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2-GOOD RINGS AND THEIR EXTENSIONS

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Abstract. P. Vámos called a ring R 2-good if every element is the sum of two units. The ring of all $n \times n$ matrices over an elementary divisor ring is 2-good. A (right) self-injective von Neumann regular ring is 2-good provided it has no 2-torsion. Some of the earlier results known to us about 2-good rings (although nobody so called at those times) were due to Ehrlich, Henriksen, Fisher, Snider, Rapharl and Badawi. We continue in this paper the study of 2-good rings by several authors. We give some examples of 2-good rings and their related properties. In particular, it is shown that if R is an exchange ring with Artinian primitive factors and 2 is a unit in R, then R is 2-good. We also investigate various kinds of extensions of 2-good rings, including the polynomial extension, Nagata extension and Dorroh extension.

1. Introduction

Throughout this paper all rings are associative with identity and all modules are unitary. We denote the multiplicative group of units (invertible elements) of the ring R by $U(R)$, the nil radical by $N(R)$ and the Jacobson radical by $J(R)$, and we write $\mathbb{Z}$ for the ring of integers, write $M_n(R)$ and $T_n(R)$ for the rings of all $n \times n$ matrices and all $n \times n$ upper triangular matrices over the ring R, respectively. Recall that a ring R is (*von Neumann*) *regular* if for each a in R there exists an x in R such that $a = axa$. A ring R is called *strongly regular* [7] if for any $a \in R$ there is an $x \in R$ such that $a^2x = a$. A ring R is *unit-regular* [10] provided that for each $x \in R$ there exists a $u \in U(R)$ such that $xux = x$. A ring R is π-*regular* [8] if for each $a \in R$ there exists an $x \in R$ and a positive integer n such that $a^n = a^nxa^n$. Call a ring R *strongly* π-*regular* [8] if for every element $a \in R$ there exists a positive integer number n (depending on a) and an element $x \in R$ such that $a^n = a^{n+1}x$. A ring is *elementary division* if square matrices can be diagonalized, that is, equivalent to a diagonal matrix (see [11], [18]).

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Vámos [18] called an element a in a ring R 2-*good* if a is the sum of two units, and called R 2-*good* if every element in R is 2-good. In [18], Vámos showed that every ring can be embedded in a 2-good ring and that a (right) self-injective Von Neumann regular ring is 2-good provided it has no 2-torsion. Some of the earlier results known to us about 2-good rings (although nobody so called at those times) were due to Ehrlich, Henriksen, Fisher, Snider and Raphael. Ehrlich [7, Theorem 7] showed that if R is unit regular and 2 is a unit in R, then R is 2-good. Henriksen [11, Theorem 11] showed that the ring of all $n \times n$ matrices over an elementary divisor ring is 2-good. Fisher-Snider [8, Theorem 3] showed that if R is strongly π-regular and 2 is a unit in R, then R is 2-good. Raphael [15, Proposition 2] showed that strongly regula rings are 2-good. 2-good rings were studied under the name $(s, 2)$-*ring* by Badawi [1, 2]. [1, Theorem 4] showed that a π-regular ring R is 2-good if and only if every idempotent element in R is a sum of two units of R, and [2, Theorem 6] showed that if R is abelian π-regular, then R is 2-good if and only if $Z/2Z$ is not a homomorphic image of R. In this paper we continue the study of 2-good rings, give some examples of 2-good rings and their related properties, and investigate various kinds of extensions of 2-good rings.

2. Examples and basic properties

Example 2.1. Every division ring R, which is not isomorphic to $\mathbb{Z}_2$, is 2-good. For $a \in R$, if $a \neq 1$, then $a = 1 + (a - 1)$; if $a = 1$, then $1 = b + (1 - b)$ for $b\ (\neq 1) \in R$.

Example 2.2. If $R\ (\neq 0)$ is local and $2 \in U(R)$, then R is 2-good. In fact, if $a \in J(R)$ we have $a = 1 + (a - 1)$, while if $a \notin J(R)$, then $a = \frac{1}{2}a + \frac{1}{2}a$.

Example 2.3. If X is a completely regular Hausdorff space, then the ring $C(X)$ of real valued continuous functions on X is 2-good. Indeed, any $f(x) \in C(X)$ can be written as $f(x) = [f(x) \vee o] + [(f(x) \wedge o) - 1]$, a sum of two units in $C(X)$ (see Gillman-Jerison [9, pp.11–15]).

Example 2.4 (Ye [22, Corollary 3.1])**.** Let $p\ (\neq 2)$ be a prime number, $\mathbb{Z}_{(p)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, (p, n) = 1\}$, and $G = (a) = \{e, a, a^2\}$ a cyclic group of order 3. Then the group ring $\mathbb{Z}_{(p)}G$ is 2-good.

Recall that a ring R is called *clean* if every element in R is the sum of an idempotent and a unit in R.

Proposition 2.5 (Camillo-Yu [4, Proposition 10])**.** *If R is clean and $2 \in U(R)$, then R is 2-good.*

It is worth noting that $2 \in U(R)$ is necessary in Proposition 2.5. Indeed, the ring $R = \mathbb{Z}/(6)$ is clean, and $\bar{2} \notin U(R)$, but it is not 2-good. For, $\bar{5} \in R$ can not be written as a sum of two units.

The concepts of clean rings and 2-good ring are independent of each other. This is illustrated by examples below.

Example 2.6. Let R be a Boolean ring with more than two elements. For any $x \in R$, we have $x = (x-1)+1$ with $(x-1)^2 = x-1$ and $1 \in U(R)$. Hence R is a clean ring. Suppose $x \in U(R)$, then $1 = xx^{-1} = x^2x^{-1} = x$. Thus $U(R) = \{1\}$, $1+1=0$, and 0 is the only element which can be written as a sum of units. So R is not 2-good.

Example 2.7. Let ring $R = \{\mathrm{diag}(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in \mathbb{Z}\}$. For any $A = \sum_{i=1}^{n} a_i E_{ii} \in R$, if $a_{i_1}, a_{i_2}, \ldots, a_{i_k} = 0$ and else where $a_j \neq 0$. Without loss of generality we may assume that $a_1, \ldots, a_k = 0$ and $a_{k+1}, \ldots, a_n \neq 0$. Put $A_1 = \sum_{i=1}^{k} 1E_{ii} + \sum_{j=k+1}^{n} b_j E_{jj}$, $A_2 = \sum_{i=1}^{k}(-1)E_{ii} + \sum_{j=k+1}^{n} c_j E_{jj}$, where $b_j \neq 0$, $c_j \neq 0$ and $b_j + c_j = a_j$ for all $k+1 \leq j \leq n$. Then $A = A_1 + A_2$ with $A_1,\ A_2 \in U(R)$. This prove that R is 2-good.

But element $B = 1E_{11} + \sum_{i=2}^{n} 0E_{ii}$ can not be expressed a sum of an idempotent and a unit of R. For, if $B = C + D$ with $C^2 = C = \mathrm{diag}(c_1, c_2, \ldots, c_n) \in R$, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n) \in R$, then from $c_1^2 = c_1$ it follows that $c_1 = 1$ and $d_1 = 0$. Thus, D is not a unit of R.

Following Goodearl-Menal [10], an associative ring R is said to satisfy *unit 1-stable range* if $aR + bR = R$ with $a, b \in R$ implies that there exists some $u \in U(R)$ such that $a + bu \in U(R)$. [10, Theorem 3.1] proved that algebraic algebras over infinite field satisfies unit 1-stable range, and Chen [5, Theorem 2.2] showed that if R satisfies unit 1-stable range, then so dose $M_n(R)$ for any integer number $n \geq 1$.

Proposition 2.8. *Every ring R satisfying unit* 1*-stable range is* 2*-good.*

Proof. For any $a \in R$, there exists $u \in U(R)$ such that $a + 1 \cdot u \in U(R)$ since $aR + 1 \cdot R = R$. It follows that $a = (-u) + v$ with $-u, v \in U(R)$. □

Corollary 2.9. *If R is an algebraic algebra over an infinite field F, then R is* 2*-good.*

Ehrlich [7] proved that regular rings satisfy the minimum condition on right (left) ideals, semisimple Artinian rings, strongly regular rings and commutative regular rings are all unit regular rings. Fisher-Snider [8, Theorem 1] showed that regular rings with primitive factor Artinian is also unit regular. Thus by [7, Theorem 7] we have the following examples of 2-good rings.

Proposition 2.10. *Let R be a ring and $2 \in U(R)$.*

(1) *If R is semisimple Artinian, then R is* 2*-good.*

(2) *If R is strongly regular, then R is* 2*-good.*

(3) *If R is commutative regular, then R is* 2*-good.*

(4) *If R is a regular ring with primitive factor Artinian, then R is* 2*-good.*

(5) *If R is a regular ring satisfies the minimum condition on right* (*left*) *ideals, then R is* 2*-good.*

Let M_R be a right R-module. Following Crawley-Jonsson [6], M_R is said to have the *exchange property* if for every module A_R and any two decompositions of A_R

$$A_R = M' \bigoplus N = \bigoplus_{i \in I} A_i,$$

where $M'_R \cong M_R$, there exist submodules $A'_i \subseteq A_i$ such that

$$A = M' \bigoplus (\bigoplus_{i \in I} A'_i).$$

Many familiar classes of modules have the exchange property, see Zimmermann-Huisgen and Zimmermann [24] for a list of these classes of modules.

Warfield [20] introduced the class of exchange rings. He called a ring R an *exchange ring* if R_R has the exchange property above and proved that this definition is left-right symmetric. The class of exchange rings is quite large. Call a ring R *semiregular* (*semi-π-regular, semi-strongly π-regular*) if $R/J(R)$ is regular (π-regular, strongly π-regular) and idempotents can be lifted modulo $J(R)$. The following classes of rings are all contained in the class of exchange rings: (1) clean rings (Nicholson [13, Proposition 1.8(1)]); (2) local rings; (3) semiperfect rings; (4) semiregular rings; (5) semistrongly π-regular rings; (6) semi-π-regular rings ((2)-(6) see, for example, Stock [16, p. 440] and Tuganbaev [17, Theorem 2.11]).

Theorem 2.11. *Let R be an exchange ring with Artinian primitive factors. If $2 \in U(R)$, then R is 2-good.*

Proof. Assume that R is not a 2-good ring, then there exists $a \in R$ which cannot be expressed as a sum of two units of R. Put

$$\Omega = \{I \mid I \lhd R, \overline{a} \text{ cannot be expressed as a sum of two units of } R/I\},$$

then Ω is nonempty. Let $\{I_\alpha\}$ be a chain in Ω and set $I = \underset{\alpha}{\cup} I_\alpha$. Then $I \lhd R$. If $I \notin \Omega$, then $\overline{a}$ is a sum of two units of R/I. Then there exist $u+I, v+I \in U(R/I)$, such that $a+I = (u+I)+(v+I)$. Hence, $\{a-(u+v), uu'-1, u'u-1, vv'-1, v'v-1\} \subseteq I$ for some $u', v' \in R$. Thus there exits β such that $\{a-(u+v), uu'-1, u'u-1, vv'-1, v'v-1\} \subseteq I_\beta$. So $\overline{a}$ is a sum of two units of R/I_β. This contradicts the choice of I_β, so $I \in \Omega$. By Zorn's Lemma, Ω contain a maximal element A. Let $S = R/A$. The maximality of $A \in \Omega$ implies that S is indecomposable as a ring.

If $J(S) \neq 0$, then $J(S) = B/A$ with $B \supset A$. By the maximality of A, $a = a + A \in S$ is the sum of two units of R/A. From $S/J(S) \cong R/B$, we have

$$(a+A) + J(S) = [(u_1 + A) + J(S)] + [(v_1 + A) + J(S)]$$

with $(u_1 + A) + J(S), (v_1 + A) + J(S) \in U(S/J(S))$. Since units lift modulo the Jacobson radical of S, so that $\bar{u_1} = u_1 + A, \bar{v_1} = v_1 + A \in U(S)$. Thus

$$\bar{a} = (\bar{u_1} + \bar{v_1}) + \bar{r}$$

for some $\bar{r} = r + A \in J(S)$. Note that $\bar{v_1} + \bar{r} = \bar{v_1}(1 + \bar{v_1}^{-1}\bar{r}) \in U(S)$, hence $\bar{a} = \bar{u_1} + \bar{v_1}(1 + \bar{v_1}^{-1}\ \bar{r})$ is a sum of two units of S. In this final case we have contradicted the choice of A. Thus, we see that $J(S) = 0$.

Since R is an exchange ring with Artinal primitive factors, by virtue of Yu [23, Lemma 3.7], S is simple Artinian. By Proposition 2.10(1), $\bar{a}$ can be expressed as a sum of two units of S, and it yields a contradiction. Therefor R is 2-good. □

Corollary 2.12. *Let R be a regular* (*resp. π-regular, semiregular, clean, local, semiperfect, semistrongly π-regular, semi-π-regular*) *ring with primitive factor rings Artinian. If $2 \in U(R)$, then R is 2-good.*

Proposition 2.13. *Let F be a field and $2 \neq 0$, G a finite group and* $\mathrm{char}(F)$ *do not divide $|G|$. Then group ring FG is 2-good.*

Proof. In virtue of Kelarev [12, Theorem 3.1] (Maschke's Theorem) FG is an semisimple Artinian ring. The result follows by Proposition 2.10. □

Recall that a semigroup S is called *t.u.p.* (*two-unique-product*) *semigroup* if, for any nonempty finite subsets X, Y with $|X| + |Y| > 2$, there exist at least two elements in S that have unique presentations as xy, for some $x \in X$, $y \in Y$ (see [14]).

Proposition 2.14. *Let K be a finite field and $2 \neq 0$, S be a finite t.u.p. semigroup. Then semigroup ring KG is 2-good.*

Proof. Let $a \in KG$, say $a = k_1 s_1 + \cdots + k_n s_n$ where $k_i \in K$ and $s_i \in S$ for each i. Thus $a \in K_0 S_0$ where K_0 is the subfield of K generated by $\{k_1, \ldots, k_n\}$, and S_0 is the sub-semigroup of S generated by $\{s_1, \ldots, s_n\}$. By hypothesis, $R_0 S_0$ is a finite ring. Hence $R_0 S_0$ is Artinian. According to Okninski [14, Corollary 10.5], $J(KS) = 0$ and hence KS is semisimple Artinian ring. The result follows by Proposition 2.10. □

Let $S(R)$ be the nonempty set of all the proper ideals of R generated by central idempotents. Recall that the factor ring R/P is called a *Pierce stalk* (see Tuganbaev [17]) of R if P is a maximal element in $S(R)$.

Proposition 2.15. *For a ring R, the following are equivalent*:

(1) *R is 2-good.*

(2) *All homomorphism images of R are 2-good.*

(3) *All indecomposable factor rings of R are 2-good.*

(4) *R/I is 2-good for some ideal I of R contained in $J(R)$.*

(5) *A/I is 2-good for every proper ideal I of R generated by central idempotents of R.*

(6) *All Pierce stalks of R are 2-good.*

Proof. (1)⇒(2)⇒(3), (1)⇒(2)⇒(4) and (2)⇒(5)⇒(6) are trivial. (4)⇒(1) is a corollary of [18, Lemma 2(a)].

(6)$\Rightarrow$ (1) If R is not 2-good. Put

$$\Omega = \{I \lhd R \mid I \text{ is a proper ideal generated by central idempotents of } R \text{ such that } R/I \text{ is not 2-good}\}.$$

Then $\Omega \neq \phi$ since $\{0\} \in \Omega$. It is easily verified that the union of every chain of ideals from Ω is contained in Ω. By Zorn's Lemma, Ω contains a maximal element J. We next prove that J is generated by central idempotents. Assume the contrary, then there is a central idempotent e such that $J+eR$ and $J+(1-e)R$ are proper ideals of R and properly contain J. Since $(J+eR)\cap(J+(1-e)R) = J$, $(J+eR)+(J+(1-e)R) = R$ and $R^2+(J+eR) = R = R^2+(J+(1-e)R)$, by Chinese Remainder Theorem, $R/J \cong R/(J+eR)\times R/(J+(1-e)R)$. The maximality of $J \in \Omega$ implies that $J+eR$ and $J+(1-e)R$ are not in Ω, hence $R/(J+eR)$ and $R/(J+(1-e)R)$ are 2-good. It follows that R/J is 2-good, and it yields a contradiction. Thus we see that R/J is a Pierce stalk. By hypothesis, R/J is 2-good, a contradiction. Therefore R is 2-good.

(3)$\Rightarrow$(1) It is similar to (6)$\Rightarrow$(1), we omit the proof. □

Corollary 2.16. *Let $e^2 = e \in R$. Then eR is 2-good if and only if so is eRe.*

Proof. Put $\sigma : eR \to eRe, \sigma(x) = xe$. It is easy to see that σ is a epimorphism of rings, $\ker\sigma = eR(1-e)$, $eR/\ker\sigma \cong eRe$, and $\ker\sigma \subseteq J(eR)$ since $(\ker\sigma)^2 = 0$. If eRe is 2-good, by Proposition 2.15(4), eR is 2-good. Conversely, if eR is 2-good, then eRe is 2-good since it is σ-homomorphic image of eR. □

Proposition 2.17. *The class of all 2-good rings is an Amitsur-Kurosh radical class.*

Proof. Let $P = \{R \mid R \text{ is a 2-good ring}\}$. By Proposition 2.15, P is homomorphism closed. If I is an ideal of $R \in P$, and if I, R/I are in P, then $1 \in I$, hence $R = I$ is 2-good. It is easy to see that a union of a chain of P-ideals related to a ring $R \in P$ is again a P-ideal of R. □

Finally, we recall that Henriksen [10] called a ring R (S,n)-*ring* if every element of R is a sum of no more than n units. 2-good rings are $(S,2)$-ring, But the converse is not true. Indeed, $R = \mathbb{Z}/(4) = \{\bar{0},\bar{1},\bar{2},\bar{3}\}$ is a $(S,2)$-ring but not 2-good.

3. The extensions of 2-good rings

Proposition 3.1. (1) *A direct product $R = \prod R_\alpha$ of 2-good rings $\{R_\alpha\}$ is 2-good if and only if so is each R_α.*

(2) *A finite direct sum $R = \bigoplus_{i=1}^n R_i$ of 2-good rings $\{R_i\}$ is 2-good if and only if so is each R_i.*

(3) *The direct limit $\varinjlim_I R_i$ is 2-good if and only if so is each $R_i, i \in I$.*

Proof. These assertions are directly verified. □

If R is a ring and $\alpha : R \to R$ is a ring endomorphism. Let $R[[x, \alpha]]$ denote the ring of *skew formal power series* over R, that is all formal power series in x with coefficients from R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, 1_R]]$ is the ring of formal power series over R.

Proposition 3.2. *Let R be a ring. Then the ring $R[[x, \alpha]]$ is 2-good if and only if R is 2-good. In particular, $R[[x]]$ is 2-good if and only if R is 2-good.*

Proof. By Proposition 2.15, $R[[x, \alpha]]$ is 2-good, and this gives that

$$R = R[[x, \alpha]]/(x)$$

is 2-good.

Conversely, suppose that R is 2-good. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x, a]]$. Write $a_0 = u + v \in U(R)$. Then $f(x) = (u + a_1 x + a_2 x^2 + \cdots) + v$, where $u + a_1 x + a_2 x^2 + \cdots$, $v \in U(R[[x, \alpha]])$. Thus $R[[x, \alpha]]$ is 2-good. □

Recall that a ring R is called *semicommutative* if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$. Commutative rings, symmetric rings, reversible rings and one-sided duo rings are all semicommutative (see [3]).

Proposition 3.3. *If R is semicommutative, then the polynomial ring $R[x]$ is not 2-good.*

Proof. Note that by Xiao-Tong [21, Lemma 3.5],

$U(R[x]) = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_0 \in U(R),\ a_1, \ldots, a_n \in N(R),\ n \in N\}$.

If x is 2-good, then $x = u(x) + v(x)$, where $u(x) = u_0 + u_1 x + \cdots + u_n x^n$, $v(x) = v_0 + v_1 x + \cdots + v_m x^m \in U(R[x])$, and $u_1, \ldots, u_n$, $v_1, \ldots, v_m \in N(R)$, $n, m \in N$. It follows that $1 = u_1 + v_1 \in N(R) \subseteq J(R)$, a contradiction. Thus $R[x]$ is not 2-good. □

Corollary 3.4. *If R is commutative (symmetric, reversible, one-side duo), then the polynomial ring $R[x]$ is not 2-good.*

Remark 3.5. (1) If $R[x]$ is 2-good, then so is R.

(2) The subring of 2-good ring need not inherit the property.

(3) The polynomial ring $R[x]$ over a 2-good ring R need not be 2-good.

Indeed, if Q is the rational number field, then Q and $Q[[x]]$ are both 2-good by Proposition 3.2, but the polynomial ring $Q[x]$ over Q, a subring of $Q[[x]]$, is not 2-good by Corollary 3.4.

Proposition 3.6. (1) *Let $e^2 = e \in R$. If eRe and $(1 - e)R(1 - e)$ are both 2-good, then R is 2-good.*

(2) *Let $e^2 = e$ is a central idempotent of R. Then R is 2-good if and only if so are eR and $(1 - e)R$.*

Proof. (1) For convenience write $\bar{e} = 1 - e$. We use the Pierce decomposition of the ring R:

$$R = \begin{pmatrix} eRe & eR\bar{e} \\ \bar{e}Re & \bar{e}R\bar{e} \end{pmatrix}.$$

Let $A = \left(\begin{smallmatrix} a & x \\ y & b \end{smallmatrix}\right) \in R$ with $a \in eRe$, $b \in \bar{e}Re$. By hypothesis, there exist $u_1, u_2 \in U(eRe)$ with inverse u_1^{-1} and u_2^{-1} such that $a = u_1 + u_2$. Thus $b - yu_2^{-1}x \in \bar{e}R\bar{e}$, so we can write $b - yu_2^{-1}x = v_1 + v_2$ where $v_1, v_2 \in U(\bar{e}R\bar{e})$ with inverse v_1^{-1} and v_2^{-1}. Hence

$$A = \begin{pmatrix} u_1 + u_2 & x \\ y & v_1 + v_2 + yu_2^{-1}x \end{pmatrix} = \begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix} + \begin{pmatrix} u_2 & x \\ y & v_2 + yu_2^{-1}x \end{pmatrix}$$

and it is sufficient to show that $\begin{pmatrix} u_2 & x \\ y & v_2+yu_2^{-1}x \end{pmatrix}$ is a unit in R. To this end compute

$$\begin{aligned} &\begin{pmatrix} e & 0 \\ -yu_2^{-1} & \bar{e} \end{pmatrix} \begin{pmatrix} u_2 & x \\ y & v_2 + yu_2^{-1}x \end{pmatrix} \begin{pmatrix} e & -u_2^{-1}x \\ 0 & \bar{e} \end{pmatrix} \\ &= \begin{pmatrix} u_2 & x \\ 0 & v_2 \end{pmatrix} \begin{pmatrix} e & -u_2^{-1}x \\ 0 & \bar{e} \end{pmatrix} = \begin{pmatrix} u_2 & 0 \\ 0 & v_2 \end{pmatrix}. \end{aligned}$$

Since $\begin{pmatrix} e & 0 \\ -yu_2^{-1} & \bar{e} \end{pmatrix}$, $\begin{pmatrix} e & -u_2^{-1}x \\ 0 & \bar{e} \end{pmatrix}$ and $\left(\begin{smallmatrix} u_2 & 0 \\ 0 & v_2 \end{smallmatrix}\right)$ are all units in R, the proof is complete.

(2) Suppose R is 2-good. For any $er \in eR$, we have $r = u + v$, where $u, v \in U(R)$ with inverse u^{-1} and v^{-1}. It follows that $eueu^{-1} = e = evev^{-1}$. Thus $eu, ev \in U(eR)$, $er = eu + ev$. Hence eR is 2-good. Note that $e^{'} = 1 - e$ is also a central idempotent of R. We know that $(1-e)R$ is 2-good. The converse is clear by (1). ☐

Vaserstein [19, Theorem 2.8] showed that if R satisfies unit 1-stable rang, then so does eRe for any idempotent $e \in R$. This combined with Proposition 2.8 gives:

Proposition 3.7. *If R satisfies unit 1-stable rang, and e is any idempotent in R, then eRe is 2-good.*

Corollary 3.8. *If $1 = e_1 + e_2 + \cdots + e_m$ in a ring R where the e_i are orthogonal idempotents and each e_iRe_i is 2-good, then so is R.*

Proof. By Proposition 3.6(1) and induction. ☐

Corollary 3.9. *If R is 2-good, then so is the matrix ring $M_n(R)$ for any positive integer n.*

Corollary 3.10. *If $M = M_1 \bigoplus M_2 \bigoplus \cdots \bigoplus M_n$ are modules and* $\mathrm{End}(M_i)$ *is 2-good for each i, then* $\mathrm{End}(M)$ *is 2-good.*

By Proposition 3.6(1) and Proposition 2.15, we obtain:

Corollary 3.11. *If A and B are rings and $M = {}_BM_A$ is a bimodule, the formal triangular matrix ring $T = \left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$ is 2-good if and only if both A and B are 2-good.*

Recall that a *Morita Context* denote by (A, B, M, N, Ψ, Φ) consists of two rings A, B, two bimodules ${}_AN_B$, ${}_BM_A$ and a pair of bimodule homomorphisms $\Psi : N \bigotimes_B M \to A$ and $\Phi : M \bigotimes_A N \to B$ which satisfy the following associativity: $\Psi(v, w)v' = v\Phi(w, v')$ and $\Phi(w, v)w' = w\Psi(v, w')$. These conditions will insure that the set C of generalized matrices

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix}, \ a \in A, \ b \in B, \ m \in M, \ n \in N.$$

will form a ring $C = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$, called *Morita ring*.

Corollary 3.12. *If both A and B are 2-good, then so is C.*

Proof. Take $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\bar{e} = 1 - e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we have that $A \cong eCe$ and $B \cong \bar{e}C\bar{e}$. The result follows by Proposition 3.6. □

Remark 3.13. The converses of Proposition 3.6(1), Corollary 3.9 and Corollary 3.12 are all not true. For example, by Vámos [18, Proposition 6], $R = M_2(\mathbb{Z})$ is 2-good. Taking $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, we have $eRe \cong \mathbb{Z}$, which is not 2-good. It also shows that the property of 2-good is not a Morita invariant.

Proposition 3.14. *Let $e_1, \ldots, e_n$ be idempotents of a ring R. If $e_1Re_1, \ldots, e_nRe_n$ are all 2-good, then so is the ring $\left(\begin{smallmatrix} e_1Re_1 & \cdots & e_1Re_n \\ \cdots & \cdots & \cdots \\ e_nRe_1 & \cdots & e_nRe_n \end{smallmatrix}\right)$.*

Proof. By Proposition 3.6(1), the result holds for $n = 2$. Assume inductively that the result holds for $n = k \geq 2$. Let $n = k + 1$, and let

$$B = \begin{pmatrix} e_2Re_2 & \cdots & e_2Re_{k+1} \\ \cdots & \cdots & \cdots \\ e_{k+1}Re_2 & \cdots & e_{k+1}Re_{k+1} \end{pmatrix}_{k \times k},$$

$$M = \begin{pmatrix} e_2Re_1 \\ \vdots \\ e_{k+1}Re_1 \end{pmatrix}_{k \times 1}, \ N = \begin{pmatrix} e_1Re_2 & \cdots & e_1Re_{k+1} \end{pmatrix}_{1 \times k}.$$

Then B is 2-good. Given $\left(\begin{smallmatrix} a & n \\ m & b \end{smallmatrix}\right) \in \left(\begin{smallmatrix} e_1Re_1 & N \\ M & B \end{smallmatrix}\right)$, similar to the proof of Proposition 3.6(1), we can show that it is a sum of two units. □

Corollary 3.15. *Let R be a ring. Then the following are equivalent:*

(1) *R is 2-good.*

(2) *R has a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents such that all e_iRe_i are 2-good.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Construct a map

$$\Theta : R \to \begin{pmatrix} e_1Re_1 & \cdots & e_1Re_n \\ \cdots & \cdots & \cdots \\ e_nRe_1 & \cdots & e_nRe_n \end{pmatrix}$$

given by $\Theta(r) = \begin{pmatrix} e_1re_1 & \cdots & e_1re_n \\ \cdots & \cdots & \cdots \\ e_nre_1 & \cdots & e_nre_n \end{pmatrix}$. Since $\{e_1, \ldots, e_n\}$ is a complete set of orthogonal idempotents, Θ is a ring isomorphism, we get the result by Proposition 3.14. $\square$

Let R be a ring. Put $QM_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b = c + d,\ a, b, c, d \in R \right\}$. Then $QM_2(R)$ is a subring of $M_2(R)$.

Theorem 3.16. *Let R is 2-good. Then the following statements hold:*

(1) *For any $n \in N$, the ring $T_n(R)$ of $n \times n$ upper triangular matrices over R is 2-good.*

(2) *$QM_2(R)$ is 2-good.*

(3) *For any $n \in N$, $S_n(R) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \cdots & & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \mid a_i \in R, i = 0, 1, \ldots, n-1 \right\}$ is 2-good.*

(4) *For any $n \in N$, $R[x]/(x^n)$ is 2-good, where (x^n) is the ideal generated by x^n.*

Proof. (1) Let $A = (a_{ij}) \in T_n(R)$, where $a_{ij} = 0$ if $i > j$. By hypothesis there exist $u_i, v_i \in U(R)$ such that $a_{ii} = u_i + v_i$ for each $1 \leq i \leq n$. Then $A = \mathrm{diag}(u_1, u_2, \ldots, u_n) + B$, where $B = (b_{ij})$ with $b_{ii} = v_i (1 \leq i \leq n)$ and $b_{ij} (i \neq j) = a_{ij}$. It is clear that $\mathrm{diag}(u_1, \ldots, u_n),\ B \in U(T_n(R))$.

(2) Put Θ: $QM_2(R) \to T_2(R)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b & b \\ 0 & d-b \end{pmatrix}$. Then Θ is a monomorphism of rings. Also, for any $\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \in T_2(R)$, we have

$$\Theta\left(\begin{pmatrix} x-y & z \\ x-y-z & y+z \end{pmatrix}\right) = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix}.$$

Hence Θ is an isomorphism of rings. This completes the proof by (1).

(3) The proof is similar to that of (1).

(4) Note that $R[x]/(x^n) \cong S_n(R)$, we obtain the result by (3). $\square$

Given a ring R and a (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \bigoplus M$ with the usual addition and the following multiplication:

$$(r_1,\ m_1)(r_2,\ m_2) = (r_1r_2,\ r_1m_2 + m_1r_2).$$

Proposition 3.17. *$T(R, M)$ is 2-good if and only if R is 2-good.*

Proof. Suppose $T(R, M)$ is 2-good. For any $x \in R$, we have $(x, 0) = (u, m) + (v, n)$, where $(u, m),\ (v, n) \in U(T(R, M))$ with inverses (u_1, m_1) and (v_1, n_1). Note that $1_T = (1_R, 0)$, by $(u, m)(u_1, m_1) = (1_R, 0) = (v_1, n_1)(v, n)$ we obtain

$uu_1 = 1_R = u_1u,\ vv_1 = 1_R = v_1v$. Hence $x = u + v$ with $u, v \in U(R)$. So R is 2-good.

Conversely, suppose R is 2-good. For any $(x, m) \in T(R, M)$, by hypothesis, there exist $u, v \in U(R)$ such that $x = u + v$. Thus $(x, m) = (u, m) + (v, 0)$. Since $(u, m)(u^{-1}, -u^{-1}mu) = (1, 0)$ and $(v, 0)(v^{-1}, 0) = (1, 0)$, $(u, m),\ (v, 0) \in U(T(R, M))$. Hence $T(R, M)$ is 2-good. □

Let R be a commutative ring, M be an R-module, and σ be an endomorphism of R. Give $R \bigoplus M$ a ring structure with multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, \sigma(r_1)m_2 + r_2m_1)$, where $r_i \in R$ and $m_i \in M$. We call this extension the *Nagata extension* of R by M and σ and denote it by $N(R, M, \sigma)$.

Proposition 3.18. *$N(R, M, \sigma)$ is 2-good if and only if R is 2-good.*

Proof. Suppose R is 2-good. Then for any $(x, m) \in N(R, M, \sigma)$ there exist $u, v \in U(R)$ such that $(x, m) = (u, m) + (v, 0)$. Since

$$(u, m)(u^{-1}, -\sigma(u^{-1})mu^{-1}) = (1_R, 0)$$

and $(v, 0)(v^{-1}, 0) = (1, 0)$, $(u, m), (v, 0) \in U(N(R, M, \sigma))$. Hence $N(R, M, \sigma)$ is 2-good.

The converse is similar to Proposition 3.17. □

A ring R is called *right ore* if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is a well-known fact that R is a right ore ring if and only if the classical right quotient ring of R exists.

Proposition 3.19. *Let R be a right ore ring and Q be the classical right quotient ring of R. If R is 2-good, then so is Q.*

Proof. For any $r = ab^{-1} \in Q$, where $a, b \in R$ with b regular. By hypothesis there exist $u, v \in U(R)$ such that $a = u + v$. Hence $r = ub^{-1} + vb^{-1}$. It is clear that $(ub^{-1})^{-1} = bu^{-1},\ (vb^{-1})^{-1} = bv^{-1}$, thus $ub^{-1},\ vb^{-1} \in U(Q)$. □

The converse of Proposition 3.19 is not true. For example, the rational number field Q is the classical right quotient ring of $\mathbb{Z}$, but $\mathbb{Z}$ is not 2-good.

Let R be an algebra over a commutative ring S. Recall that the *Dorroh extension* of R by S denoted $D(R, S)$, is the ring $R \times S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$.

Proposition 3.20. *The Dorroh extension $D(R, S)$ of R by S is 2-good if the following conditions are satisfied*:

(1) *S is 2-good*;

(2) *R is right quasi-regular.*

Proof. Assume that (1), (2) are satisfied. Let $d = (r, s) \in D(R, S)$. Then by (1), we can write $s = u + v$ with $u, v \in U(S)$. Thus $d = (r, u) + (0, v)$ and $(0, v)$ is unit since $(0, v)(0, v^{-1}) = (0, 1)$. Now we have $(r, u) = (0, u)(u^{-1}r, 1)$, and

$(u^{-1}r, 1) = (0, 1) + (u^{-1}r, 0)$ is a unit of $D(R, S)$ because $(R, 0) \subseteq J(D(R, S))$ by (2). Hence $(r, u) \in U(D(R, S))$, so d is 2-good, as required. $\square$

References

[1] A. Badawi, *On abelian π-regular rings*, Comm. Algebra **25** (1997), no. 4, 1009–1021.
[2] ———, *On semicommutative π-regular rings*, Comm. Algebra **22** (1994), no. 1, 151–157.
[3] V. Camillo and P. P. Nielsen, *McCoy rings and zero-divisors*, J. Pure Appl. Algebra **212** (2008), no. 3, 599–615.
[4] V. P. Camillo and H. P. Yu, *Exchange rings, Units and idempotents*, Comm. Algebra **22** (1994), no. 12, 4737–4749.
[5] H. Chen, *Units, idempotents, and stable range conditions*, Comm. Algebra **29** (2001), no. 2, 703–717.
[6] P. Crawley and B. Jonsson, *Refinements for infinite direct decompositions of algebraic systems*, Pacific J. Math. **14** (1964), 797–855.
[7] G. Ehrlich, *Unit-regular rings*, Port. Math. **27** (1968), 209–212.
[8] J. W. Fisher and R. L. Snider, *Rings generated by their units*, J. Algebra **42** (1976), no. 2, 363–368.
[9] L. Gillman and M. Jerison, *Rings of Continuous Functions*, The University Series in Higher Mathematics, D. Van Nostrand Co., Princeton, N. J., 1960.
[10] K. R. Goodearl and P. Menal, *Stable range one for ring with many units*, J. Pure Appl. Algebra **54** (1988), no. 2-3, 261–287.
[11] M. Henriksen, *Two classes of rings generated by their units*, J. Algebra **31** (1974), 182–193.
[12] A. V. Kelarev, *Ring Constructions and Applications*, World Scientific, New Jersey, 2002.
[13] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269–278.
[14] J. Okninski, *Semigroup Algebras*, Marcel Dekker, INC, New York, 1991.
[15] R. Raphael, *Rings which are generated by their units*, J. Algebra **28** (1974), 199–205.
[16] J. Stock, *On rings whose projective modules have the exchange property*, J. Algebra **103** (1986), no. 2, 437–453.
[17] A. A. Tuganbaev, *Rings and modules with exchange properties*, J. Math. Sci. (N.Y.) **110** (2002), no. 1, 2348–2421.
[18] P. Vámos, *2-good rings*, Q. J. Math. **56** (2005), no. 3, 417–430.
[19] L. N. Vaserstein, *Bass's first stable range condition*, J. Pure Appl. Algebra **34** (1984), no. 2-3, 319–330.
[20] R. B. Warfield, *Exchange rings and decomposition of modules*, Math. Ann. **199** (1972), 31–36.
[21] G. S. Xiao and W. T. Tong, *n-clean rings*, Algebra Colloq. **13** (2006), no. 4, 599–606.
[22] Y. Q. Ye, *Semiclean rings*, Comm. Algebra **31** (2003), no. 11, 5609–5625.
[23] H. P. Yu, *On the structure of exchange rings*, Comm. Algebra **25** (1997), no. 2, 661–670.
[24] B. Zimmermann-Huisgen and W. Zimmermann, *Classes of modules with the exchange property*, J. Algebra **88** (1984), no. 2, 416–434.

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