* p *is trivial for* $k \equiv 0,2,4,6 \pmod 8$ *and is the same as*

$$\pi_k(U(n+m)/U(n) \times U(m))$$

for $k \equiv 1,3,5,7 \pmod 8$.

Proof. Consider the homotopy exact sequence induced by a fiber bundle

$$U(n) \times U(m) \hookrightarrow U(n+m) \xrightarrow{p} U(n+m)/U(n) \times U(m):$$

$$\cdots \to \pi_k(U(n)\times U(m)) \xrightarrow{i_*} \pi_k(U(n+m)) \xrightarrow{p_*} \pi_k(U(n+m)/U(n)\times U(m)) \to \cdots .$$

For $k \equiv 0, 2, 4, 6 \pmod 8$, $\pi_k(U(n+m)) = 0$. So

$$p_*\pi_*(U(n+m)) = 0.$$

For $k \equiv 1, 3, 5, 7 \pmod 8$, the sequence

$$\pi_k(U(n+m)) \xrightarrow{p_*} \pi_k(U(n+m)/U(n) \times U(m)) \xrightarrow{\partial} \pi_k(U(n) \times U(m))$$

is same as

$$\mathbb{Z} \xrightarrow{p_*} \pi_k(U(n+m)/U(n) \times U(m)) \xrightarrow{\partial} 0.$$

Thus, p_* is surjective. □

By Theorem 3.5, we have the following corollary.

Corollary 3.6. *For* $k \equiv 1, 3, 5, 7 \pmod 8$*, we have*

$$G_k(U(n+m)/U(n) \times U(m)) = \pi_k(U(n+m)/U(n) \times U(m)).$$

For Type (4), we have the following theorem.

Theorem 3.7. *Let* $p : SO(n+m) \to SO(n+m)/SO(n) \times SO(m)$ *be the natural projection and* k *be an integer such that* $0 < k \leq \min\{n-2, m-2\}$. *Then*

$$p_*\pi_k(SO(n+m)) = 0$$

for $k \equiv 2, 4, 5, 6 \pmod 8$ *and*

$$p_*\pi_k(SO(n+m)) = \pi_k(SO(n+m)/SO(n) \times SO(m))$$

for $k \equiv 3, 7 \pmod 8$.

In particular, if $m = n$*, then we have*

$$p_*\pi_k(SO(2n)) = 0$$

for all k *such that* $0 < k \leq n-2$.

Proof. Consider the homotopy exact sequence induced by a fiber bundle

$$SO(n) \times SO(m) \hookrightarrow SO(n+m) \xrightarrow{p} SO(n+m)/SO(n) \times SO(m):$$

$$\cdots \to \pi_k(SO(n) \times SO(m)) \xrightarrow{i_*} \pi_k(SO(n+m)) \xrightarrow{p_*}$$
$$\pi_k(SO(n+m)/SO(n) \times SO(m)) \to \cdots .$$

Since $\pi_k(SO(n+m))$ is trivial for $k \equiv 2, 4, 5, 6 \pmod 8$,

$$p_*\pi_k(SO(n+m)) = 0.$$

For $k \equiv 3, 7 \pmod 8$, we have an exact sequence

$$\mathbb{Z} \xrightarrow{p_*} \pi_k(SO(n+m)/SO(n) \times SO(m)) \xrightarrow{\partial} 0.$$

Thus, p_* is an epimorphism and

$$p_*\pi_k(SO(n+m)) = \pi_k(SO(n+m)/SO(n)\times SO(m)).$$

Suppose $m = n$. Then, since $\pi_k(SO(2n)/SO(n)\times SO(n)) = 0$ for $k \equiv 3,7 \pmod 8$, $p_*\pi_k(SO(2n)) = 0$.

For $k \equiv 0 \pmod 8$, we have an exact sequence

$$\mathbb{Z}_2 \xrightarrow{p_*} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}\oplus\mathbb{Z}.$$

Thus p_* is a trivial map and $p_*\pi_*(SO(2n)) = 0$.

For $k \equiv 1 \pmod 8$, we have an exact sequence

$$\mathbb{Z}_2 \xrightarrow{p_*} \mathbb{Z}_2 \xrightarrow{\partial} \mathbb{Z}_2\oplus\mathbb{Z}_2 \xrightarrow{i_*} \mathbb{Z}_2.$$

Thus, $p_*\pi_k(SO(2n)) = 0$ or $\mathbb{Z}_2$.

Assume that $p_*\pi_k(SO(2n)) = \mathbb{Z}_2$. Then p_* is an isomorphism. This implies $\partial = 0$ and i_* is a monomorphism. However, this is a contradiction. Thus $p_*\pi_k(SO(2n)) = 0$. □

Corollary 3.8. *For $k \equiv 3,7 \pmod 8$, we have*

$$G_k(SO(n+m)/SO(n)\times SO(m)) = \pi_k(SO(n+m)/SO(n)\times SO(m)).$$

In particular,

$$G_k(SO(2n)/SO(n)\times SO(n)) = 0$$

for $k \equiv 3,5,6,7 \pmod 8$.

Since $SO(n+1)/SO(n) \equiv S^n$ and $\pi_k(S^n) = 0$ for $k = 1,2,\ldots,n-1$, we have the following theorem for Type (5).

Theorem 3.9. *Let $p : SO(n+1) \to SO(n+1)/SO(n)$ be the natural projection and k be an integer such that $0 < k \le n-1$, $p_*\pi_k(SO(n+1))$ is trivial.*

For Type (6), we have the following theorem.

Theorem 3.10. *Let $p : Sp(n) \to Sp(n)/U(n)$ be the natural projection and k be an integer such that $0 < k \le 2n-1$.*

$$p_*\pi_k(Sp(n)) = \begin{cases} 0 & \text{for } k \equiv 0,1,2,5,6,7 \ (\mathrm{mod} 8), \\ \mathbb{Z}_2 & \text{for } k \equiv 3,4 \pmod 8. \end{cases}$$

Proof. Consider the homotopy exact sequence induced by a fiber bundle

$$U(n) \xrightarrow{i} Sp(n) \xrightarrow{p} Sp(n)/U(n):$$

$$\cdots \to \pi_k(U(n)) \xrightarrow{i_*} \pi_k(Sp(n)) \xrightarrow{p_*} \pi_k(Sp(n)/U(n)) \to \cdots.$$

Since $\pi_k(Sp(n))$ is trivial for $k \equiv 0,1,2,6 \pmod 8$, $p_*\pi_k(Sp(n)) = 0$. Moreover, since $\pi_k(Sp(n)/U(n)) = 0$ for $k \equiv 0,1,5,7 \pmod 8$, $p_*\pi_k(Sp(n)) = 0$.

For $k \equiv 3 \pmod 8$, the homotopy exact sequence

$$\to \pi_k(Sp(n)) \xrightarrow{p_*} \pi_k(Sp(n)/U(n)) \to \pi_{k-1}(U(n))$$

becomes

$$\mathbb{Z} \xrightarrow{p_*} \mathbb{Z}_2 \to 0.$$

Therefore, p_* is an epimorphism and thus $p_*\pi_k(Sp(n)) = \mathbb{Z}_2$.

For $k \equiv 4$, the homotopy exact sequence

$$\to \pi_k(U(n)) \xrightarrow{i_*} \pi_k(Sp(n)) \xrightarrow{p_*} \pi_k(Sp(n)/U(n)) \xrightarrow{\partial} \pi_{k-1}(U(n)) \to$$

becomes

$$0 \xrightarrow{i_*} \mathbb{Z}_2 \xrightarrow{p_*} \mathbb{Z}_2 \to 0.$$

Thus p_* is an isomorphism and as a result, $p_*\pi_k(Sp(n)) = \mathbb{Z}_2$. □

From Theorem 3.10, we have the following corollary.

Corollary 3.11. *For $1 < k \leq 2n-1$,*

$$G_k(Sp(n)/U(n)) = \begin{cases} 0 & \text{for } k \equiv 0,1,5,7 \pmod 8, \\ \mathbb{Z}_2 & \text{for } k \equiv 3,4 \pmod 8). \end{cases}$$

For Type (7), we have the following theorem.

Theorem 3.12. *Let $p : Sp(n+m) \to Sp(n+m)/Sp(n) \times Sp(m)$ be the natural projection and k be an integer such that $0 < k \leq \min\{4n+1, 4m+1\}$. Then*

$$p_*\pi_k(Sp(n+m)) = 0$$

for $k \equiv 0,1,2,4,5,6 \pmod 8$ and

$$p_*\pi_k(Sp(n+m)) = \pi_k((Sp(n+m)/Sp(n) \times Sp(m))$$

for $k \equiv 3,7 \pmod 8$.

In particular, if $m = n$, then we have

$$p_*\pi_k(Sp(2n)) = 0$$

for all k such that $0 < k \leq 4n+1$.

Proof. Consider the homotopy exact sequence induced by a fiber bundle

$$Sp(n) \times Sp(m) \hookrightarrow Sp(n+m) \xrightarrow{p} Sp(n+m)/Sp(n) \times Sp(m):$$

$$\cdots \to \pi_k(Sp(n) \times Sp(m)) \xrightarrow{i_*} \pi_k(Sp(n+m)) \xrightarrow{p_*} \\ \pi_k(Sp(n+m)/Sp(n) \times Sp(m)) \to \cdots.$$

Since $\pi_k(Sp(n+m)) = 0$ for $k \equiv 0,1,2,6 \pmod 8$,

$$p_*\pi_k(Sp(n+m)) = 0.$$

The sequence

$$\pi_k(Sp(n+m)) \to \pi_k(Sp(n+m)/Sp(n) \times Sp(m)) \to \pi_{k-1}(Sp(n) \times Sp(m))$$

becomes

$$\mathbb{Z} \xrightarrow{p_*} \pi_k(Sp(n+m)/Sp(n) \times Sp(m)) \xrightarrow{\partial} 0$$

for $k \equiv 3,7 \pmod 8$. Thus p_* is an epimorphism and hence,

$$p_*\pi_k(Sp(n+m)) = \pi_k(Sp(n+m)/Sp(n) \times Sp(m)).$$

Suppose $m = n$. Then, since $\pi_k(Sp(2n)/Sp(n) \times Sp(n)) = 0$ for $k \equiv 3, 7 \pmod 8$, $p_*\pi_k(Sp(2n)) = 0$.

For $k \equiv 4 \pmod 8$, we have an exact sequence

$$\mathbb{Z}_2 \xrightarrow{p_*} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i_*} \mathbb{Z}.$$

Thus p_* is trivial map and $p_*\pi_*(Sp(2n)) = 0$.

For $k \equiv 5 \pmod 8$, we have an exact sequence

$$\mathbb{Z}_2 \xrightarrow{p_*} \mathbb{Z}_2 \xrightarrow{\partial} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{i_*} \mathbb{Z}_2.$$

Thus $p_*\pi_k(Sp(2n)) = 0$ or $\mathbb{Z}_2$.

Assume that $p_*\pi_k(Sp(2n)) = \mathbb{Z}_2$. Then p_* is an isomorphism. This implies $\partial = 0$ and i_* is a monomorphism. However, this is a contradiction. Thus, $p_*\pi_k(Sp(2n)) = 0$. □

Corollary 3.13. *For $k \equiv 3, 7 \pmod 8$, we have*

$$G_k(Sp(n+m)/Sp(n) \times Sp(m)) = \pi_k(Sp(n+m)/Sp(n) \times Sp(m)).$$

In particular, $G_k(Sp(2n)/Sp(n) \times Sp(n)) = 0$ for $k \equiv 1, 2, 3, 7 \pmod 8$.

For Type (8), we have the following theorem.

Theorem 3.14. *Let $p : SO(2n) \to SO(n)/Sp(n)$ be the natural projection and k be an integer such that $k \leq 2n - 2$. Then*

$$p_*\pi_k(SO(2n)) = \begin{cases} 0 & \text{for } k \equiv 2, 4, 5, 6 \pmod 8, \\ \mathbb{Z}_2 & \text{for } k \equiv 0, 1 \pmod 8, \\ \pi_k(SO(2n)/Sp(n)) & \text{for } k \equiv 3, 7 \pmod 8. \end{cases}$$

Proof. Consider the homotopy exact sequence induced by a fiber bundle

$$Sp(n) \xrightarrow{i} SO(2n) \xrightarrow{p} SO(2n)/Sp(n):$$

$$\cdots \to \pi_k(Sp(n)) \xrightarrow{i_*} \pi_k(SO(2n)) \xrightarrow{p_*} \pi_k(SO(2n)/Sp(n)) \to \cdots.$$

Since $\pi_k(SO(2n)) = 0$ for $k \equiv 2, 4, 5, 6 \pmod 8$, $p_*\pi_k(SO(2n)) = 0$. Furthermore, since $\pi_k(Sp(n)) = 0$ and $\pi_k(SO(2n)) = \mathbb{Z}_2$ for $k \equiv 0, 1 \pmod 8$, the homotopy sequence becomes

$$\cdots \to 0 \to \mathbb{Z}_2 \xrightarrow{p_*} \pi_k(SO(2n)/Sp(n)) \to \cdots.$$

Thus p_* is a monomorphism and

$$p_*\pi_k(SO(2n)) \cong \pi_k(SO(2n)) \cong \mathbb{Z}_2.$$

For $k \equiv 3, 7 \pmod 8$, we have an exact sequence

$$\pi_k(SO(2n)) \xrightarrow{p_*} \pi_k(SO(2n)/Sp(n)) \xrightarrow{\partial} 0.$$

Thus p_* is an epimorphism and hence,

$$p_*\pi_k(SO(2n)) = \pi_k(SO(2n)/Sp(n)).$$ □

From Theorem 3.14, we have following corollary.

Corollary 3.15. *For* $k \equiv 3, 7 \pmod 8$*, we have*

$$G_k(SO(2n)/Sp(n)) = \pi_k(SO(2n)/Sp(n)).$$

In particular, $\mathbb{Z}_2 \subseteq G_k(SO(2n)/Sp(n))$ *for* $k \equiv 0, 1 \pmod 8$.

Through theorems and corollaries, we obtain the following two tables for image subgroups and for Gottlieb groups of symmetric spaces of classical type respectively. For the image subgroups, we have:

Type, $p_*\pi_k \backslash\ k_i \pmod 8$	0	1	2	3	4	5	6	7
(1) $p_*\pi_{k_1}(SU(n))$	0	$\mathbb{Z}$	0	=	0	$\mathbb{Z}$	0	=
(2) $p_*\pi_{k_2}(SU(2n))$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
(3) $p_*\pi_{k_3}(U(n+m))$	0	=	0	=	0	=	0	=
(4) $p_*\pi_{k_4}(SO(2n))$	0	0	0	0	0	0	0	0
(5) $p_*\pi_{k_5}(SO(n+1))$	0	0	0	0	0	0	0	0
(6) $p_*\pi_{k_6}(Sp(n))$	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0
(7) $p_*\pi_{k_7}(Sp(n+m))$	0	0	0	=	0	0	0	=
(8) $p_*\pi_{k_8}(SO(2n))$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	=	0	0	0	=

where k_i's are integers such that $1 < k_1 \leq n-2$, $1 < k_2 \leq 4n-1$, $1 < k_3 \leq \min\{2n-1, 2m-1\}$, $1 < k_4 \leq n-2$, $1 < k_5 \leq n-1$, $1 < k_6 \leq 2n-1$, $1 < k_7 \leq \min\{4n+1, 4m+1\}$, $1 < k_8 \leq 2n-2$ and '=' means that $p_*\pi_{k_i}(G) = \pi_{k_i}(G/H)$ for compact Lie group G , its closed subgroup H and the natural projection $p : G \to G/H$.

For the Gottlieb groups, we have:

$G_{k_i} \backslash\ k_i \pmod 8$	0	1	2	3	4	5	6	7
$G_{k_1}(SU(n)/SO(n))$	0	$\mathbb{Z}$		$\mathbb{Z}_2$	0	∞	0	0
$G_{k_2}(SU(2n)/Sp(n))$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$		$\mathbb{Z}_2$
$G_{k_3}(U(n+m)/U(n) \times U(m))$		=		=		=		=
$G_{k_4}(SO(2n)/SO(n) \times SO(n)))$				0		0	0	0
$G_{k_5}(Sp(2n)/Sp(n) \times Sp(n))$		0	0	0				0
$G_{k_6}(Sp(n)/U(n))$	0	0		$\mathbb{Z}_2$	$\mathbb{Z}_2$	0		0
$G_{k_7}(SO(2n)/Sp(n))$	$2 \subseteq$	$2 \subseteq$	=					=

where k_i's are integers such that $1 < k_1 \leq n-2$, $1 < k_2 \leq 4n-1$, $1 < k_3 \leq \min\{2n-1, 2m-1\}$, $1 < k_4 \leq n-2$, $1 < k_5 \leq 4n+1$, $1 < k_6 \leq 2n-1$, $1 < k_7 \leq 2n-2$ and '=' means that $G_{k_i}(G/H) = \pi_{k_i}(G/H)$ for compact Lie group G and its closed subgroup H and '$2 \subseteq$' means that $\mathbb{Z}_2 \subseteq G_k$.

References

[1] D. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. **87** (1965), 840–856.

[2] ______, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91** (1969), 729–756.
[3] ______, *Covering transformations and universal fibrations*, Illinois J. Math. **13** (1969), 432–437.
[4] M. Golasinski and J. Mukai, *Gottlieb groups of spheres*, Topology, **47** (2008), no. 6, 399–430.
[5] S. Hu, *Homotopy Theory*, Acdemic press 1959.
[6] B. Jiang, *Estimation of the Nielsen numbers*, Acta Math. Sinica **14** (1964), 304–312.
[7] ______, *Lectures on Nielsen fixed point theory*, Contemporary Mathematics, 14. American Mathematical Society, Providence, R.I., 1983.
[8] T. H. Kiang, *The Theory of Fixed Point Classes*, Springer-Verlag, Berlin/Science Press, Beijing.
[9] K. Lee, M. Mimura, and M. Woo, *Gottlieb groups of homogenous spaces*, Topology Appl. **145** (2004), no. 1-3, 147–155.
[10] K. Lee and M. Woo, *The G-sequence and the ω - homology of CW-pair*, Topology Appl. **52** (1993), no. 3, 221–236.
[11] M. Mimura, *Topolgy of Lie groups. I and II*, Tran. Math. Monographs **91**, Amer. Math. Soc., Providence, RI, 1991.
[12] J. Pak and M. Woo, *A remark on G-sequences*, Math. Japan. **46** (1997), no. 3, 427–432.
[13] J. Pan, X. Shen, and M. Woo, *The G-sequence of a map and its exactness*, J. Korean Math. Soc. **35** (1998), no. 2, 281–294.
[14] J. Siegel, *G-spaces, H-spaces and W-spaces*, Pacific J. Math. **31** (1969), no. 1, 209–214.
[15] M. Woo and J. Kim, *Certain subgroups of homotopy groups*, J. Korean Math. Soc. **21** (1984), no. 2, 109–120.
[16] M. Woo and K. Lee, *On the relative evaluation subgroups of a CW-pair*, J. Korean Math. Soc. **25** (1988), no. 1, 149–160.

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