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MINIMAX PROBLEMS OF UNIFORMLY SAME-ORDER SET-VALUED MAPPINGS

Yu Zhang and Shengjie Li

Abstract. In this paper, a class of set-valued mappings is introduced, which is called uniformly same-order. For this sort of mappings, some minimax problems, in which the minimization and the maximization of set-valued mappings are taken in the sense of vector optimization, are investigated without any hypotheses of convexity.

1. Introduction

Minimax problems are important in the areas of optimization theory and game theory. As for optimization theory, the main motivation of studying saddle point has been their connection with characterizing solutions to minimax dual problems. Also, as for game theory, the main motivation has been the determination of two-person zero-sum games based on the minimax principle. Li [15] obtained a minimax theorem involving separable homogeneous polynomials and established a Lagrangian duality theorem for the nonconvex separable homogeneous polynomial programming problem with bounded constraints. Park [21] obtained a generalization of Nash equilibrium theorem by using the Ky Fan minimax inequality. Because of its wide applications, minimax theorems relative to scalar functions have been studied extensively (see [7, 8, 9, 14, 26] and references therein).

In recent years, based on the development of vector optimization, a great deal of papers have devoted to the study of minimax problems of vector-valued mappings. Nieuwenhuis [20] introduced the notion of cone saddle points for vector-valued functions in finite-dimensional spaces and obtained a cone saddle point theorem for general vector-valued mappings. Besides, Nieuwenhuis proved a minimax theorem when the vector-valued function is of the form $f(x, y) = x + y$. Tanaka [24] obtained a minimax theorem of the separated

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vector-valued function. Shi and Ling [22] proved a minimax theorem and a cone saddle point theorem for a class of vector-valued functions which includes separated functions as its proper subset. Chen [4] obtained a Ky Fan minimax inequality for a vector-valued function on H-spaces by using a generalized Fan's section theorem. Chang et al. [3] proved a Ky Fan minimax inequality for a vector-valued function on W-spaces. Yang et al. [25] established minimax theorems for vector-valued mappings in abstract convex spaces. Gong [11] established a strong minimax theorem and a strong cone saddle point theorem of vector-valued functions. Li et al. [17] investigated a minimax theorem and a saddle point theorem for vector-valued functions in the sense of lexicographic order.

There are also many papers to investigate minimax problems of set-valued mappings under some hypotheses of convexity. Luc and Vargas [19] obtained a cone loose saddle point theorem for general set-valued mappings by using a fixed point theorem and scalarization functions. Under weaker hypotheses, Tan et al. [23] established a cone loose saddle point theorem for general set-valued mappings. Kim et al. [13] proved a cone loose saddle point theorem for general set-valued mappings by using the Fan-Browder fixed point theorem and scalarization functions. Some other types of existence results on cone loose saddle points for set-valued mappings can be found in [18] and [27]. Li et al. [16] obtained some minimax inequalities for set-valued mappings by using a section theorem and a linear scalarization function. Zhang et al. [28, 29] obtained some minimax problems for general set-valued mappings by using some fixed point theorems. Motivated by earlier work [13, 16, 17, 19, 22, 23, 27], we introduce a class of set-valued mappings, which is called uniformly same-order. For this sort of mappings, we investigate some minimax problems without any hypotheses of convexity.

The rest of the paper is organized as follows. In Section 2, we introduce notations and preliminary results. In Section 3, we investigate some problems for uniformly same-order set-valued mappings without any hypotheses of convexity.

2. Preliminaries

Let X, Y and V be real Hausdorff topological vector spaces. Assume that S is a pointed closed convex cone in V with its interior $\mathrm{int}S \neq \emptyset$. Some fundamental terminologies are presented as follows.

Definition 2.1. ([6, 12]) Let $A \subset V$ be a nonempty subset.

(i) A point $z \in A$ is said to be a minimal point of A if $A \bigcap (z - S) = \{z\}$, and $\mathrm{Min}A$ denotes the set of all minimal points of A.

(ii) A point $z \in A$ is said to be a weakly minimal point of A if $A \bigcap (z - \mathrm{int}S) = \emptyset$, and $\mathrm{Min}_w A$ denotes the set of all weakly minimal points of A.

(iii) A point $z \in A$ is said to be a maximal point of A if $A \bigcap (z + S) = \{z\}$, and $\mathrm{Max}A$ denotes the set of all maximal points of A.

(iv) A point $z \in A$ is said to be a weakly maximal point of A if $A \bigcap (z + \mathrm{int} S) = \emptyset$, and Max_w denotes the set of all weakly maximal points of A.

It is easy to verity that

$$\mathrm{Min} A \subset \mathrm{Min}_w A \quad and \quad \mathrm{Max} A \subset \mathrm{Max}_w A.$$

Definition 2.2. ([1]) Let $F : X \to 2^V$ be a set-valued mapping with nonempty values.

(i) F is said to be upper semicontinuous (u.s.c.) at $x_0 \in X$, if for any neighborhood $N(F(x_0))$ of $F(x_0)$, there exists a neighborhood $N(x_0)$ of x_0 such that

$$F(x) \subset N(F(x_0)), \quad \forall x \in N(x_0).$$

(ii) F is said to be lower semicontinuous (l.s.c.) at $x_0 \in X$, if for any open neighborhood N in V satisfying $F(x_0) \bigcap N \neq \emptyset$, there exists a neighborhood $N(x_0)$ of x_0 such that

$$F(x) \bigcap N \neq \emptyset, \qquad \forall x \in N(x_0).$$

(iii) F is said to be continuous at $x_0 \in X$ if F is both u.s.c. and l.s.c. at x_0.

Proposition 2.1 ([2]). *Let $F : X \to 2^V$ be a set-valued mapping with nonempty values.*

(i) *F is said to be u.s.c. on X if and only if for any closed subset G of V, the inverse image of G*

$$F^{-1}(G) = \{x \in X \mid F(x) \bigcap G \neq \emptyset\}$$

is closed.

(ii) *F is said to be l.s.c. on X if and only if for any closed subset G of V, the core of G*

$$F^{+1}(G) = \{x \in X \mid F(x) \subset G\}$$

is closed.

Definition 2.3. ([19]) Let X_0 and Y_0 be two nonempty subsets of X and Y, respectively, and $F : X_0 \times Y_0 \to 2^V$ be a set-valued mapping. A point $(x, y) \in X_0 \times Y_0$ is said to be a S-loose saddle point of F on $X_0 \times Y_0$ if

$$F(x,y) \bigcap \mathrm{Min} \bigcup_{y \in Y_0} F(x,y) \neq \emptyset \quad and \quad F(x,y) \bigcap \mathrm{Max} \bigcup_{x \in X_0} F(x,y) \neq \emptyset.$$

Lemma 2.1. *Let X_0 and Y_0 be two nonempty compact subsets in X and Y respectively. Suppose that $F : X_0 \times Y_0 \to 2^V$ is a continuous set-valued mapping and for each $(x, y) \in X_0 \times Y_0$, $F(x, y)$ is a nonempty compact set. Then, $\Gamma(x) = \mathrm{Min}_w \bigcup_{y \in Y_0} F(x, y)$ and $\Phi(y) = \mathrm{Max}_w \bigcup_{x \in X_0} F(x, y)$ are u.s.c. and compact-valued on X_0 and Y_0, respectively.*

Proof. It follows from Lemma 2.2 in [16] that Γ and Φ are u.s.c.. By the compactness of X_0 and Y_0, and the closeness of weakly minimal (maximal) point sets, Γ and Φ are compact-valued. □

Lemma 2.2 ([1]). *Let X_0 be a nonempty subset of X, and let $F : X_0 \to 2^V$ be a set-valued mapping. If X_0 is compact and F is upper semicontinuous and compact-valued, then $F(X_0) = \bigcup_{x\in X_0} F(x)$ is compact.*

Lemma 2.3 ([10]). *Let $A \subset V$ be a nonempty compact subset. Then* (i) $\mathrm{Min}A \neq \emptyset$; (ii) $A \subset \mathrm{Min}A + S$; (iii) $\mathrm{Max}A \neq \emptyset$; *and* (iv) $A \subset \mathrm{Max}A - S$.

3. Main results

From [17, 22], we introduce the definition of uniformly same-order set-valued mappings.

Definition 3.1. Let X_0 and Y_0 be two nonempty subsets of X and Y, respectively. Let $F : X_0 \times Y_0 \to 2^V$ be a set-valued mapping with nonempty values. $F(x, y)$ is said to be $S(\mathrm{int}S)$-uniformly same-order on X_0 with respect to $y_0 \in Y_0$, if there exists $x_0 \in X_0$ such that

$$F(x_0, y_0) \subset F(x_0, Y_0) + S\backslash\{0_V\}(\mathrm{int}S),$$

then for all $x \in X_0$,

$$F(x, y_0) \subset F(x, Y_0) + S\backslash\{0_V\}(\mathrm{int}S).$$

F is said to be $S(\mathrm{int}S)$-uniformly same-order on X_0 if F is $S(\mathrm{int}S)$-uniformly same-order on X_0 with respect to any $y_0 \in Y_0$. The definition that $F(x, y)$ is said to be $S(\mathrm{int}S)$-uniformly same-order on Y_0 is similar.

The following example is given to show the rationality of the notation of uniformly same-order set-valued mappings.

Example 3.1. Let $X_0 = \{(x_1, x_2) \mid 1 \leq x_i \leq 2(i = 1, 2)\} \subset \mathbb{R}^2$, $Y_0 = \{(y_1, y_2) \mid 1 \leq y_i \leq 2(i = 1, 2)\} \subset \mathbb{R}^2$, $V = \mathbb{R}^2$, $S = \mathbb{R}^2_+$ and $M = \{(u, v) \mid u^2 + v^2 \leq 1\}$. Let

$$f(x, y) = (x_1 y_1, x_2 y_2)$$

and

$$F(x, y) = f(x, y) + M.$$

It is easy to show that F is $S(\mathrm{int}S)$-uniformly same order on $X_0 \times Y_0$.

Remark 3.1. Let u and v be two vector-valued mappings and M be a nonempty subset of V. Obviously, the set-valued mapping $F(x, y) = u(x) + v(y) + M$ is $S(\mathrm{int}S)$-uniformly same-order on X_0 and Y_0, respectively.

Remark 3.2. In [22], Shi and Ling gave a definition of S-uniformly same-order on X_0 for a vector-valued map f. When F reduces to a vector-valued mapping, i.e., $F \equiv f$, we have that Definition 3.1 is a weaker concept than Definition 2.1 in [22]. In fact, if there exists $x_0 \in X_0$ such that

$$f(x_0, y) \in f(x_0, Y_0) + S\backslash\{0_V\}, \quad \forall y \in Y_0.$$

Then, there exists $\bar{y} \in Y_0$ such that

$$f(x_0, y) \in f(x_0, \bar{y}) + S\backslash\{0_V\}, \quad \forall y \in Y_0.$$

By Definition 2.1 in [22], we have that for all $x \in X_0$,

$$f(x,y) \in f(x,\bar{y}) + S\backslash\{0_V\}, \quad \forall y \in Y_0.$$

Naturally, for all $x \in X_0$,

$$f(x,y) \in f(x,Y_0) + S\backslash\{0_V\}, \quad \forall y \in Y_0.$$

So, if f is S-uniformly same-order on X_0 (Y_0) defined in [22], then f also satisfies Definition 3.1. However, the converse may not hold. The following example explains the case.

Example 3.2. Let $X_0 = [1,2] \subset \mathbb{R}$, $Y_0 = [0,1] \subset \mathbb{R}$, $V = \mathbb{R}^2$ and $S = \mathbb{R}^2_+$. Let $f : X_0 \times Y_0 \to V$,

$$f(x,y) = \begin{cases} (1,-4), & \text{if } x = 1, y \in (0,1]; \\ (x^2, -x^2 y), & \text{otherwise.} \end{cases}$$

Obviously, $f(x,y)$ is S-uniformly same-order on Y_0 (Definition 3.1). However, $f(x,y)$ is not S-uniformly same-order on Y_0 defined in [22]. Indeed, let $y_0 = 0$, $x_1 = 2$ and $x_2 = \frac{3}{2}$. By directly computing,

$$f(x_1,y_0) - f(x_2,y_0) = (4,0) - (\frac{9}{4},0) = (\frac{7}{4},0) \in S\backslash\{0_{\mathbb{R}^2}\},$$

and for every $y \in (0,1]$,

$$f(x_1,y) - f(x_2,y) = (4,-4y) - (\frac{9}{4},-\frac{9}{4}y) = (\frac{7}{4},-\frac{7}{4}y) \notin S\backslash\{0_{\mathbb{R}^2}\}.$$

Lemma 3.1. *Let X_0 and Y_0 be two nonempty subsets of X and Y, respectively, and $F : X_0 \times Y_0 \to 2^V$ be a set-valued mapping with nonempty values.*

(i) *If $F(x,y)$ is S-uniformly same-order on X_0 and $F(\hat{x},\hat{y}) \bigcap \mathrm{Min}F(\hat{x},Y_0) \neq \emptyset$, then for each $x \in X_0$, $F(x,\hat{y}) \bigcap \mathrm{Min}F(x,Y_0) \neq \emptyset$.*

(ii) *If $-F(x,y)$ is S-uniformly same-order on Y_0 and $F(\hat{x},\hat{y}) \bigcap \mathrm{Max}F(X_0,\hat{y}) \neq \emptyset$, then for each $y \in Y_0$, $F(\hat{x},y) \bigcap \mathrm{Max}F(X_0,y) \neq \emptyset$.*

Proof. (i) Let $z \in F(\hat{x},\hat{y}) \bigcap \mathrm{Min}F(\hat{x},Y_0)$. If there exists $\bar{x} \in X_0$ such that

$$F(\bar{x},\hat{y}) \bigcap \mathrm{Min}F(\bar{x},Y_0) = \emptyset. \tag{1}$$

By (1), we have that for all $\bar{z} \in F(\bar{x},\hat{y})$, $\bar{z} \notin \mathrm{Min}F(\bar{x},Y_0)$. Thus, there exists $z' \in F(\bar{x},Y_0)$ such that $\bar{z} \in z' + S\backslash\{0_V\}$. Since $F(x,y)$ is S-uniformly same-order on X_0, we have

$$F(\hat{x},\hat{y}) \subset F(\hat{x},Y_0) + S\backslash\{0_V\}. \tag{2}$$

For $z \in F(\hat{x},\hat{y})$, by (2), there exists $\hat{z} \in F(\hat{x},Y_0)$ such that $z \in \hat{z} + S\backslash\{0_V\}$, which contradicts $z \in \mathrm{Min}F(\hat{x},Y_0)$.

(ii) Let $z \in F(\hat{x},\hat{y}) \bigcap \mathrm{Max}F(X_0,\hat{y})$. If there exists $\bar{y} \in Y_0$ such that

$$F(\hat{x},\bar{y}) \bigcap \mathrm{Max}F(X_0,\bar{y}) = \emptyset. \tag{3}$$

By (3), we have that for all $\bar{z} \in F(\hat{x}, \bar{y})$, $\bar{z} \notin \text{Max}F(X_0, \bar{y})$. Thus, there exists $z' \in F(X_0, \bar{y})$ such that $\bar{z} \in z' - S\backslash\{0_V\}$. Since $-F(x, y)$ is S-uniformly same-order on Y_0, we have

$$F(\hat{x}, \hat{y}) \subset F(X_0, \hat{y}) - S\backslash\{0_V\}. \tag{4}$$

For $z \in F(\hat{x}, \hat{y})$, by (4), there exists $\hat{z} \in F(X_0, \hat{y})$ such that $z \in \hat{z} - S\backslash\{0_V\}$, which contradicts $z \in \text{Max}F(X_0, \hat{y})$. □

Similar to the proof of Lemma 3.1, we can get the following lemma.

Lemma 3.2. *Let X_0 and Y_0 be two nonempty subsets of X and Y, respectively, and $F : X_0 \times Y_0 \to 2^V$ be a set-valued mapping with nonempty values.*

(i) *If $F(x, y)$ is* int*S-uniformly same-order on X_0 and*

$$F(\hat{x}, \hat{y}) \bigcap \text{Min}_w F(\hat{x}, Y_0) \neq \emptyset,$$

then for each $x \in X_0$, $F(x, \hat{y}) \bigcap \text{Min}_w F(x, Y_0) \neq \emptyset$.

(ii) *If $-F(x, y)$ is* int*S-uniformly same-order on Y_0 and*

$$F(\hat{x}, \hat{y}) \bigcap \text{Max}_w F(X_0, \hat{y}) \neq \emptyset,$$

then for each $y \in Y_0$, $F(\hat{x}, y) \bigcap \text{Max}_w F(X_0, y) \neq \emptyset$.

In order to obtain existence theorems for cone loose saddle points of S-uniformly same-order set-valued mappings, we introduce the following symbols:

$$A = \{y \in Y_0 \mid F(x, y) \bigcap \text{Min}F(x, Y_0) \neq \emptyset, \forall x \in X_0\};$$

$$B = \{x \in X_0 \mid F(x, y) \bigcap \text{Max}F(X_0, y) \neq \emptyset, \forall y \in Y_0\};$$

$$A_w = \{y \in Y_0 \mid `F(x, y) \bigcap \text{Min}_w F(x, Y_0) \neq \emptyset, \forall x \in X_0\};$$

$$B_w = \{x \in X_0 \mid F(x, y) \bigcap \text{Max}_w F(X_0, y) \neq \emptyset, \forall y \in Y_0\}.$$

The set of all (weakly) S-loose saddle points of the S(int$S$)-uniformly same-order set-valued mapping F with respect to $X_0 \times Y_0$ is denoted by $SP(SP_w)$.

Theorem 3.1. *Let X_0 and Y_0 be two nonempty compact subsets of X and Y, respectively. Suppose that $F : X_0 \times Y_0 \to 2^V$ is a set-valued mapping and the following conditions are satisfied*:

(i) *F is u.s.c. with nonempty compact values*;

(ii) *$F(\cdot, y)$ is S-uniformly same-order on X_0*;

(iii) *$-F(x, \cdot)$ is S-uniformly same-order on Y_0.*

Then, F has a S-loose saddle point and $SP = B \times A$.

Proof. First, we prove that $A \neq \emptyset$ and $B \neq \emptyset$. Since $F(x_0, \cdot)$ is u.s.c. with compact values and Y_0 is compact, by Lemma 2.1, $F(x_0, Y_0)$ is a compact set, for every $x_0 \in X_0$. By Lemma 2.2, $\text{Min}F(x_0, Y_0) \neq \emptyset$. Let $z_0 \in \text{Min}F(x_0, Y_0)$. Then, there exists $y_0 \in Y_0$ such that $z_0 \in F(x_0, y_0)$. Thus, $F(x_0, y_0) \bigcap \mathrm{Min}F(x_0, Y_0) \neq \emptyset$. By the condition (ii) and Lemma 3.1 (i), for all $x \in X_0$,

$$F(x, y_0) \bigcap \mathrm{Min}F(x, Y_0) \neq \emptyset.$$

Therefore, $y_0 \in A$ and $A \neq \emptyset$. Similarly, $B \neq \emptyset$.

Next, we show that $SP = B \times A$. Clearly, $B \times A \subset SP$. Let $(x_0, y_0) \in SP$. Then, $F(x_0, y_0) \bigcap \mathrm{Min}F(x_0, Y_0) \neq \emptyset$ and $F(x_0, y_0) \bigcap \mathrm{Max}F(X_0, y_0) \neq \emptyset$. Therefore, by the conditions (ii) and (iii), and Lemma 3.1, we have that for all $x \in X_0$,

$$F(x, y_0) \bigcap \mathrm{Min}F(x, Y_0) \neq \emptyset$$

and for all $y \in Y_0$,

$$F(x_0, y) \bigcap \mathrm{Max}F(X_0, y) \neq \emptyset.$$

Thus, $y_0 \in A$ and $x_0 \in B$; that is, $SP \subset B \times A$. Hence, $SP = B \times A \neq \emptyset$. This completes the proof. □

Remark 3.3. In [13, 19, 23, 27], some existence results on cone loose saddle points of general set-valued mappings are investigated by applying various fixed point theorems and scalarization functions. However, the method of the proof and the conditions of Theorem 3.1 are different from the corresponding ones in [13, 19, 23, 27], respectively.

Similar to the proof of Theorem 3.1, we can get the following theorem.

Theorem 3.2. *Let X_0 and Y_0 be two nonempty compact subsets of X and Y, respectively. Suppose that $F : X_0 \times Y_0 \to 2^V$ is a set-valued mapping and the following conditions are satisfied*:

(i) *F is u.s.c. with nonempty compact values*;

(ii) *$F(\cdot, y)$ is $\mathrm{int}S$-uniformly same-order on X_0*;

(iii) *$-F(x, \cdot)$ is $\mathrm{int}S$-uniformly same-order on Y_0.*

Then, F has a weakly S-loose saddle point and $SP_w = B_w \times A_w$.

Theorem 3.3. *Let X_0 and Y_0 be two nonempty compact subsets of X and Y, respectively. Suppose that $F : X_0 \times Y_0 \to 2^V$ is a set-valued mapping and the following conditions are satisfied*:

(i) *F is continuous with nonempty compact values*;

(ii) *$F(\cdot, y)$ is $\mathrm{int}S$-uniformly same-order on X_0*;

(iii) *$-F(x, \cdot)$ is $\mathrm{int}S$-uniformly same-order on Y_0.*

Then, there exists $(\bar{x}, \bar{y}) \in X_0 \times Y_0$ such that

$$F(\bar{x}, \bar{y}) \bigcap (\mathrm{Max} \bigcup_{x \in X_0} \mathrm{Min}_w F(x, Y_0) - S) \neq \emptyset$$

and

$$F(\bar{x}, \bar{y}) \bigcap (\mathrm{Min} \bigcup_{y \in Y_0} \mathrm{Max}_w F(X_0, y) + S) \neq \emptyset.$$

Proof. By assumptions and Lemmas 2.1-2.3,

$$\text{Min}\bigcup_{y\in Y_0}\text{Max}_w F(X_0,y)\neq\emptyset \text{ and } \text{Max}\bigcup_{x\in X_0}\text{Min}_w F(x,Y_0)\neq\emptyset.$$

By Theorem 3.2, there exists $(\bar{x},\bar{y})\in X_0\times Y_0$ such that

$$F(\bar{x},\bar{y})\bigcap\text{Min}_w F(\bar{x},Y_0)\neq\emptyset \text{ and } F(\bar{x},\bar{y})\bigcap\text{Max}_w F(X_0,\bar{y})\neq\emptyset.$$

Then,

$$F(\bar{x},\bar{y})\bigcap(\bigcup_{x\in X_0}\text{Min}_w F(x,Y_0))\neq\emptyset \text{ and } F(\bar{x},\bar{y})\bigcap(\bigcup_{y\in Y_0}\text{Max}_w F(X_0,y))\neq\emptyset.$$

By Lemma 2.3, we have

$$F(\bar{x},\bar{y})\bigcap(\text{Max}\bigcup_{x\in X_0}\text{Min}_w F(x,Y_0)-S)\neq\emptyset$$

and

$$F(\bar{x},\bar{y})\bigcap(\text{Min}\bigcup_{y\in Y_0}\text{Max}_w F(X_0,y)+S)\neq\emptyset.$$

This completes the proof. □

Remark 3.4. When F is a real-valued function and $S=R_+$, the conclusions of Theorem 3.3 reduce to

$$\min\bigcup_{y\in Y_0}\max F(X_0,y)\leq\max\bigcup_{x\in X_0}\min F(x,Y_0);$$

namely,

$$\min\bigcup_{y\in Y_0}\max F(X_0,y)=\max\bigcup_{x\in X_0}\min F(x,Y_0).$$

So, Theorem 3.3 is an extension for the minimax theorem of real-valued functions.

In all the sequel of this section, we suppose that X and Y are two metric spaces, and S is a point closed convex cone in R^n with its interior $\text{int}S\neq\emptyset$.

Definition 3.2. ([5]) Let M be a nonempty subset of R^n. M is said to be S-bounded if there exist $z_1,z_2\in V$ such that

$$M\subset(z_1+S)\bigcap(z_2-S).$$

Lemma 3.3. *Let X_0 be a nonempty compact subset of X. Let $F:X_0\to 2^{R^n}$ be a set-valued mapping satisfying $F(x)=f(x)+M$, where $f:X_0\to R^n$ is a vector-valued mapping.*

(i) *If f is a continuous vector-valued mapping on X_0, and M is a nonempty S-bounded closed subset of R^n, then F is u.s.c. on X_0.*

(ii) *If f is a continuous vector-valued mapping on X_0, then F is l.s.c. on X_0.*

Proof. (i) By Proposition 2.1, we only need to prove that for any closed subset G of R^n, the inverse image of G

$$F^{-1}(G) = \{x \in X_0 \mid (f(x) + M) \bigcap G \neq \emptyset\}$$

is closed. Let $x_n \in F^{-1}(G)$ and $x_n \to x_0 \in X_0$. By the definition of $F^{-1}(G)$, $(f(x_n) + M) \bigcap G \neq \emptyset$. Then, for any n, there exists $m_n \in M$ such that $f(x_n) + m_n \in G$. Since M is a S-bounded closed set, there exists a converging subsequence $\{m_{n_k}\}$ of $\{m_n\}$ and $m_{n_k} \to m_0 \in M$. By the closedness of G, $f(x_0) + m_0 \in G$, i.e.,

$$x_0 \in F^{-1}(G) = \{x \in X_0 \mid (f(x) + M) \bigcap G \neq \emptyset\}$$

and hence F is u.s.c. on X_0.

(ii) By Proposition 2.1, we only need to prove that for any closed subset G of R^n, the core of G

$$F^{+1}(G) = \{x \in X_0 \mid f(x) + M \subset G\}$$

is closed. Let $x_n \in F^{+1}(G)$ and $x_n \to x_0 \in X_0$. By the definition of $F^{+1}(G)$, $f(x_n) \in G - m$, for all $m \in M$. Since G is a closed set, $f(x_0) \in G - m$, for all $m \in M$. By the arbitrary of m, $f(x_0) + M \subset G$, i.e.,

$$x_0 \in F^{+1}(G) = \{x \in X_0 \mid f(x) + M \subset G\}$$

and hence F is l.s.c. on X_0. □

The following simple example shows that if M is not S-bounded, Lemma 3.3(i) is not true.

Example 3.3. Let $X = \mathbb{R}$, $V = \mathbb{R}^2$, $X_0 = [-1, 1] \subset X$, $S = \mathbb{R}^2_+$, and $M = \{(0, t) \mid \forall t \in \mathbb{R}\}$. Let $f : X_0 \to V$ and $F : X_0 \to 2^V$,

$$f(x) = (x, 0) \qquad \text{for } x \in [-1, 1]$$

and

$$F(x) = f(x) + M.$$

Obviously, f is continuous, and M is a closed set. But M is not $\mathbb{R}^2_+$-bounded. We claim that F is not u.s.c. for every $x \in X_0$. In fact, for every $t \in [-1, 1]$,

$$N(F(t)) = \{(x, y) : |y| < \frac{1}{|x - t|}\}$$

is a neighborhood of $F(t)$. For all the neighborhood $N(t)$ of t, there exists $t_0 \in N(t)$ such that $F(t_0) \not\subset N(F(t))$. Clearly, by the definition of the u.s.c. of set-valued mappings, F is not u.s.c. for each $t \in [-1, 1]$.

Similarly, in order to obtain existence theorems for cone loose saddle points of the class of set-valued mappings $F(x, y) = u(x) + v(y) + M$, where u and v are two vector-valued mappings and M is a fixed set, we also introduce the following symbols:

$$A'_w = \{y \in Y_0 \mid (v(y) + M) \bigcap \mathrm{Min}_w(v(Y_0) + M) \neq \emptyset\};$$

$$B'_w = \{x \in X_0 \mid (u(x) + M) \bigcap \text{Max}_w(u(X_0) + M) \neq \emptyset\}.$$

The set of all weakly S-loose saddle points of $F(x, y) = u(x) + v(y) + M$ with respect to $X_0 \times Y_0$ is denoted by SP'_w.

Theorem 3.4. *Let X_0 and Y_0 be two nonempty compact subsets of X and Y, respectively. Suppose that $F : X_0 \times Y_0 \to 2^{\mathbb{R}^n}$, $F(x, y) = u(x) + v(y) + M$ and the following conditions are satisfied*:

(i) *u and v are two continuous vector-valued mappings on X_0 and Y_0, respectively.*

(ii) *M is a nonempty S-bounded closed subset of $\mathbb{R}^n$.*

Then, F has a weakly S-loose saddle point and $SP'_w = B'_w \times A'_w$.

Proof. Since M is a nonempty S-bounded closed subset of $\mathbb{R}^n$, M is a compact subset of $\mathbb{R}^n$. Then, by assumptions and Lemma 3.3(i), the condition (i) of Theorem 3.2 holds. By Remark 3.1, the conditions (ii) and (iii) of Theorem 3.2 also hold. Therefore, by Theorem 3.2, the conclusion follows readily. □

Remark 3.5. When $M = \{0_{\mathbb{R}^n}\}$, Theorem 3.4 reduces to Proposition 4.2 in [24].

Similar to the proof of Theorems 3.3 and 3.4, we can get the following theorem.

Theorem 3.5. *Let X_0 and Y_0 be two nonempty compact subsets of X and Y, respectively. Suppose that $F : X_0 \times Y_0 \to 2^{\mathbb{R}^n}$, $F(x, y) = u(x) + v(y) + M$ and the following conditions are satisfied*:

(i) *u and v are two continuous vector-valued mappings on X_0 and Y_0, respectively.*

(ii) *M is a nonempty S-bounded closed subset of $\mathbb{R}^n$.*

Then, there exists $(\bar{x}, \bar{y}) \in X_0 \times Y_0$ such that

$$(u(\bar{x}) + v(\bar{y}) + M) \bigcap ((\text{Max} \bigcup_{x \in X_0} \text{Min}_w F(x, Y_0) - S)) \neq \emptyset$$

and

$$(u(\bar{x}) + v(\bar{y}) + M) \bigcap (\text{Min} \bigcup_{y \in Y_0} \text{Max}_w F(X_0, y) + S) \neq \emptyset.$$

Remark 3.6. When $M = \{0_{R^n}\}$, Theorem 3.5 reduces to the next following result:

$$\exists\ z_1 \in \text{Min} \bigcup_{y \in Y_0} \text{Max}_w F(X_0, y) \quad \text{and} \quad \exists\ z_2 \in \text{Max} \bigcup_{x \in X_0} \text{Min}_w F(x, Y_0)$$

such that

$$z_1 \in z_2 - S.$$

4. Concluding remarks

In this paper, we first introduce a class of set-valued mappings. Then, for this sort of mappings, we investigate the cone loose saddle point theorem and minimax theorem without any hypotheses of convexity, which generalize existing results in the literatures.

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Yu Zhang
College of Mathematics and Statistics
Chongqing University
Chongqing 401331, P. R. China
and
College of Statistics and Mathematics
Yunnan University of Finance and Economics
Kunming, Yunnan, 650221, P. R. China
E-mail address: zhangyu198606@sina.com

Shengjie Li
College of Mathematics and Statistics
Chongqing University
Chongqing 401331, P. R. China
E-mail address: lisj@cqu.edu.cn