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ON PERIODIC P-CONTINUED FRACTION HAVING PERIOD LENGTH ONE

Amara Chandoul, Hela Ben Amar, and Mohamed Mkaouar

Abstract. The aim of this paper is to prove that every quadratic formal power series ω can be expressed as a periodic non-simple continued fraction having period length one.

1. Introduction

Let p be a prime, q be a power of p and $\mathbb{F}_q$ be a field with q elements. Let $\mathbb{F}_q((X^{-1}))$ denote the field of all formal power series $\omega = \sum_{n\geq v} \omega_n X^{-n}$ in an indeterminate X, with ω_n all lying in the field $\mathbb{F}_q$. Recall that $\mathbb{F}_q[X]$ denote the ring of polynomials in X with coefficients in $\mathbb{F}_q$.

For the above formal Laurent series ω, we may assume that $v \neq 0$. Then the integer $v = v(\omega)$ is called the *order* of ω. The *valuation* of ω is defined to be $\mid \omega \mid= q^{-v(\omega)}$. It is well known that $\mid \cdot \mid$ is a non-archimedean valuation on the field $\mathbb{F}_q((X^{-1}))$ and $\mathbb{F}_q((X^{-1}))$ is a complete metric space under the metric ρ defined by $\rho(\omega - \psi) =\mid \omega - \psi \mid$.

For $\omega = \sum_{n\geq n_0} \omega_n X^{-n} \in \mathbb{F}_q((X^{-1}))$, let $[\omega] = \sum_{v\leq n\leq 0} \omega_n X^{-n} \in \mathbb{F}_q[X]$. We call $[\omega]$ the polynomial part of ω. It is evident that the integer $-v(\omega) := -v$ is equal to the degree $\deg [\omega]$ of the polynomial $[\omega]$ provided $v \leq 0$, i.e., $\omega \neq 0$.

Let $D(0,1)$ denote the valuation ideal $X^{-1}\mathbb{F}_q[[X^{-1}]]$ in the ring of formal power series $\mathbb{F}_q[[X^{-1}]]$. It consists of all formal series $\sum_{n\geq 1} \omega_n X^{-n}$.

Let $P \in \mathbb{F}_q[X]$. Consider the following transformation from $D(0,1)$ to $D(0,1)$ defined by

$$T_P(\omega) := \left\{\frac{P}{\omega}\right\},\ T_P(0) := 0.$$

This map describes the P-continued fraction over the field of Laurent series. As in the classical theory, every $\omega \in D(0,1)$ has the following P-continued

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fraction expansion

$$\omega = C_0 + \cfrac{P}{C_1 + \cfrac{P}{C_2 + \cfrac{P}{C_3 + \ddots + \cfrac{P}{C_i + \ddots}}}} = [0; C_1(\omega), C_2(\omega), \ldots]_P, \tag{1.1}$$

where the digits $C_i(\omega)$ are polynomials of strictly positive degree and are defined by

$$\forall i \geq 1,\ C_i(\omega) = \left[\frac{P}{T_P^i \left(\frac{P}{\omega} \right)} \right].$$

It is clear that

$$T_P[0, C_1, \ldots, C_i, \ldots]_P = [0, C_2, \ldots, C_i, \ldots]_P. \tag{1.2}$$

It is easy to convert a simple continued fraction to a P-continued fraction as follows

$$[D_0, D_1, D_2, D_3, \ldots] = [D_0, D_1, D_2, D_3, \ldots]_1 = [D_0, PD_1, D_2, PD_3, \ldots]_P. \tag{1.3}$$

In [1] Burger et al. prove that every real quadratic irrational α can be expressed as a periodic non-simple continued fraction having period length one. Moreover, it is proved that the sequence of rational numbers generated by successive truncations of this expansion is a sequence of convergents of α, For further references on the subject, see also [3], [2] and [4].

In this paper, we extend their results over the field of formal power series by proving that every quadratic formal power series ω can be expressed as a non-simple continued fraction having period length one. We establish our result by extending some of the arguments of [1], in this new context and making appropriate adjustments.

2. Results

Each $\omega \in \mathbb{F}_q((X^{-1}))$ has a continued fraction representation given by $\omega = [a_0; a_1, \ldots]$, $\deg a_i \geq 1$, $\forall i \geq 1$, we refer to the $(\frac{A_n}{B_n})_{n \geq 0}$ as convergents to ω.

A_n and B_n satisfy the recurrence relation:

$$A_{-1} = 1,\ A_0 = D_0,\ A_n = D_n A_{n-1} + A_{n-2} \quad \text{for } n = 1, 2, \ldots, \tag{2.1}$$

$$B_{-1} = 0,\ B_0 = 1,\ \ B_n = D_n B_{n-1} + B_{n-2} \quad \text{for } n = 1, 2, \ldots, \tag{2.2}$$

$$A_n B_{n-1} - B_n A_{n-1} = (-1)^n \quad \text{for } n = 1, 2, \ldots. \tag{2.3}$$

Now, we give the main result of this section.

Theorem 2.1. *Let ω an arbitrary quadratic formal power series written as*

$$\omega = [a_0, \dots, a_{s-1}, \overline{D_0, D_1, \dots, D_{t-1}}].$$

Then ω can be expressed as the period one $(-1)^t R_{t+1}^2$-continued fraction

$$\begin{aligned}\omega = \ &\big[a_0, (-1)^t R_{t+1}^2 a_1, \dots, (-1)^t R_{t+1}^2 a_{s-1},\\ &D_0, (-1)^{t+1} R_{t+1} B_{t+1}, \overline{(-1)^{t+1} R_{t+1} P}\big]_{(-1)^t R_{t+1}^2},\end{aligned}$$

where $P = A_t + B_{t-1}$ and $R_{t+1} = D_0 B_{t+1} - A_{t+1}$.

Remark 2.2. If the preperiod of ω is odd, then

$$\omega = [a_0, \dots, a_{s-1}, \overline{D_0, D_1, \dots, D_{t-1}}] = [a_0, \dots, a_{s-1}, D_0, \overline{D_1, \dots, D_{t-1}, D_0}].$$

Consequently, we can suppose that s is even in Theorem 2.1.

In order to prove this theorem, we need the following lemmas.

Lemma 2.3. *Let $R_i = D_0 B_i - A_i$ for all $i \geq 0$. For all t, the following assertions are satisfied*:

(i) $R_{t-1} A_t + R_{t-2} B_t = (B_{t-2} + A_{t-1}) R_t$.
(ii) $B_{t-1} R_{t-2} - R_{t-1} B_{t-2} = (-1)^t$.
(iii) $A_{t-1} R_{t-2} - R_{t-1} A_{t-2} = D_0 (-1)^t$.
(iv) $D_0 R_{t-1} + R_{t-2} = R_t$.

Proof. Replacing R_i by $D_0 B_i - A_i$ and using (2.3), we get.

(i)
$$\begin{aligned}&R_{t-1} A_t + R_{t-2} B_t - (B_{t-2} + A_{t-1}) R_t\\ &= D_0 (A_t B_{t-1} - A_{t-1} B_t) + (A_t B_{t-2} - A_{t-2} B_t)\\ &= D_0 (-1)^{t-1} + D_0 (-1)^{t-2}\\ &= 0.\end{aligned}$$

(ii) $B_{t-1} R_{t-2} - R_{t-1} B_{t-2} = -B_{t-1} A_{t-2} + A_{t-1} B_{t-2} = (-1)^t$.
(iii) $A_{t-1} R_{t-2} - R_{t-1} A_{t-2} = D_0 (A_{t-1} B_{t-2} - B_{t-1} A_{t-2}) = D_0 (-1)^t$.
(iv)
$$\begin{aligned}D_0 R_{t-1} + R_{t-2} &= D_0 (D_0 B_{t-1} + B_{t-2}) - (D_0 A_{t-1} + A_{t-2}),\\ &= D_0 B_t - A_t\\ &= R_t.\end{aligned}$$
□

Lemma 2.4. *Let $\omega = [\overline{D_0, D_1, \dots, D_{t-1}}]$ be a quadratic formal power series having a purely simple continued fraction of period of length t. Let $(\frac{A_n}{B_n})_n$ be the sequence of the convergents of ω, $P = A_t + B_{t-1}$ and*

$$R_i = D_0 B_i - A_i \quad \text{for all } i \geq 0. \tag{2.4}$$

Then

$$\omega = \frac{D_0 \varphi_P - R_{t-1}}{\varphi_P + R_t},$$

where $\varphi_P = \varphi(P, t)$ is the root of $Y^2 - PY + (-1)^{t+1} = 0$, with $[\varphi_P] = P$.

Proof. Let $\omega = [\overline{D_0, D_1, \ldots, D_{t-1}}]$. Then ω is the unique root of the equation

$$\Lambda(Y) = B_t Y^2 + (B_{t-1} - A_t)Y - A_{t-1}$$

such that $\mid \omega \mid > 1$. Let $\gamma = \frac{D_0\varphi_P - R_{t-1}}{\varphi_P + R_t}$, since $\mid \gamma \mid > 1$, then it is sufficient to show that $\Lambda(\gamma) = 0$ in order to prove that $\gamma = \omega$. We have

$$\begin{aligned}\Lambda(\gamma) &= B_t\Big(\frac{D_0\varphi_P - R_{t-1}}{\varphi_P + R_t}\Big)^2 + (B_{t-1} - A_t)\Big(\frac{D_0\varphi_P - R_{t-1}}{\varphi_P + R_t}\Big) - A_{t-1}\\ &= \frac{\alpha\varphi_P^2 + \beta\varphi_P + \lambda}{(\varphi_P + R_t)^2},\end{aligned}$$

where, according to (2.2),

$$\begin{aligned}\alpha &= B_t D_0^2 + D_0(B_{t-1} - A_t) - A_{t-1}\\ &= D_0(D_0 B_t + B_{t-1}) - (D_0 A_t + A_{t-1})\\ &= D_0 B_{t+1} - A_{t+1}\\ &= R_{t+1},\\ \beta &= -2D_0 R_{t-1} B_t + (B_{t-1} - A_t)(D_0 R_t - R_{t-1}) - 2A_{t-1} R_t\\ &= R_{t-1}(-2D_0 B_t - B_{t-1} + A_t) + R_t(D_0(B_{t-1} - A_t) - 2A_{t-1})\\ &= R_{t-1}(-B_{t+1} - R_t) + R_t(R_{t-1} - A_{t+1})\\ &= -R_{t-1}B_{t+1} - R_t A_{t+1}.\end{aligned}$$

Using Lemma 2.3(i), we obtain

$$\beta = -(B_{t-1} + A_t)R_{t+1} = -PR_{t+1},$$

and

$$\begin{aligned}\lambda &= B_t R_{t-1}^2 - (B_{t-1} - A_t)R_t R_{t-1} - A_{t-1}R_t^2,\\ &= R_{t-1}(B_t R_{t-1} - R_t B_{t-1}) + R_t(A_t R_{t-1} - A_{t-1}R_t).\end{aligned}$$

Lemma 2.3(ii), (iii) and (iv) implies that

$$\begin{aligned}\lambda &= R_{t-1}(-1)^{t+1} + R_t(-1)^{t+1}D_0,\\ &= (-1)^{t+1}(D_0 R_t + R_{t-1}),\\ &= (-1)^{t+1}R_{t+1}.\end{aligned}$$

Finally,

$$\Lambda(\gamma) = \frac{R_{t+1}(\varphi_P^2 - P\varphi_P + (-1)^{t+1})}{(\varphi_P + R_t)^2} = 0. \qquad \square$$

Lemma 2.5. *Let $\omega \in \mathbb{F}_q((X^{-1}))$ such that $\omega = [\overline{D_0, D_1, \ldots, D_{t-1}}]$ and $(\frac{A_n}{B_n})_n$ be the sequence of the convergents of ω. Then*

$$\omega = [D_0, B_{t+1}, \overline{(-1)^{t+1}R_{t+1}P, P}]_{-R_{t+1}} \tag{2.5}$$

and

$$\omega = \Big[D_0, (-1)^{t+1}R_{t+1}B_{t+1}, \overline{(-1)^{t+1}R_{t+1}P}\Big]_{(-1)^t R_{t+1}^2}, \tag{2.6}$$

where $P = A_t + B_{t-1}$.

Proof. Let $\omega = [\overline{D_0, D_1, \ldots, D_{t-1}}]$. Then it is clear, by Lemma 2.4 that

$$\omega = \frac{D_0\varphi_P - R_{t-1}}{\varphi_P + R_t},$$

where φ_P is the root of $Y^2 - PY + (-1)^{t+1} = 0$, with $P = A_t + B_{t-1}$ and $[\varphi_P] = P$. Then, we have

$$\omega = D_0 + \cfrac{-R_{t+1}}{B_{t+1} + \cfrac{(-1)^t}{\varphi_P}}.$$

As $\varphi_P = P + \frac{(-1)^t}{\varphi_P}$, we get

$$\begin{aligned}
\omega &= D_0 + \cfrac{-R_{t+1}}{B_{t+1} + \cfrac{(-1)^t}{\varphi_P}}\\
&= D_0 + \cfrac{-R_{t+1}}{B_{t+1} + \cfrac{-R_{t+1}}{(-1)^{t+1}R_{t+1}P + \cfrac{-R_{t+1}}{P + \cfrac{(-1)^t}{\varphi_P}}}}\\
&= D_0 + \cfrac{-R_{t+1}}{B_{t+1} + \cfrac{-R_{t+1}}{(-1)^{t+1}R_{t+1}P + \cfrac{-R_{t+1}}{P + \cfrac{(-1)^t}{\varphi_P}}}}\\
&= D_0 + \cfrac{-R_{t+1}}{B_{t+1} + \cfrac{-R_{t+1}}{(-1)^{t+1}R_{t+1}P + \cfrac{-R_{t+1}}{P + \cfrac{-R_{t+1}}{(-1)^{t+1}R_{t+1}P + \cfrac{-R_{t+1}}{\varphi_P}}}}}\\
&= \left[D_0, B_{t+1}, \overline{(-1)^{t+1}R_{t+1}P, P}\right]_{-R_{t+1}}.
\end{aligned}$$

In addition, we have

$$\omega = D_0 + \cfrac{-R_{t+1}}{B_{t+1} + \cfrac{-R_{t+1}}{(-1)^{t+1}R_{t+1}P + \cfrac{-R_{t+1}}{P + \cfrac{-R_{t+1}}{(-1)^{t+1}R_{t+1}P + \cfrac{-R_{t+1}}{\varphi_P}}}}}$$

$$= D_0 + \cfrac{(-1)^t R_{t+1}^2}{(-1)^{t+1} R_{t+1} B_{t+1} + \cfrac{(-1)^t R_{t+1}^2}{(-1)^{t+1} R_{t+1} P + \cfrac{(-1)^t R_{t+1}^2}{(-1)^{t+1} R_{t+1} P + \cfrac{(-1)^t R_{t+1}^2}{\varphi_P}}}}$$

$$= \left[D_0, (-1)^{t+1} R_{t+1} B_{t+1}, \overline{(-1)^{t+1} R_{t+1} P}\right]_{(-1)^t R_{t+1}^2}. \qquad \square$$

Proof of Theorem 2.1. Let

$$\omega = [a_0, \dots, a_{s-1}, \overline{D_0, D_1, \dots, D_{t-1}}] \text{ and } \omega^* = [\overline{D_0, D_1, \dots, D_{t-1}}].$$

By Lemma 2.5 and (2.6), we conclude that

$$\omega^* = \left[D_0, (-1)^{t+1} R_{t+1} B_{t+1}, \overline{(-1)^{t+1} R_{t+1} P}\right]_{(-1)^t R_{t+1}^2}. \tag{2.7}$$

Applying (1.3) to ω, we obtain

$$\omega = \left[a_0, (-1)^t R_{t+1}^2 a_1, \dots, (-1)^t R_{t+1}^2 a_{s-1}, \omega^*\right]. \tag{2.8}$$

We have immediately from (2.7) and (2.8),

$$\begin{aligned}\omega = \ &\left[a_0, (-1)^t R_{t+1}^2 a_1, \dots, (-1)^t R_{t+1}^2 a_{s-1},\right.\\ &\left. D_0, (-1)^{t+1} R_{t+1} B_{t+1}, \overline{(-1)^{t+1} R_{t+1} P}\right]_{(-1)^t R_{t+1}^2}.\end{aligned} \qquad \square$$

3. Examples

Example 3.1. Let $\omega \in \mathbb{F}_2((X^{-1}))$ be a quadratic formal power series solution of the equation

$$(X^2 + X + 1)Y^2 + (X^3 + X^2 + X + 1)Y + X^2 + X + 1 = 0.$$

We can prove that the continued fraction expansion of ω is defined as follows

$$\omega = [\overline{X, X+1, X}].$$

Applying Lemma 2.5, we obtain

$$\omega = \left[X, R_4 B_4, \overline{R_4 P}\right]_{-R_4^2},$$

where $P = A_3 + B_2$.

Using the induction formulas (2.1) and (2.2), we obtain

$$\begin{aligned}A_1 &= X^2 + X + 1\\ A_2 &= X^2(X+1)\\ A_3 &= X^3(X+1) + X(X+1) + 1\\ A_4 &= (X^3+1)(X^2+1)\end{aligned}$$

and

$$\begin{aligned} B_1 &= X + 1 \\ B_2 &= X^2 + X + 1 \\ B_3 &= X^3 + X^2 + 1 \\ B_4 &= X^4. \end{aligned}$$

Then

$$\begin{aligned} R_4 &= D_0 B_4 - A_4 \\ &= X^3 + X^2 + 1 \end{aligned}$$

and

$$P = X^3(X + 1).$$

Finally

$$\omega = \Big[X, X^7 + X^6 + X^4, \overline{X^7 + X^6 + X^3 + X}\Big]_{X^6+X^4+1}.$$

Example 3.2. Let $\psi \in \mathbb{F}_2((X^{-1}))$ be a quadratic formal power series solution of the equation

$$(X^2 + X + 1)Y^2 + X^3 Y + X^3 + X^2 + 1 = 0.$$

We can prove that the continued fraction expansion of ψ is defined as follows

$$\psi = [X, X^2 + 1, X^2, \overline{X + 1, X, X + 1}].$$

The preperiod of ψ is odd, so by Remark 2.2

$$\psi = [X, X^2 + 1, X^2, X + 1, \overline{X, X + 1, X}].$$

Let $\psi^* = [\overline{X, X + 1, X}]$. Then $\psi = [X, X^2 + 1, X^2, X + 1, \psi^*]$, and by Example 3.1

$$\psi^* = \Big[X, X^7 + X^6 + X^4, \overline{X^7 + X^6 + X^3 + X}\Big]_{X^6+X^4+1}.$$

Applying Theorem 2.1, we obtain

$$\psi = \Big[X, (X^2 + 1)L, X^2 L, (X + 1)L, X, X^7 + X^6 + X^4, \overline{X^7 + X^6 + X^3 + X}\Big]_L,$$

where $L = X^6 + X^4 + 1$.

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Amara Chandoul
Faculté des Sciences de Sfax
BP 1171, Sfax 3000, Tunisia
E-mail address: amarachandoul@yahoo.fr

Hela Ben Amar
Faculté des Sciences de Sfax
BP 1171, Sfax 3000, Tunisia
E-mail address: Hela.BenAmar@issatgb.rnu.tn

Mohamed Mkaouar
Faculté des Sciences de Sfax
BP 1171, Sfax 3000, Tunisia
E-mail address: mohamed.mkaouar@fss.rnu.tn