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CERTAIN HYPERGEOMETRIC IDENTITIES DEDUCIBLE BY USING THE BETA INTEGRAL METHOD

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Abstract. The main objective of this paper is to show how one can obtain eleven new and interesting hypergeometric identities in the form of a single result from the old ones by mainly employing the known beta integral method which was recently introduced and used in a systematic manner by Krattenthaler and Rao [6]. The results are derived with the help of a generalization of a well-known hypergeometric transformation formula due to Kummer. Several identities including one obtained earlier by Krattenthaler and Rao [6] follow as special cases of our main results.

1. Introduction and preliminaries

In the usual notation, let $\mathbb{C}$ denote the set of *complex* numbers. For

$$\alpha_j \in \mathbb{C}\ (j = 1, \ldots, p) \ \text{ and } \ \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \ \left(\mathbb{Z}_0^- := \mathbb{Z} \cup \{0\} = \{0, -1, -2, \ldots\}\right),$$

the *generalized hypergeometric function* ${}_pF_q$ with p numerator parameters α_1, $\ldots$, α_p and q denominator parameters $\beta_1, \ldots, \beta_q$ is defined by (see, *e.g.* [1, Chapter II], [9, Chapter 4]; see also [10, pp. 71–72]):

(1.1)

$$ {}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p\,; \\ \beta_1, \ldots, \beta_q\,;\end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!} = {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)$$

$$\Big(p, q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\};\ p \leqq q+1;\ p \leqq q \ \text{ and } \ |z| < \infty;$$
$$p = q+1 \ \text{ and } \ |z| < 1;\ p = q+1,\ |z| = 1 \ \text{ and } \ \Re(\omega) > 0\Big),$$

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where

(1.2)
$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \quad \bigl(\alpha_j \in \mathbb{C}\,(j=1,\ldots,p);\ \beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-\ (j=1,\ldots,q)\bigr)$$

and $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$), in terms of the familiar Gamma function Γ, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}). \end{cases} \tag{1.3}$$

We now recall the following interesting formula due to Kummer (*cf.* [7, p. 81, Entry 72]):

$$\begin{aligned} &{}_2F_1\left[\begin{matrix} a,\, b; \\ \frac{1}{2}(a+b+1); \end{matrix}\ \frac{1}{2}(1+z)\right] \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left[\frac{1}{2}(a+b+1)\right]}{\Gamma\left[\frac{1}{2}(a+1)\right]\Gamma\left[\frac{1}{2}(b+1)\right]}\,{}_2F_1\left[\begin{matrix} \frac{a}{2},\, \frac{b}{2}; \\ \frac{1}{2}; \end{matrix}\ z^2\right] \\ &\quad + \frac{2z\,\Gamma\left(\frac{1}{2}\right)\Gamma\left[\frac{1}{2}(a+b+1)\right]}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}\,{}_2F_1\left[\begin{matrix} \frac{a}{2}+\frac{1}{2},\, \frac{b}{2}+\frac{1}{2}; \\ \frac{3}{2}; \end{matrix}\ z^2\right] \quad (z\in\mathbb{U}), \end{aligned} \tag{1.4}$$

where $\mathbb{U}$ denotes the *open* unit disk, that is, $\mathbb{U} = \{z : z\in\mathbb{C} \text{ and } |z|<1\}$.

The formula (1.4) was proved independently by Ramanujan [2, p. 64, Entry 21] and is also recorded by Erdélyi *et al.* [5, p. 65, Equation 2.1.5(28)]. An interesting special case of Kummer's formula (1.4) when $a=b=\frac{1}{2}$ would yield the following identity due to Ramanujan [2, p. 96, Entry 34(1)]:

$$\begin{aligned} &{}_2F_1\left[\begin{matrix} \frac{1}{2},\, \frac{1}{2}; \\ 1\ ; \end{matrix}\ \frac{1}{2}(1+z)\right] \\ &= \mu\,{}_2F_1\left[\begin{matrix} \frac{1}{4},\, \frac{1}{4}; \\ \frac{1}{2}\ ; \end{matrix}\ z^2\right] + \eta\, z\,{}_2F_1\left[\begin{matrix} \frac{3}{4},\, \frac{3}{4}; \\ \frac{3}{2}\ ; \end{matrix}\ z^2\right], \end{aligned} \tag{1.5}$$

where, for later use, the coefficients μ and η are given by

$$\mu = \frac{\Gamma\left(\frac{1}{2}\right)}{\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2} \quad \text{and} \quad \eta = \frac{\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2}{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^3}. \tag{1.6}$$

It is noted in passing that Choi and Rathie [3], very recently, presented two interesting formulas contiguous to a quadratic transformation due to Kummer.

By using the beta integral method, which was developed recently by Krattenthaler and Rao [6], one of the new identities (among others) which they obtained by employing the Kummer's formula (1.4) (by first replacing z by

$-z$) is given by [6, p. 170, Equation (3.34)]:

(1.7)

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\, b,\, e-d; & \dfrac{1}{2} \\ \frac{1}{2}(a+b+1),\, e; & 2 \end{matrix}\right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left[\frac{1}{2}(a+b+1)\right]}{\Gamma\left[\frac{1}{2}(a+1)\right]\,\Gamma\left[\frac{1}{2}(b+1)\right]}\,{}_4F_3\left[\begin{matrix} \frac{a}{2},\, \frac{b}{2},\, \frac{1}{2}+\frac{d}{2},\, \frac{d}{2}; \\ \frac{1}{2},\, \frac{1}{2}+\frac{e}{2},\, \frac{e}{2}\ ; \end{matrix}\ 1\right] \\
&\quad + \frac{d}{e}\,\frac{\Gamma\left(-\frac{1}{2}\right)\,\Gamma\left[\frac{1}{2}(a+b+1)\right]}{\Gamma\left(\frac{a}{2}\right)\,\Gamma\left(\frac{b}{2}\right)}\,{}_4F_3\left[\begin{matrix} \frac{a}{2}+\frac{1}{2},\, \frac{b}{2}+\frac{1}{2},\, \frac{1}{2}+\frac{d}{2},\, \frac{d}{2}+1; \\ \frac{3}{2},\, \frac{1}{2}+\frac{e}{2},\, \frac{e}{2}+1\ ; \end{matrix}\ 1\right].
\end{aligned}
$$

Here, in this paper, we show how one can obtain eleven identities including the Krattenthaler-Rao result (1.7) in the form of a single unified result by employing the beta integral method developed by Krattenthaler and Rao [6]. The results are derived with the help of a generalization of the Kummer's formula (1.4) which was recently presented by Choi *et al.* [4]. Several interesting special cases of our main result including (1.7) are also explicitly demonstrated. We note that the results obtained in this paper are simple, interesting and (potentially) useful.

For our purpose, we need to recall the following generalization of Kummer's formula (1.4) due to Choi *et al.* [4, Equation (2.1)]:

(1.8)

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a,\quad b\ ; & \dfrac{1}{2}(1+z) \\ \frac{1}{2}(a+b+\ell+1)\ ; & \end{matrix}\right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\ell+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}|\ell|\right)} \\
&\quad \cdot\left[\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}a\right)_j\,\left(\frac{1}{2}a+\frac{1}{2}\right)_j\,\left(\frac{1}{2}b\right)_j\,\left(\frac{1}{2}b+\frac{1}{2}\right)_j}{\left(\frac{1}{2}\right)_j\ j!}\,z^{2j}\right. \\
&\quad \cdot\left\{\frac{C_\ell}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2}-\left[\frac{\ell+1}{2}\right]\right)\,\left(\frac{1}{2}a+\frac{1}{2}\right)_j\,\left(\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2}-\left[\frac{\ell+1}{2}\right]\right)_j}\right. \\
&\quad \left.+\frac{D_\ell}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)\,\left(\frac{1}{2}a\right)_j\,\left(\frac{1}{2}b+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)_j}\right\} \\
&\quad +\frac{abz}{2}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}a+\frac{1}{2}\right)_j\,\left(\frac{1}{2}a+1\right)_j\,\left(\frac{1}{2}b+\frac{1}{2}\right)_j\,\left(\frac{1}{2}b+1\right)_j}{\left(\frac{3}{2}\right)_j\ j!}\,z^{2j} \\
&\quad \cdot\left\{\frac{E_\ell}{\Gamma\left(\frac{1}{2}a+1\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\ell+1-\left[\frac{\ell+1}{2}\right]\right)\,\left(\frac{1}{2}a+1\right)_j\,\left(\frac{1}{2}b+1+\frac{1}{2}\ell-\left[\frac{\ell+1}{2}\right]\right)_j}\right. \\
&\quad \left.\left.+\frac{F_\ell}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)\,\left(\frac{1}{2}a+\frac{1}{2}\right)_j\,\left(\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)_j}\right\}\right]
\end{aligned}
$$

$(z \in \mathbb{U};\ \ell = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 5)$,

where the coefficients C_ℓ, D_ℓ and E_ℓ, F_ℓ are given in Tables 1 and 2 below, respectively.

2. Main result

Our eleven main identities are given in the form of a single unified result asserted in the following theorem.

Theorem. *The following generalization of the Krattenthaler-Rao formula* (1.7) *holds true*:

(2.1)

$$
\begin{aligned}
&{}_3F_2\left[\begin{array}{r} a,\ b,\ e-d\ ;\ \dfrac{1}{2} \\ \frac{1}{2}(a+b+\ell+1),\ e\ ;\ \phantom{\dfrac{1}{2}} \end{array}\right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\ell+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}|\ell|\right)} \\
&\quad\cdot\left[\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}a\right)_j\,\left(\frac{1}{2}a+\frac{1}{2}\right)_j\,\left(\frac{1}{2}b\right)_j\,\left(\frac{1}{2}b+\frac{1}{2}\right)_j}{\left(\frac{1}{2}\right)_j\,j!}\,\frac{\left(\frac{d}{2}\right)_j\,\left(\frac{d}{2}+\frac{1}{2}\right)_j}{\left(\frac{e}{2}\right)_j\,\left(\frac{e}{2}+\frac{1}{2}\right)_j}\right. \\
&\quad\cdot\left\{\frac{C_\ell}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2}-\left[\frac{\ell+1}{2}\right]\right)\,\left(\frac{1}{2}a+\frac{1}{2}\right)_j\,\left(\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2}-\left[\frac{\ell+1}{2}\right]\right)_j}\right. \\
&\quad\left.+\frac{D_\ell}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)\,\left(\frac{1}{2}a\right)_j\,\left(\frac{1}{2}b+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)_j}\right\} \\
&\quad-\frac{abd}{2e}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}a+\frac{1}{2}\right)_j\,\left(\frac{1}{2}a+1\right)_j\,\left(\frac{1}{2}b+\frac{1}{2}\right)_j\,\left(\frac{1}{2}b+1\right)_j}{\left(\frac{3}{2}\right)_j\,j!}\,\frac{\left(\frac{d}{2}+\frac{1}{2}\right)_j\,\left(\frac{d}{2}+1\right)_j}{\left(\frac{e}{2}+\frac{1}{2}\right)_j\,\left(\frac{e}{2}+1\right)_j} \\
&\quad\cdot\left\{\frac{E_\ell}{\Gamma\left(\frac{1}{2}a+1\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\ell+1-\left[\frac{\ell+1}{2}\right]\right)\,\left(\frac{1}{2}a+1\right)_j\,\left(\frac{1}{2}b+1+\frac{1}{2}\ell-\left[\frac{\ell+1}{2}\right]\right)_j}\right. \\
&\quad\left.\left.+\frac{F_\ell}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)\,\left(\frac{1}{2}a+\frac{1}{2}\right)_j\,\left(\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)_j}\right\}\right] \\
&\quad(\ell = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 5),
\end{aligned}
$$

where the coefficients C_ℓ, D_ℓ *and* E_ℓ, F_ℓ *are given in Tables 1 and 2 below, respectively.*

Proof. We first replace z by $-z$ in (1.8) and then multiply both sides of the resulting identity by $z^{d-1}\,(1-z)^{e-d-1}$. Then, upon integrating the resulting equation with respect to z from 0 to 1, expressing the involved ${}_2F_1$ on the left-hand side as series and changing the order of integration and summation, which is justified due to the uniform convergence of the involved series, we make use of the Beta function $B(\alpha,\,\beta)$ defined by the first integral and known to be evaluated as the second one as follows:

$$B(\alpha,\,\beta)=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha)>0;\ \Re(\beta)>0)\\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-).\end{cases} \tag{2.2}$$

Finally, after a little simplification, we arrive at the desired result (2.1). This completes the proof of our main theorem. □

Remark 1. Taking $a=b=\frac{1}{2}$ in (1.7), we obtain the following identity:

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}\frac{1}{2},\,\frac{1}{2},\,e-d;\\ 1,\,e;\end{matrix}\ \frac{1}{2}\right]\\ &=\mu\,{}_4F_3\left[\begin{matrix}\frac{1}{4},\,\frac{1}{4},\,\frac{d}{2},\,\frac{d}{2}+\frac{1}{2};\\ \frac{1}{2},\,\frac{e}{2},\,\frac{e}{2}+\frac{1}{2}\ ;\end{matrix}\ 1\right]-\eta\,\frac{d}{e}\,{}_4F_3\left[\begin{matrix}\frac{3}{4},\,\frac{3}{4},\,\frac{d}{2}+\frac{1}{2},\,\frac{d}{2}+1;\\ \frac{3}{2},\,\frac{e}{2}+\frac{1}{2},\,\frac{e}{2}+1\ ;\end{matrix}\ 1\right],\end{aligned} \tag{2.3}$$

where μ and η are the same as given in (1.6).

3. Special cases

Here we consider some of the very interesting special cases of our main result (2.1).

Corollary 1. *Each of the following formulas holds true*:

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}a,\,b,\,e-d;\\ \frac{1}{2}(a+b+2),\,e;\end{matrix}\ \frac{1}{2}\right]\\ &=\frac{2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{b}{2}+1\right)}{a-b}\Bigg(\frac{1}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}\,{}_4F_3\left[\begin{matrix}\frac{a}{2}+\frac{1}{2},\,\frac{b}{2},\,\frac{d}{2},\,\frac{d}{2}+\frac{1}{2};\\ \frac{1}{2},\,\frac{e}{2},\,\frac{e}{2}+\frac{1}{2}\ ;\end{matrix}\ 1\right]\\ &\qquad-\frac{1}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}\right)}\,{}_4F_3\left[\begin{matrix}\frac{a}{2},\,\frac{b}{2}+\frac{1}{2},\,\frac{d}{2},\,\frac{d}{2}+\frac{1}{2};\\ \frac{1}{2},\,\frac{e}{2},\,\frac{e}{2}+\frac{1}{2}\ ;\end{matrix}\ 1\right]\\ &\qquad+\frac{abd}{2e}\,\frac{1}{\Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}\,{}_4F_3\left[\begin{matrix}\frac{a}{2}+\frac{1}{2},\,\frac{b}{2}+1,\,\frac{d}{2}+\frac{1}{2},\,\frac{d}{2}+1;\\ \frac{3}{2},\,\frac{e}{2}+\frac{1}{2},\,\frac{e}{2}+1\ ;\end{matrix}\ 1\right]\\ &\qquad-\frac{abd}{2e}\,\frac{1}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+1\right)}\,{}_4F_3\left[\begin{matrix}\frac{a}{2}+1,\,\frac{b}{2}+\frac{1}{2},\,\frac{d}{2}+\frac{1}{2},\,\frac{d}{2}+1;\\ \frac{3}{2},\,\frac{e}{2}+\frac{1}{2},\,\frac{e}{2}+1\ ;\end{matrix}\ 1\right]\Bigg)\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}\frac{1}{2},\,\frac{3}{2},\,e-d;\\ 2,\,e;\end{matrix}\ \frac{1}{2}\right]\\ &=2\mu\,{}_4F_3\left[\begin{matrix}\frac{1}{4},\,\frac{5}{4},\,\frac{d}{2},\,\frac{d}{2}+\frac{1}{2};\\ \frac{1}{2},\,\frac{e}{2},\,\frac{e}{2}+\frac{1}{2}\ ;\end{matrix}\ 1\right]-4\eta\,{}_4F_3\left[\begin{matrix}\frac{3}{4},\,\frac{3}{4},\,\frac{d}{2},\,\frac{d}{2}+\frac{1}{2};\\ \frac{1}{2},\,\frac{e}{2},\,\frac{e}{2}+\frac{1}{2}\ ;\end{matrix}\ 1\right]\end{aligned} \tag{3.2}$$

$$+\,\mu\,\frac{d}{e}\,{}_4F_3\left[\begin{matrix}\frac{5}{4},\frac{5}{4},\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\\ \frac{3}{2},\frac{e}{2}+\frac{1}{2},\frac{e}{2}+1\;;\end{matrix}\;1\right]-6\mu\frac{d}{e}\,{}_4F_3\left[\begin{matrix}\frac{3}{4},\frac{7}{4},\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\\ \frac{3}{2},\frac{e}{2}+\frac{1}{2},\frac{e}{2}+1\;;\end{matrix}\;1\right],$$

where the coefficients μ *and* η *are given in* (1.6).

Proof. Setting $\ell = 1$ in (2.1) and simplifying the resulting identity, we are led to the formula (3.1). Taking $a = \frac{1}{2}$ and $b = \frac{3}{2}$ in (3.1), we get the identity (3.2). □

Corollary 2. *Each of the following formulas holds true*:

(3.3)

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}a,\,b,\,e-d;\\ \frac{1}{2}(a+b),\,e;\end{matrix}\;\frac{1}{2}\right]\\
&=\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{a}{2}+\tfrac{b}{2}\right)\left(\frac{1}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}\right)}\,{}_4F_3\left[\begin{matrix}\frac{a}{2},\,\frac{b}{2}+\frac{1}{2},\,\frac{d}{2},\,\frac{d}{2}+\frac{1}{2};\\ \frac{1}{2},\,\frac{e}{2},\,\frac{e}{2}+\frac{1}{2}\;;\end{matrix}\;1\right]\right.\\
&\qquad+\frac{1}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}\,{}_4F_3\left[\begin{matrix}\frac{a}{2}+\frac{1}{2},\,\frac{b}{2},\,\frac{d}{2},\,\frac{d}{2}+\frac{1}{2};\\ \frac{1}{2},\,\frac{e}{2},\,\frac{e}{2}+\frac{1}{2}\;;\end{matrix}\;1\right]\\
&\qquad-\frac{abd}{2e}\frac{1}{\Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}\,{}_4F_3\left[\begin{matrix}\frac{a}{2}+\frac{1}{2},\,\frac{b}{2}+1,\,\frac{d}{2}+\frac{1}{2},\,\frac{d}{2}+1;\\ \frac{3}{2},\,\frac{e}{2}+\frac{1}{2},\,\frac{e}{2}+1\;;\end{matrix}\;1\right]\\
&\qquad\left.-\frac{abd}{2e}\frac{1}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+1\right)}\,{}_4F_3\left[\begin{matrix}\frac{a}{2}+1,\,\frac{b}{2}+\frac{1}{2},\,\frac{d}{2}+\frac{1}{2},\,\frac{d}{2}+1;\\ \frac{3}{2},\,\frac{e}{2}+\frac{1}{2},\,\frac{e}{2}+1\;;\end{matrix}\;1\right]\right)\end{aligned}$$

and

(3.4)

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}\frac{1}{2},\,\frac{3}{2},\,e-d;\\ 2,\,e;\end{matrix}\;\frac{1}{2}\right]\\
&=\mu\,{}_4F_3\left[\begin{matrix}\frac{1}{4},\frac{5}{4},\frac{d}{2},\frac{d}{2}+\frac{1}{2};\\ \frac{1}{2},\frac{e}{2},\frac{e}{2}+\frac{1}{2}\;;\end{matrix}\;1\right]+2\eta\,{}_4F_3\left[\begin{matrix}\frac{3}{4},\frac{3}{4},\frac{d}{2},\frac{d}{2}+\frac{1}{2};\\ \frac{1}{2},\frac{e}{2},\frac{e}{2}+\frac{1}{2}\;;\end{matrix}\;1\right]\\
&\quad-\frac{\mu}{2}\,\frac{d}{e}\,{}_4F_3\left[\begin{matrix}\frac{5}{4},\frac{5}{4},\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\\ \frac{3}{2},\frac{e}{2}+\frac{1}{2},\frac{e}{2}+1\;;\end{matrix}\;1\right]-3\eta\,\frac{d}{e}\,{}_4F_3\left[\begin{matrix}\frac{3}{4},\frac{7}{4},\frac{d}{2}+\frac{1}{2},\frac{d}{2}+1;\\ \frac{3}{2},\frac{e}{2}+\frac{1}{2},\frac{e}{2}+1\;;\end{matrix}\;1\right],\end{aligned}$$

where the coefficients μ *and* η *are given in* (1.6).

Proof. Setting $\ell = -1$ in (2.1) and simplifying the resulting identity, we are led to the formula (3.3). Taking $a = \frac{1}{2}$ and $b = \frac{3}{2}$ in (3.3), we get the identity (3.4). □

Remark 2. If we take $\ell = 0$ in (2.1) and simplify the resulting identity, we get the Krattenthaler-Rao formula (1.7).

Remark 3. It is seen that the identities (3.1) and (3.3) are closely related to the Krattenthaler-Rao formula (1.7) and the identities (3.2) and (3.4) are closely related to (2.3).

Many other specialized cases of our main result (2.1) can also be deduced. It is indeed interesting to compare the results (3.1) and (3.3) with (1.7) and, similarly, (3.2) and (3.4) with (2.3).

Table 1

ℓ	C_ℓ	D_ℓ
5	$-(b+a+4j+6)^2+\frac{1}{4}(b-a+6)^2$ $+\frac{1}{2}(b-a+6)(b+a+4j+6)$ $+11(b+a+4j+6)$ $-\frac{13}{2}(b-a+6)-20$	$(b+a+4j+6)^2-\frac{1}{4}(b-a+6)^2$ $+\frac{1}{2}(b-a+6)(b+a+4j+6)$ $-17(b+a+4j+6)$ $-\frac{1}{2}(b-a+6)+62$
4	$\frac{1}{2}(b+a+4j+1)(b+a+4j-3)$ $-\frac{1}{4}(b-a+3)(b-a-3)$	$-2(b+a+4j-1)$
3	$-\frac{1}{2}(3a+b+8j-2)$	$\frac{1}{2}(3b+a+8j-2)$
2	$\frac{1}{2}(b+a++4j-1)$	-2
1	-1	1
0	1	0
-1	1	1
-2	$\frac{1}{2}(b+a+4j-1)$	2
-3	$\frac{1}{2}(3a+b+8j-2)$	$\frac{1}{2}(3b+a+8j-2)$
-4	$\frac{1}{2}(b+a+4j+1)(b+a+4j-3)$ $-\frac{1}{2}(b-a+3)(b-a-3)$	$2(b+a+4j-1)$
-5	$(b+a+4j-4)^2-\frac{1}{4}(b-a-4)^2$ $-\frac{1}{2}(b+a+4j-4)(b-a-4)$ $+4(b+a+4j-4)-\frac{7}{2}(b-a-4)$	$(b+a+4j-4)^2-\frac{1}{4}(b-a-4)^2$ $+\frac{1}{2}(b+a+4j-4)(b-a-4)$ $+8(b+a+4j-4)-\frac{1}{2}(b-a-4)$ $+12$

Table 2

ℓ	E_ℓ	F_ℓ
5	$-(b+a+4j+8)^2+\frac{1}{4}(b-a+6)^2$ $+\frac{1}{2}(b-a+6)(b+a+4j+8)$ $+11(b+a+4j+8)$ $-\frac{13}{2}(b-a+6)-20$	$(b+a+4j+8)^2-\frac{1}{4}(b-a+6)^2$ $+\frac{1}{2}(b-a+6)(b+a+4j+8)$ $-17(b+a+4j+8)$ $-\frac{1}{2}(b-a+6)+62$
4	$\frac{1}{2}(b+a+4j+3)(b+a+4j-1)$ $-\frac{1}{2}(b-a+3)(b-a-3)$	$-2(b+a+4j+1)$
3	$-\frac{1}{2}(3a+b+8j+2)$	$\frac{1}{2}(3b+a+8j+2)$
2	$\frac{1}{2}(b+a+4j+1)$	-2
1	-1	1
0	1	0
-1	1	1
-2	$\frac{1}{2}(b+a+4j+1)$	2
-3	$\frac{1}{2}(3a+b+8j+2)$	$\frac{1}{2}(3b+a+8j+2)$
-4	$\frac{1}{2}(b+a++4j+3)(b+a+4j-1)$ $-\frac{1}{4}(b-a+3)(b-a-3)$	$2(b+a+4j+1)$
-5	$(b+a+4j-2)^2-\frac{1}{4}(b-a-4)^2$ $-\frac{1}{2}(b+a+4j-2)(b-a-4)$ $+4(b+a+4j-2)-\frac{7}{2}(b-a-4)$	$(b+a+4j-2)^2-\frac{1}{4}(b-a-4)^2$ $+\frac{1}{2}(b+a+4j-2)(b-a-4)$ $+8(b+a+4j-2)-\frac{1}{2}(b-a-4)$ $+12$

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